

RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

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of x -free modules**

Rendiconti del Seminario Matematico della Università di Padova,
tome 77 (1987), p. 161-176

[<http://www.numdam.org/item?id=RSMUP_1987__77__161_0>](http://www.numdam.org/item?id=RSMUP_1987__77__161_0)

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Chains of Free Modules and Construction of κ -Free Modules.

RADOSLAV DIMITRIĆ (*)

SUMMARY - Examined are special classes $\mathcal{F}_n$ of torsion free modules of projective dimension one over valuation domains, defined inductively as ω_n -smooth unions of $\aleph_n$ -generated free modules where successive quotients belong to $\mathcal{F}_{n-1}$ and with the initial class $\mathcal{F}_0$ defined to consists of all countably generated torsion free modules. Results on the unions of free modules are connected to κ -free modules through the fact that every module in the class $\mathcal{F}_n$ is $\aleph_n$ -free. One of the most important consequences is that there are $\aleph_n$ -free but not $\aleph_{n+1}$ -free modules for every integer n .

1. Introduction.

Throughout the paper our modules are torsion free modules over commutative domains and the majority of the results are established for commutative valuation domains that are not fields. Q as usual denotes the quotient field of R .

$gen\ M$ and $rk\ M$ will denote a generating set of minimal cardinality and a maximal independent set of M respectively.

By a *slice* of a free R -module $F = \bigoplus_{i \in I} Rx_i$ we mean a submodule H of F such that $H = \bigoplus_{i \in J} Rx_i$, $J \subset I$.

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H is a pure submodule of M (denoted $H \leq_* M$) if every system $\sum_{j=1}^m r_{ij}x_j = h_i$ ($h_i \in H, i \in I$) over H with a finite number m of unknowns solvable in M is also solvable in H . In case of Prüfer domains this is equivalent to RD -purity (see Fuchs and Salce [7]) i.e. that every equation $rx = h$ over H having a solution x in M must have a solution in H too.

A pure ascending chain of R -modules M

$$0 = M_0 \leq_* M_1 \leq_* \dots \leq_* M_\alpha \leq_* \dots \leq_* M, \quad \alpha < \mu$$

is called *smooth* or *continuous* if $M = \bigcup_{\alpha < \mu} M_\alpha$ and, for every limit ordinal α , $M_\alpha = \bigcup_{i < \alpha} M_i$. If in addition $\mu = \kappa$ is a regular cardinal and every *gen* $M_\alpha < \kappa$, then we talk about a κ -filtration of M .

If, for a regular κ , $rk M = \{x_\alpha\}_{\alpha < \kappa}$, the κ -filtration obtained by setting $M_\alpha = \langle \{x_i\}_{i < \alpha} \rangle_*$ for every α is called the *rank filtration* of M (it is easy to check the smoothness). Notice that $rk M_{\alpha+1}/M_\alpha = 1$, and that $rk M_\alpha = |\alpha| < \kappa$. Here $\langle X \rangle_*$ denotes the purification of $\langle X \rangle$ in M .

If in a filtration of free modules $\{F_\alpha\}$ every F_α is a slice of $F_{\alpha+1}$, we talk about the *slice filtration*.

By a widespread terminology, call a subset C of a regular cardinal κ a *cub* in κ if C is closed (with respect to sup of ordinals) and unbounded in κ . It is well known that the intersection of fewer than κ cubs in κ is a cub in κ as well as that there is one-to-one correspondence between cubs in κ and normal functions $\varphi: \kappa \rightarrow \kappa$ (strictly increasing and continuous).

The following definition divides all torsion free modules into two disjoint groups:

DEFINITION. A torsion free R -module M is *small* if $|rk M| < |gen M|$. It is *large* if $|rk M| = |gen M|$.

It is easily seen that if M is a small R -module then $|M| = |R|$; that explains the name.

PROPOSITION. There are at most $2^{|R|}$ non-isomorphic small R -modules.

PROOF. A small R -module M has a free resolution $0 \rightarrow H \rightarrow \rightarrow F \rightarrow M \rightarrow 0$, where $|F| = |R|$ (use the remark that $|M| = |R|$).

The total number of subsets of F and the cardinality of all operations making every subset into an R -module, do not exceed $|R|^{|R|} = 2^{|R|}$, so the cardinality of all small modules $F/H \cong M$ is $2^{|R|}$.

A very good study on projective dimension of small modules over valuation domains is [6], where the projective dimension of a small module has been completely determined by the cardinality of its generating system.

Studies on the class of large modules are difficult and require work in several different directions: pure submodules of free modules (over valuation domains) were studied in [2], [3], [7]; κ -free modules in [1], [2]. Before we go into the study of chains of free modules and the classes $\mathcal{F}_\kappa$ we mention a couple more remarks on large modules.

It is easy to show that for the class of R -modules M satisfying $|R| \leq |\text{gen } M|$, every pure submodule N of M satisfies $|\text{gen } N| \leq |\text{gen } M|$. The real problem arises in case of modules M with $|\text{gen } M| < |R| = |M|$ (as in the last proposition, there are $2^{|R|}$ of them). For the case of valuation domains it has been shown to hold true in [6] and we shall use this fact whenever necessary, without special reference.

LEMMA. If a torsion free module over valuation domain is generated by $\leq \kappa$ elements, then so is its every pure submodule.

A very thorough treatment of modules over valuation domains is to be found in an excellent newly published book (Fuchs and Salce [7]).

2. κ -free and $\mathcal{F}_\kappa$ -modules.

From now on R denotes a fixed commutative valuation domain (with unit), not equal to its quotient field Q .

We repeat the definition given in [2]:

DEFINITION 1. For an infinite cardinal κ , an R -module M is κ -free if every (pure) submodule K of M of rank $< \kappa$ can be embedded in a free pure submodule F of M .

The following two lemmas are generalizations of Lemmas 11 and 8 respectively in [1], and were suggested to me by Laszlo Fuchs.

LEMMA 2. If $M = \bigoplus_{i \in I} M_i$, where all M_i ($i \in I$) are κ -free, then M is likewise κ -free.

PROOF. Let $K \leq_* M$, $|rk K| < \kappa$. Since $\pi_i K \leq M_i$ (here π_i are the projections), there are free modules F_i such that $\pi_i K \leq F_i \leq_* M_i$. Now $K \leq \bigoplus \pi_i K \leq \bigoplus F_i = F \leq_* \bigoplus M_i = M$.

LEMMA 3. Every direct summand of a κ -free module is κ -free.

PROOF. Let $M = A \oplus B$ be a κ -free module with $K \leq_* A$ and $|rk K| < \kappa$. By κ -freeness of M there is a free module F_0 such that $K \leq_* F_0 \leq_* M$ and $rk F_0 = rk K$. Inductively define a sequence F_n of free pure submodules of M of rank less than κ , such that $K \leq F_n \leq \pi_A F_n \oplus \pi_B F_n \leq F_{n+1} \leq_* M$ (the first inequality being obvious and the other obtained by the use of κ -freeness of M). By Corollary 3 [3] $F = \bigcup_{n \in \mathbb{N}} F_n$ is a free pure submodule of M of rank $< \kappa$ such that $K \leq F = \pi_A F \oplus \pi_B F$; $\pi_A K = K \leq \pi_A F_n \leq \pi_A F$, $\pi_A F$ is a direct summand of F so $\pi_A F$ is free.

κ -free modules have uniquely determined κ -filtrations, coinciding with rank filtrations:

LEMMA 4. For a regular uncountable cardinal κ and a κ -free module M of rank κ , every κ -filtration on M agrees with its rank filtration.

PROOF. Let

$$0 = M_0 \leq_* M_1 \leq_* \dots \leq_* M_\alpha \leq_* \dots \leq_* M, \quad \alpha < \omega_\nu, \quad |\omega_\nu| = \kappa,$$

be a rank filtration (so every $|rk M_\alpha| < \kappa$). By κ -freeness of M and Proposition 2 [2] $|gen M_\alpha| < \kappa$ so the rank filtration is a κ -filtration. By Lemma 9 [1], every filtration agrees on a cub with it.

The unions of free modules are κ -free:

PROPOSITION 5. For every regular cardinal κ and every M represented as a smooth union of free modules

$$(*) \quad 0 = F_0 \leq_* F_1 \leq_* \dots \leq_* F_\alpha \leq_* \dots \leq_* M, \quad \alpha < \kappa$$

M is κ -free.

PROOF. If $K \leq_* M$ with $|rk K| < \kappa$, then all rank elements of K are contained in an F_α , $\alpha < \kappa$ since $cf \kappa = \kappa > |rk K|$.

LEMMA 6. If, for given $n \geq 1$, M is represented as the smooth union of free modules $(*)$ (where $\kappa = \omega_n$) in a way that there is no cub $C \subset \omega_n$

with the property that, for every $i \in \mathbb{C}$, F_i is a slice of F_{i+1} , then M is $\aleph_n$ -free, but not $\aleph_{n+1}$ -free.

PROOF. By Proposition 5, M is $\aleph_n$ -free. If it were $\aleph_{n+1}$ -free, it would have to be free with the slice filtration. This filtration would have to agree with $(*)$ on a cub (Lemma 9 in [1]), which would be a contradiction.

It is the time to introduce the notion of $\mathcal{F}_n$ modules

DEFINITION 7. The classes $\mathcal{F}_n$ are defined inductively: $\mathcal{F}_0$ consists of all countably generated torsion free R -modules. For $n \geq 1$ $\mathcal{F}_n$ consists of all modules M represented as the smooth union

$$0 = F_0 \leq_* F_1 \leq_* \dots \leq_* F_\alpha \leq_* \dots \leq_* M, \quad \alpha < \omega_n \quad (*)$$

of free modules of rank at most $\aleph_{n-1}$ where, for every $\alpha < \beta < \omega_n$ $F_\beta/F_\alpha \in \mathcal{F}_{n-1}$.

Note that free modules of rank $\leq \aleph_n$ are in $\mathcal{F}_n$, and the question is whether there are some nonfree ones contained in the class. It is also evident that if $M \in \mathcal{F}_n$, then $\text{pd } M \leq 1$, and for $n \geq 1$ $|\text{gen } M| = |\text{rk } M| \leq \aleph_n$ (by Theorem 2.4 in [6]). By Proposition 5, modules in $\mathcal{F}_n$ are $\aleph_n$ -free.

A few results that follow give some properties of $\mathcal{F}_n$ -modules.

PROPOSITION 8. If $A, B \in \mathcal{F}_n$, then $A \oplus B \in \mathcal{F}_n$, for every $n \in \mathbb{N}$.

PROOF. By Definition 7, A and B can be represented as the unions:

$$0 = F_0 \leq_* F_1 \leq_* \dots \leq_* F_\alpha \leq_* \dots \leq_* A, \quad \alpha < \omega_n \text{ and } F_\beta/F_\alpha \in \mathcal{F}_{n-1};$$

$$0 = F'_0 \leq_* F'_1 \leq_* \dots \leq_* F'_\alpha \leq_* \dots \leq_* B, \quad \alpha < \omega_n \text{ and } F'_\beta/F'_\alpha \in \mathcal{F}_{n-1}.$$

The following smooth filtration is evident:

$$0 = F_0 \oplus F'_0 \leq_* F_1 \oplus F'_1 \leq_* \dots \leq_* F_\alpha \oplus F'_\alpha \leq_* \dots \leq_* A \oplus B, \quad \alpha < \omega_n$$

If $\alpha < \beta < \omega_n$ then $(F_\beta \oplus F'_\beta)/(F_\alpha \oplus F'_\alpha) \cong F_\beta/F_\alpha \oplus F'_\beta/F'_\alpha$ and the inductive hypothesis implies that the last module is in $\mathcal{F}_{n-1}$.

The same proof can be used to prove that the direct sum of less than $\aleph_n$ modules in $\mathcal{F}_n$ is likewise a module in $\mathcal{F}_n$.

PROPOSITION 9. Let $0 \rightarrow B \rightarrow A \rightarrow C \rightarrow 0$ be an exact sequence. Then if B and C are in the class $\mathcal{F}_n$ then so is A , for every n .

PROOF. Let us write down the representations of B and C :

(1) $0 = F_0 \leq_* F_1 \leq_* \dots \leq_* F_\alpha \leq_* \dots \leq_* B$, $\alpha < \omega_n$, where all F_α are free modules of rank $\leq \aleph_{n-1}$ and for all $\alpha < \beta < \omega_n$ $F_\beta/F_\alpha \in \mathcal{F}_{n-1}$;

(2) $0 = F'_0 \leq_* F'_1 \leq_* \dots \leq_* F'_\alpha \leq_* \dots \leq_* C$, $\alpha < \omega_n$, with all F'_α free of rank $\leq \aleph_{n-1}$ and, for all $\alpha < \beta < \omega_n$ $F'_\beta/F'_\alpha \in \mathcal{F}_{n-1}$.

Since B and C are at most $\aleph_n$ -generated, so is A and therefore A has the rank filtration

$$(3) \quad 0 = A_0 \leq_* A_1 \leq_* \dots \leq_* A_\alpha \leq_* \dots \leq_* A, \alpha < \omega_n$$

where every $|gen A_\alpha| \leq \aleph_{n-1}$. For $B_\alpha = B \cap A_\alpha$, B_α is at most $\aleph_{n-1}$ -generated since by Corollary 5 [2] $|gen B_\alpha| = |rk B_\alpha| \leq \aleph_{n-1}$. Now, by Lemma 1.2 in [4] $B \cap A_\alpha = F_\alpha$ on a cub $C_1 \ni \alpha$.

$$\bigcup_{\alpha \in C_1} (B + A_\alpha)/B = (B + \bigcup_{\alpha \in C_1} A_\alpha)/B = A/B = C$$

is an $\aleph_n$ -filtration, thus, by the same lemma, there is a cub $C \subset C_1$ with $F'_{\nu(\alpha)} = (B + A_\alpha)/B$ (C is here identified with a normal function $\nu: \omega_n \rightarrow \omega_n$). We can summarize: the filtration (3) satisfies (i) every $|gen A_\alpha| \leq \aleph_{n-1}$, (ii) every $B \cap A_\alpha = F'_{\nu(\alpha)}$ (iii) every $(B + A_\alpha)/B = F'_{\nu(\alpha)}$.

The exact sequence $0 \rightarrow B \cap A_\alpha \rightarrow A_\alpha \rightarrow (B + A_\alpha)/B \rightarrow 0$, is, by (ii) and (iii), an extension of free by free, so every A_α , $\alpha \in C$ is free. For every $\alpha < \beta$ the exact sequence

$$0 \rightarrow ((A_\beta \cap B) + A_\alpha)/A_\alpha \rightarrow A_\beta/A_\alpha \rightarrow A_\beta/((A_\beta \cap B) + A_\alpha) \rightarrow 0$$

is, because of $A_\beta/((A_\beta \cap B) + A_\alpha) \cong (B + A_\beta)/B/(B + A_\alpha)/B$ and $((A_\beta \cap B) + A_\alpha)/A_\alpha \cong (A_\beta \cap B)/(A_\alpha \cap B)$, the same as the following:

$$0 \rightarrow F'_{\nu(\beta)}/F'_{\nu(\alpha)} \rightarrow A_\beta/A_\alpha \rightarrow F'_{\nu(\beta)}/F'_{\nu(\alpha)} \rightarrow 0$$

and the conditions on (1) and (2) imply by the inductive hypothesis that $A_\beta/A_\alpha \in \mathcal{F}_{n-1}$, which, by definition proves that $A \in \mathcal{F}_n$.

Before we prove something about direct summands of $\mathcal{F}_n$ -modules, note that if H is a pure submodule of rank $\leq \aleph_1$ of a free module F , then H is an $\mathcal{F}_1$ -modules. Namely, if $|rk H| = \aleph_0$, then H is free

(see [2], Proposition 2); if $|rk H| = |gen H| = |rk F| = \aleph_1$, then $H \cap F_\alpha = H_\alpha$ is the desired filtration (here $\{F_\alpha\}$ denotes the slice filtration of F).

PROPOSITION 10. If $M = B \oplus C$ is a module in $\mathcal{F}_n$, then $B \in \mathcal{F}_n$.

PROOF. Let M be the union of type $(*)$ (Definition 7) satisfying the conditions thereof and define $B_\alpha = B \cap F_\alpha$, $C_\alpha = C \cap F_\alpha$. Obviously $B_\alpha \leq_* B$, $C_\alpha \leq_* C$ and we may consider the following filtration

$$0 = B_0 \oplus C_0 \leq_* B_1 \oplus C_1 \leq_* \dots \leq_* B_\alpha \oplus C_\alpha \leq_* \dots \leq_* M, \quad \alpha < \omega_n.$$

It is smooth $\aleph_n$ -filtration of $M = \bigcup_{\alpha < \omega_n} (B_\alpha \oplus C_\alpha)$ and therefore it agrees with $(*)$ on a cub $\mathcal{C}$ i.e. for every $\alpha \in \mathcal{C}$ $B_\alpha \oplus C_\alpha = F_\alpha$. Thus for every $\alpha \in \mathcal{C}$, B_α is free of rank $\aleph_{n-1}$ and, since for every $\beta > \alpha$ $F_\beta/F_\alpha = (B_\beta \oplus C_\beta)/(B_\alpha \oplus C_\alpha) \cong B_\beta/B_\alpha \oplus C_\beta/C_\alpha \in \mathcal{F}_{n-1}$, by inductive hypothesis $B_\beta/B_\alpha \in \mathcal{F}_{n-1}$, which proves that $B \in \mathcal{F}_n$.

3. Construction and uniqueness of $\mathcal{F}_n$ -modules.

The following lemma seems to be crucial for the results in this paper.

LEMMA 11. For an integer n assume that

$(*)$ $0 = M_0 \leq_* M_1 \leq_* \dots \leq_* M_\alpha \leq_* \dots \leq_* M$, $\alpha < \mu < \omega_n$ (smooth), where every M_α is a smooth union of free modules:

$(**)$ $0 = F_{\alpha_0} \leq_* F_{\alpha_1} \leq_* \dots \leq_* F_{\alpha_i} \leq_* \dots \leq_* M_\alpha$, $i < \omega_n$ (smooth) where for every i and α $|rk F_{\alpha_i}| \leq \aleph_{n-1}$ and there is a set $S(\alpha) = \{\beta | \alpha < \beta < \mu < \omega_n\} \neq \emptyset$ such that, for every $\alpha < \mu$ and every $\beta \in S(\alpha)$ M_β/M_α is a smooth union of free modules:

$$(***) \quad 0 = F_0^{\beta\alpha} \leq_* F_1^{\beta\alpha} \leq_* \dots \leq_* F_\gamma^{\beta\alpha} \leq_* \dots \leq_* M_\beta/M_\alpha,$$

$\gamma < \omega_n$ (smooth) such that, for every $\alpha < \mu$, $\beta \in S(\alpha)$ and $\gamma < \omega_n$ $|rk F_\gamma^{\beta\alpha}| \leq \aleph_{n-1}$.

Then given a submodule H of M of rank $\leq \aleph_{n-1}$, there is a free pure submodule $\bar{H}$ of M containing H such that:

(1) for every $\alpha < \mu < \omega_n$ $\bar{H} \cap M_\alpha = F_{\alpha_i}$, for some $i = i(\alpha) < \omega_n$;

(2) for every $\alpha < \mu < \omega_n$ and every $\beta \in S(\alpha)$ there is a $\gamma = \gamma(\alpha, \beta)$ such that $((M_\beta \cap \bar{H}) + M_\alpha)/M_\alpha = F_\gamma^{\beta\alpha}$;

(3) $\bar{H}$ is at most $\aleph_{n-1}$ -generated; (4) for every $\alpha < \mu < \omega_n$ $\bar{H} + M_\alpha \leq_* M$.

PROOF. Let us assume that we have already achieved (1) and (2). $H = \bigcup_{\alpha < \mu < \omega_n} (\bar{H} \cap M_\alpha)$, so by (1) and (**) it is an $\aleph_{n-1}$ -union of rank $\aleph_{n-1}$ free modules and therefore it has to be $\aleph_{n-1}$ -generated and thus we have (3). Now, for every α and $\beta \in S(\alpha)$ $(\bar{H} \cap M_\beta)/(\bar{H} \cap M_\alpha) = ((M_\beta \cap \bar{H}) + M_\alpha)/M_\alpha = F_\gamma^{\beta\alpha}$ (by (2)) and this shows that $\bar{H}$ is free. $\bar{H}$ is the ascending union of pure submodules in M : $\bar{H} = \bigcup_{\alpha < \mu} F_{\alpha i(\alpha)}$, so $\bar{H}$ has to be pure in M too. We also get (4): for every $\alpha < \mu < \omega_n$ and $\beta \in S(\alpha)$ $(M_\beta \cap \bar{H}) + M_\alpha)/M_\alpha = F_\gamma^{\alpha\beta} \leq_* M_\beta/M_\alpha$ hence

$$M_\beta/((M_\beta \cap \bar{H}) + M_\alpha) \cong (M_\beta + \bar{H})/(M_\alpha + \bar{H})$$

is torsion free therefore for every α and $\beta \in S(\alpha)$ $M_\alpha + \bar{H} \leq_* M_\beta + \bar{H}$ so there is a cub $C \subset \omega_n$ such that for every α , $M_\alpha + \bar{H} \leq_* \bigcup_{\beta \in C} (M_\beta + \bar{H}) = \cup M_\beta + \bar{H} = M + \bar{H} = M$.

Now we concentrate on two processes aimed to satisfy (1) and (2) respectively, whose combination will yield $\bar{H}$:

(a) for every $\alpha < \mu < \omega_n$, $|rk(H \cap M_\alpha)| \leq \aleph_{n-1}$, so there is an $i < \omega_n$ with $H \cap M_\alpha \leq F_{\alpha i(\alpha)}$, $H_2 = \langle H, \{F_{\alpha i(\alpha)}\}_{\alpha < \mu} \rangle$ is of rank $\aleph_{n-1}$, so we repeat the same and inductively construct a sequence of free modules of rank $\leq \aleph_{n-1}$: $H \leq H_2 \leq \dots \leq H_m \leq \dots$, $m < \omega_0$, where for every α and every m , $H_m \cap M_\alpha \leq F_{\alpha i_m(\alpha)} \leq H_{m+1} \cap M_\alpha$. The union $H_* = \bigcup_{m < \omega_0} H_m$ is of rank $\aleph_{n-1}$ and $H_* \cap M_\alpha = F_{\alpha i(\alpha)}$, for every $\alpha < \mu$ and some $i(\alpha) < \omega_n$.

(b) for every $\alpha < \mu < \omega_n$ and every $\beta \in S(\alpha)$, $((M_\beta \cap H) + M_\alpha)/M_\alpha$ is of rank at most $\aleph_{n-1}$ submodule of M_β/M_α so it is contained in an

$$F_{\gamma_1}^{\beta\alpha} = (F_{\gamma_1}'^{\beta\alpha} \oplus M_\alpha)/M_\alpha \quad (F_{\gamma_1}^{\beta\alpha} \cong F_{\gamma_1}'^{\beta\alpha} \leq_* M_\beta).$$

Define $H_2 = \langle H, \{F_{\gamma_1}'^{\beta\alpha}\}_{\beta \in S(\alpha)}^{\alpha < \mu < \omega_n} \rangle$ which is of rank at most $\aleph_{n-1}$. Now H_2 takes place of H and we have $F_{\gamma_1}^{\beta\alpha} \leq ((M_\beta \cap H_2) + M_\alpha)/M_\alpha \leq F_{\gamma_2}^{\beta\alpha} = (F_{\gamma_2}'^{\beta\alpha} \oplus M_\alpha)/M_\alpha$ and inductively $H_n = \langle H_{n-1}, \{F_{\gamma_{n-1}}'^{\beta\alpha}\}_{\beta \in S(\alpha)}^{\alpha < \mu} \rangle$, which

is a rank $\aleph_{n-1}$ submodule of M satisfying

$$F_{\gamma_{n-1}}^{\beta\alpha} \leq ((M_\beta \cap H_n) + M_\alpha)/M_\alpha \leq F_{\gamma_n}^{\beta\alpha}, \quad n < \omega_0.$$

Define $\bigcup_{n \in \mathbb{N}} H_n = H^*$, $\gamma = \sup \gamma_n$. By smoothness of the chains involved, we have $F_\gamma^{\beta\alpha} = ((M_\beta \cap H^*) + M_\alpha)/M_\alpha$. Now, the union $\bar{H}$ of the sequence $H \leq H_* \leq (H_*)^* \leq ((H_*)^*)_* \leq \dots \leq \bar{H}$ gives the desired submodule.

REMARKS. Often used cases are when $S(\alpha) = \{\alpha + 1\}$ and when $(*)$ is built up from free modules with the slice filtration in $(**)$. Note also that the module satisfying the conditions of the lemma is obviously $\aleph_n$ -free.

The following three results (Theorem 13, Proposition 12, Lemma 14) are proved all together by induction on n by the following scheme

$$\begin{aligned} P(n-1) + T(n-1) &\Rightarrow T(n), \\ P(n-1) + L(n) &\Rightarrow P(n), \\ P(n-1) + T(n-1) &\Rightarrow L(n), \end{aligned}$$

where the initial steps of induction are $T(0)$ (by Corollary 3, [3]), $P(0)$ (obvious).

THEOREM 13. If M is a smooth union of free modules

$$0 = F_0 \leq_* \dots \leq_* F_\alpha \leq_* \dots \leq_* M, \quad \alpha < \mu, \quad |\mu| \leq \aleph_n,$$

such that for every $\alpha < \mu$ $F_{\alpha+1}/F_\alpha \in \mathcal{F}_n$, then M is free.

PROOF. The proof is by induction on n : first note that by $P(n-1)$ we may assume that $\mu \leq \omega_n$. For $n = 0$ Corollary 3 in [3] suffices and for $n = 1$ we can prove it by Theorem 10 [2] and Proposition 5.

If $|\text{gen } M| < \aleph_n$, it is evident that M is free (Proposition 5). Assume $|\text{gen } M| = \aleph_n$ and distinguish two cases:

1) Case $\mu < \omega_n$

We build up a smooth chain of free modules

$$0 = E_0 \leq_* E_1 \leq_* \dots \leq_* E_\beta \leq_* \dots \leq_* M, \quad \beta < \omega_n,$$

where for every $\alpha < \mu < \omega_n$ and every $\beta < \omega_n$ we assure

(1) E_β is at most $\aleph_{n-1}$ -generated; (2) $E_\beta \cap F_\alpha = E_{\beta\alpha}$ is a slice of F_α ; (3) there is a $\gamma = \gamma(\alpha, \beta)$ such that $((F_{\alpha+1} \cap E_\beta) + F_\alpha)/F_\alpha = F_\gamma^{\beta\alpha}$ ($F_\gamma^{\beta\alpha}$ is taken from the chain (***) in Lemma 11 since, in this case $S(\alpha) = \{\alpha + 1\}$); (4) $M = \bigcup_{\beta < \omega_n} E_\beta$.

The build-up of this chain is by transfinite induction in β : if β is isolated, take $H = \langle E_{\beta-1}, a \rangle$ ($a \in M \setminus E_{\beta-1}$) and define $E_\beta = \bar{H}$ according to Lemma 11. For limit β let $E_\beta = \bigcup_{\gamma < \beta < \omega_n} E_\gamma$. Now each E_β is free since $E_\beta = \bigcup_{\alpha < \mu < \omega_n} (E_\beta \cap F_\alpha) = \bigcup_{\alpha < \mu < \omega_n} E_{\beta\alpha}$ is the union of free modules, where every $E_{\beta, \alpha+1}/E_{\beta, \alpha} \cong ((F_{\alpha+1} \cap E_\beta) + F_\alpha)/F_\alpha = F_\gamma^{\beta\alpha}$ is a free module. We still have to prove that $E_{\beta+1}/E_\beta$ is free, which will give the freeness of M :

$$E_{\beta+1}/E_\beta = \bigcup_{\alpha < \mu < \omega_n} ((E_{\beta+1} \cap F_\alpha) + E_\beta)/E_\beta,$$

where every

$$((E_{\beta+1} \cap F_\alpha) + E_\beta)/E_\beta \cong E_{\beta+1, \alpha}/E_{\beta, \alpha}$$

is free by (2). Also

$$\begin{aligned} ((E_{\beta+1} \cap F_{\alpha+1}) + E_\beta)/E_\beta / ((E_{\beta+1} \cap F_\alpha) + E_\beta)/E_\beta &\cong \\ &\cong ((E_{\beta+1} \cap F_{\alpha+1}) + F_\alpha) / ((E_\beta \cap F_{\alpha+1}) + F_\alpha) \cong F_{\gamma_1}^{\beta\alpha} / F_{\gamma_2}^{\beta\alpha} \in \mathcal{F}_{n-1}. \end{aligned}$$

Now, the inductive hypothesis for this theorem gives $E_{\beta+1}/E_\beta$ free, and this proves that M is free.

2) Case $\mu = \omega_n$

As in the case 1) we build up a smooth chain

$$0 = E_0 \leq_* E_1 \leq_* \dots \leq_* E_\beta \leq_* \dots \leq_* M, \quad \beta < \omega_n$$

of pure free submodules E_β of M such that

- (1) E_β is at most $\aleph_{n-1}$ -generated;
- (2) $E_\beta \leq_* F_\beta$; (3) if β is isolated, then E_β is a slice of F_β ;
- (4) for every $\beta < \omega_n$ and every $\alpha < \beta$, $E_\beta \cap F_\alpha = E_{\beta\alpha}$ is a slice of F_α ;

(5) for every α and every $\beta > \alpha$ there is a $\gamma(\alpha, \beta)$ such that

$$((E_\beta \cap F_{\alpha+1}) + F_\alpha)/F_\alpha = F_{\gamma'}^{\beta\alpha};$$

$$(6) \quad M = \bigcup_{\beta < \omega_n} E_\beta.$$

The construction of this chain is similar to that in 1) — by the use of Lemma 11, since for every $\beta < \omega_n$, cf $\beta \leq \omega_{n-1}$. In order to get (6), order a generating set of M : $\text{gen } M = \{a_\alpha | \alpha < \omega_n\}$ and, in the construction, include a_α in $E_{\alpha+1}$ which gives $a_\alpha \in E_\beta$ for all $\beta > \alpha$.

Using (3) and the reasoning as in 1) we prove that E_β is free. In order to show that $E_{\beta+1}/E_\beta$ is free consider the following exact sequence:

$$0 \rightarrow (E_{\beta+1} \cap F_\beta)/E_\beta \rightarrow E_{\beta+1}/E_\beta \rightarrow E_{\beta+1}/(E_{\beta+1} \cap F_\beta) \rightarrow 0$$

$(E_{\beta+1} \cap F_\beta)/E_\beta$ is free: if β is isolated it is evident by (3) and (4); if β is limit, then $(E_{\beta+1} \cap F_\beta)/E_\beta$ is an ascending union of pure submodules

$$((E_{\beta+1} \cap F_\alpha) + E_\beta)/E_\beta \quad (\alpha < \beta < \omega_n),$$

that are free by (4). As in the first case, the quotient of consecutive terms is in $\mathcal{F}_{n-1}$ so the inductive hypothesis for this theorem proves freeness of $(E_{\beta+1} \cap F_\beta)/E_\beta$.

The freeness of $E_{\beta+1}/(E_{\beta+1} \cap F_\beta)$ is proved by the following: by (3) $F_{\beta+1} = E_{\beta+1} \oplus F'_1$ (slice) and by (4) $F_\beta = (E_{\beta+1} \cap F_\beta) \oplus F'_2$ (slice); $E_{\beta+1} + F_\beta = E_{\beta+1} \oplus F'_2 \leq F_{\beta+1} = E_{\beta+1} \oplus F'_1$ thus $F'_2 \leq F'_1$.

$$F_{\beta+1}/F_\beta = (E_{\beta+1} \oplus F'_1)/((E_{\beta+1} \cap F_\beta) \oplus F'_2) \cong E_{\beta+1}/(E_{\beta+1} \cap F_\beta) \oplus F'_1/F'_2.$$

By the hypothesis $F_{\beta+1}/F_\beta \in \mathcal{F}_n$ so, by Proposition 9, $E_{\beta+1}/(E_{\beta+1} \cap F_\beta) \in \mathcal{F}_n$, which, by Proposition 6 and (1) implies the desired freeness.

Finally, we may claim that $E_{\beta+1}/E_\beta$ is free and this implies that M is free.

PROPOSITION 12. Given a smooth union

$$0 < M_0 \leq_* M_1 \leq_* \dots \leq_* M_\alpha \leq_* \dots \leq_* M, \quad \alpha < \mu, \quad |\mu| \leq \aleph_n,$$

with every M_α , $M_{\alpha+1}/M_\alpha \in \mathcal{F}_n$; then $M \in \mathcal{F}_n$.

PROOF. For $n = 0$ this is obviously true and we may suppose that we have proved it for all $k < n$. Notice that Theorem 13 holds for all $k < n$. We will first prove the lemma in the following section.

4. Existence of nonfree $\mathcal{F}_n$ -modules.

LEMMA 14. Given a smooth union of free modules

$$0 = F_0 \leq_* F_1 \leq_* \dots \leq_* F_\alpha \leq_* \dots \leq_* M, \quad \alpha < \mu, \quad |\mu| \leq \omega_n$$

such that every $F_{\alpha+1}/F_\alpha \in \mathcal{F}_{n-1}$. Then $M \in \mathcal{F}_n$.

PROOF. We may assume $\mu \leq \omega_n$, since if $\mu \leq \omega_n$ and the conditions of this lemma are retained by Proposition 12 (for $n - 1$) and Proposition 9. If $\mu < \omega_n$ then $|\mu| \leq \aleph_{n-1}$ and we employ Theorem 13 (for $n - 1$) to conclude that M is free and therefore belongs to $\mathcal{F}_n$.

Let us prove that for every $\beta > \alpha$ $F_\beta/F_\alpha \in \mathcal{F}_{n-1}$, by transfinite induction in β . If $\beta = \alpha + 1$, then this is the hypothesis. If β is another isolated ordinal, then $0 \rightarrow F_{\beta-1}/F_\alpha \rightarrow F_\beta/F_\alpha \rightarrow F_\beta/F_{\beta-1} \rightarrow 0$ and Proposition 9 give the result. If β is a limit ordinal, then F_β/F_α is an ascending union of modules $F_{\beta'}/F_\alpha$, ($\alpha < \beta' < \beta < \omega_n$) that are in $\mathcal{F}_{n-1}$, where

$$F_{\beta'+1}/F_\alpha/F_{\beta'}/F_\alpha \cong F_{\beta'+1}/F_{\beta'} \in \mathcal{F}_{n-1}.$$

Since $|\beta| \leq \aleph_{n-1}$, we apply Proposition 12 (for $n - 1$) to get $F_\beta/F_\alpha \in \mathcal{F}_{n-1}$ i.e. $M \in \mathcal{F}_n$.

We go back now to the proof of Proposition 12. As usual we will distinguish two cases since we may again assume $\mu \leq \omega_n$ (namely, if this proposition is false and μ is the smallest ordinal for which this proposition is not true (so $M \notin \mathcal{F}_n$), then any smooth cofinal subchain of the given chain has the conditions on the chain retained).

Given conditions imply the existence of the following smooth chains of free modules, for every α :

$$0 = F_{\alpha_0} \leq_* F_{\alpha_1} \leq_* \dots \leq_* F_{\alpha_i} \leq_* \dots \leq_* M_\alpha \quad i < \omega_n$$

where all F_{α_i} are free of rank at most $\aleph_{n-1}$ and every $F_{\alpha_{i+1}}/F_{\alpha_i} \in \mathcal{F}_{n-1}$, thus, by Lemma 14 (for n) $F_{\alpha_j}/F_{\alpha_i} \in \mathcal{F}_{n-1}$. We have yet another chain

$$0 = F_0^\alpha \leq_* F_1^\alpha \leq_* \dots \leq_* F_i^\alpha \leq_* \dots \leq_* M_{\alpha+1}/M_\alpha, \quad i < \omega_n, \quad \text{for all } \alpha,$$

where all F_i^α are free of rank at most $\aleph_{n-1}$ and $F_{i+1}^\alpha/F_i^\alpha \in \mathcal{F}_{n-1}$ and, similarly, $F_j^\alpha/F_i^\alpha \in \mathcal{F}_{n-1}$ for every $j > i$.

Case 1) $\mu < \omega_n$.

As in the proof of Theorem 13 for this case, we build up a smooth chain of free modules $0 = E_0 \leq_* E_1 \leq_* \dots \leq_* E_\beta \leq_* \dots \leq_* M$, $\beta < \omega_n$ with

- (1) $E_\beta \cap M_\alpha = F_{\alpha_i}$ for some $i = i(\alpha, \beta) < \omega_n$
- (2) there is a $\gamma = \gamma(\alpha, \beta) < \omega_n$ such that

$$((M_{\alpha+1} \cap E_\beta) + M_\alpha)/M_\alpha = F_\gamma^\alpha$$
- (3) E_β is at most $\aleph_{n-1}$ -generated (4) $M = \bigcup_{\beta < \omega_n} E_\beta$.

Now $E_{\beta+1}/E_\beta$ is an $\aleph_{n-1}$ -union of free modules

$$((E_{\beta+1} \cap M_\alpha) + E_\beta)/E_\beta, \quad \alpha < \mu < \omega_n,$$

where all

$$((E_{\beta+1} \cap M_{\alpha+1}) + E_\beta)/E_\beta / ((E_{\beta+1} \cap M_\alpha) + E_\beta)/E_\beta \cong F_j^\alpha/F_i^\alpha \in \mathcal{F}_{n-1}.$$

We derive by Theorem 13 for $n-1$ that $E_{\beta+1}/E_\beta$ is free and so is M i.e. $M \in \mathcal{F}_n$ (or, better yet, by $P(n-1)$, $E_{\beta+1}/E_\beta \in \mathcal{F}_{n-1}$ and use of $L(n)$ get $M \in \mathcal{F}_n$ and, by Proposition 5, M has to be free).

Case 2) $\mu = \omega_n$.

This one is again very similar with the corresponding one in the proof of Theorem 13: a smooth chain of pure free submodules is built.

$$0 = E_0 \leq_* E_1 \leq_* \dots \leq_* E_\beta \leq_* \dots \leq_* M, \quad \beta < \omega_n$$

with (1) $|gen E_\beta| \leq \aleph_{n-1}$ (2) $E_\beta \leq_* M_\beta$ (3) if β is isolated, then $E_\beta = F_{\beta_i} \leq_* M_\beta$ (4) for every $\beta < \omega_n$, $\alpha < \beta$ $E_\beta \cap M_\alpha = F_{\alpha_j} \leq_* M_\alpha$ (5) for every $\beta < \omega_n$, $\alpha < \beta$ there is a γ such that

$$((E_\beta \cap M_{\alpha+1}) + M_\alpha)/M_\alpha = F_\gamma^\alpha \leq_* M_{\alpha+1}/M_\alpha$$

$$(6) \quad M = \bigcup_{\beta < \omega_n} E_\beta.$$

The construction and the proof that E_β are free are the same as in Theorem 13. To prove that $E_{\beta+1}/E_\beta \in \mathcal{F}_{n-1}$ used is the exact sequence

$$0 \rightarrow (E_{\beta+1} \cap M_\beta)/E_\beta \rightarrow E_{\beta+1}/E_\beta \rightarrow E_{\beta+1}/(E_{\beta+1} \cap M_\beta) \rightarrow 0.$$

We prove that the end terms belong to $\mathcal{F}_{n-1}$.

$$(E_{\beta+1} \cap M_\beta)/E_\beta \in \mathcal{F}_{n-1}:$$

If β is an isolated ordinal, it is evident; if β is a limit ordinal, then $(E_{\beta+1} \cap M_\beta)/E_\beta$ is an $\aleph_{n-1}$ -union of modules $((E_{\beta+1} \cap M_\alpha) + E_\beta)/E_\beta$ ($\alpha < \beta < \omega_n$) that are in $\mathcal{F}_{n-1}$. The quotient of the successive terms is in $\mathcal{F}_{n-1}$ and it is enough to apply $P(n-1)$ to get $(E_{\beta+1} \cap M_\beta)/E_\beta \in \mathcal{F}_{n-1}$. $E_{\beta+1}/(E_{\beta+1} \cap M_\beta)$ is free since, by substituting $\beta \rightarrow \beta + 1$ and $\alpha \rightarrow \beta$ in (5) and using (2) one gets

$$\begin{aligned} ((E_{\beta+1} \cap M_{\beta+1}) + M_\beta)/M_\beta &\cong (E_{\beta+1} \cap M_{\beta+1})/(E_{\beta+1} \cap M_\beta) = \\ &= E_{\beta+1}/(E_{\beta+1} \cap M_\beta) \cong F_\gamma^\beta. \end{aligned}$$

Thus every $E_{\beta+1}/E_\beta \in \mathcal{F}_{n-1}$.

Now, by $L(n)$ we get $M \in \mathcal{F}_n$.

Finally, we give the result we were looking for:

THEOREM 15. For every $n \geq 0$ there is an $M \in \mathcal{F}_n$ that is not free and therefore not $\aleph_{n+1}$ -free.

PROOF. The proof runs by induction on n . If M is any countably generated submodule of the quotient field Q , then M is of rank one and therefore $\text{pd } M = 1$ by Theorem 2.1 in [6]. Suppose we have the claim for all integers less than n . We build up a chain of free modules of rank $\aleph_{n-1}$: $0 = F_0 \leq_* \dots \leq_* F_\alpha \leq_* \dots \leq_* \bigcup F_\alpha = M$, $\alpha < \mu$ $|\mu| \leq \aleph_n$, such that for every $\alpha < \mu$, $F_{\alpha+1}/F_\alpha \in \mathcal{F}_{n-1}$ but not free. The construction is by transfinite induction in $\alpha < \omega_n$.

1) If α is isolated, take $M_\alpha \in \mathcal{F}_{n-1}$ which is not free and consider its free resolution:

$$0 \rightarrow F'_{\alpha-1} \rightarrow F_\alpha \rightarrow M_\alpha \rightarrow 0;$$

$F'_{\alpha-1}$ is free since $\text{pd } M_\alpha = 1$. $|\text{gen } M_\alpha| = |\text{gen } F_\alpha| = \aleph_{n-1}$. But also $|\text{gen } F'_{\alpha-1}| = \aleph_{n-1}$; namely if $|\text{gen } F'_{\alpha-1}| \leq \aleph_{n-2}$, then it is contained in

an at most $\aleph_{n-2}$ -generated slice F' of F_α therefore

$$M_\alpha \cong F_\alpha / F'_{\alpha-1} = (F' \oplus F'') / F'_{\alpha-1} \cong (F' / F'_{\alpha-1}) \oplus F'',$$

where $|\text{gen}(F' / F'_{\alpha-1})| < \aleph_{n-1}$ and Lemma 3 (or Propositions 5 and 10) implies that $F' / F'_{\alpha-1}$ is free, which contradicts the choice of M_α . Thus $F_{\alpha-1}$ may as well be identified with a submodule $F'_{\alpha-1}$ of F_α .

2) For a limit ordinal α define $F_\alpha = \bigcup_{\nu < \alpha} F_\nu$. This is an $\aleph_{n-1}$ -union of free modules where every $F_{\nu+1} / F_\nu \in \mathcal{F}_{n-1}$ and, by Theorem 13, F_α must be free. Now $M = \bigcup_{\alpha < \omega_n} F_\alpha \in \mathcal{F}_n$ by Lemma 14, but it cannot be free since, in that case, its $\aleph_n$ -filtration $\{F_\alpha\}_{\alpha < \omega_n}$ would have to agree with the slice filtration on a cub, which contradicts the choice of $M_\alpha \cong F_{\alpha+1} / F_\alpha$.

At the end we improve the result in Proposition 8

PROPOSITION 16. The direct sum of at most $\aleph_n$ modules that belong to $\mathcal{F}_n$, also belongs to $\mathcal{F}_n$.

PROOF. Let $M = \bigoplus_{i < \omega_n} M_i$, where every $M_i \in \mathcal{F}_n$ and let $H_\alpha = \bigoplus_{i < \alpha} M_i$. By Proposition 8 every $H_\alpha \in \mathcal{F}_n$, since $|\alpha| \leq \aleph_{n-1}$ and $H_{\alpha+1} / H_\alpha \cong M_\alpha \in \mathcal{F}_n$ and the use of Proposition 12 implies $M \in \mathcal{F}_n$.

It is worth pointing out that the results of this kind are the same as those obtained for abelian groups by Paul Hill in [8] and Paul Eklof in [5] (by the use of set theoretic methods and «nonclassical» axioms). Though the case of modules over valuation domains is quite different from that of abelian groups (to mention only κ -free modules to support this), the work with torsion free modules and pure submodules as well as modules of projective dimension one gives a great chance of inheritance of abelian group properties to modules over valuation domains.

At the end we note that, while we explore a notion of κ -freeness in terms of purity and rank, there exist other definitions of κ -freeness (specially in case of abelian groups); we refer to A. L. POPE, *Almost-Free Groups in Varieties*, J. Algebra, — (1984), pp. 36-52.

I would like to thank the referee for making several useful suggestions.

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Manoscritto pervenuto in redazione il 3 dicembre 1985.