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The Dirichlet Problem in Half-Space for Elliptic Equations with Unbounded Coefficients.

J. H. CHABROWSKI (*)

Introduction.

Let $R_n^+ = \{x; x \in R_n, x_n > 0\}$, for a point $x \in R_n^+$ we write $x = (x', x_n)$, where $x' \in R_{n-1}$. In this paper we consider the Dirichlet problem

$$(1) \quad Lu + \lambda u = \sum_{i,j=1}^n D_i(a_{ij}(x) D_j u) + \sum_{i=1}^n b_i(x) D_i u + \\ + (c(x) + \lambda) u = f(x) \quad \text{in } R_n^+,$$

$$(2) \quad u(x', 0) = \varphi(x'),$$

where $\varphi \in L^2_{\mathcal{P}}(R_{n-1})$ (for the definition of $L^2_{\mathcal{P}}(R_{n-1})$ see Section 2).

In recent years the Dirichlet problem with L^2 -boundary data has been studied by several authors (see [2], [3], [5], [6], [8], [9] and the references given there). In particular, the author has investigated this problem in R_n^+ (see [3]) and established the existence theorem in the case $L^2_{\mathcal{P}}(R_{n-1}) = L^2(R_{n-1})$. The aim of this work is to weaken the assumptions from [3] on the coefficients b_i and c . More specifically, in [3] it is assumed that

$$b_i \in L^\infty(R_n^+) \cap L^n(R_n^+) \quad \text{and} \quad c \in L^n(R_n^+) + L^\infty(R_n^+)$$

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($i = 1, \dots, n$). Here we assume that for every $\delta > 0$

$$b_i \in L^n(R_{n-1} \times (\delta, \infty)) \cap L^\infty(R_{n-1} \times (\delta, \infty)) \quad (i = 1, \dots, n)$$

and

$$c \in L^n(R_{n-1} \times (\delta, \infty)) + L^\infty(R_{n-1} \times (\delta, \infty))$$

($i = 1, \dots, n$) and moreover we allow b_i and c grow to ∞ in certain way as $x_n \rightarrow 0$, namely

$$b_i(x) = 0 \ (x_n^{-\beta_1}) \quad (i = 1, \dots, n) \quad \text{and} \quad c(x) = 0 \ (x_n^{-\beta_2})$$

for x_n close to 0, where $0 < \beta_1 < 1$ and $0 < \beta_2 < 2$.

The paper is organized as follows. We derive basic properties of traces of solutions in $W_{\text{loc}}^{1,2}(R_n^+)$ of (1) in Sections 1 and 2. The results of Sections 1 and 2 justify the formulation of the Dirichlet problem adopted in this work. In Section 3 we examine solutions u_δ of (1), (2) in $R_{n-1} \times (\delta, \infty)$ with boundary condition $u_\delta(x', \delta) = 0$ on R_{n-1} . In particular we establish an energy estimate for u_δ , which is used to show that the limit of u_δ , as $\delta \rightarrow 0$, exists and solves the Dirichlet problem (1), (2) with $\varphi \equiv 0$ on R_{n-1} . By a standard argument we use this result to solve the problem (1), (2) with $\varphi \not\equiv 0$ and $\varphi \in L^2_\Psi(R_{n-1})$ (see Section 4). We note that the methods of proofs here are not new and have appeared in [3]. Bibliographic information not given in this paper is available in [2] and [3]. Finally we point out that similar problem in the case of the Dirichlet problem in bounded domains has been considered by the author in [4].

1. Assumptions and preliminaries.

Let Ψ be a positive function in $C^1(R_{n-1})$ such that

$$(\Psi) \quad |D\Psi(x')| \leq N\Psi(x') \quad \text{in } R_{n-1}$$

for certain constant N .

We put

$$L^2_\Psi(R_n^+) = \left\{ u; L^2_{\text{loc}}(R_n^+), \int_{R_n^+} u(x)^2 \Psi(x')^2 dx < \infty \right\},$$

$$L^2_{\Psi}(R_{n-1}) = \left\{ u; u \in L^2_{\text{loc}}(R_{n-1}), \int_{R_{n-1}} u(x')^2 \Psi(x')^2 dx' < \infty \right\},$$

and

$$W^{1,2}_{\Psi}(R_n^+) = \left\{ u; u \in W^{1,2}_{\text{loc}}(R_n^+), \int_T^{\infty} dx_n \int_{R_{n-1}} u(x)^2 \Psi(x')^2 dx' < \infty \right. \\ \left. \text{for all } T > 0 \right\},$$

where $W^{1,2}_{\text{loc}}(R_n^+)$ denotes a Sobolev space, i.e., $W^{1,2}_{\text{loc}}(R_n^+)$ is the space of all functions in $L^2_{\text{loc}}(R_n^+)$ with first order weak derivatives also in $L^2_{\text{loc}}(R_n^+)$.

Throughout this article we make the following assumptions about the operator L :

(A) L is uniformly elliptic in R_n^+ , i.e., there exists a positive constant γ such that

$$\gamma |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j$$

for all $x \in R_n^+$ and $\xi \in R_n$ and moreover $a_{ij} \in L^{\infty}(R_n^+)$ ($i, j = 1, \dots, n$).

(B) (i) There exist positive constant K and $0 < \alpha < 1$ such that

$$|a_{nn}(x', x_n) - a_{nn}(x', \bar{x}_n)| \leq K |x_n - \bar{x}_n|^{\alpha}$$

for all $x' \in R_{n-1}$ and all $x_n, \bar{x}_n \in (0, \infty)$.

(ii) The coefficients a_{in} ($i = 1, \dots, n-1$) have partial derivatives $D_i a_{in}$ satisfying the inequality

$$|D_i a_{in}(x)| \leq K_1 x_n^{-\beta} \quad (i = 1, \dots, n-1)$$

for all $x' \in R_{n-1}$ and $x_n \in (0, b]$, where K_1 , b and β are positive constants with $0 < \beta < 1$ and moreover

$$D_i a_{in} \in L^{\infty}(R_{n-1} \times [b, \infty)) \quad (i = 1, \dots, n-1).$$

(iii) $b_i \in L^{\infty}(R_{n-1} \times (\delta, \infty))$ ($i = 1, \dots, n$) for all $\delta > 0$ and

$$|b_i(x)| \leq B x_n^{-\beta_1} \quad \text{in } R_{n-1} \times (0, b)$$

where B and β_1 are positive constants with $0 \leq \beta_1 < 1$.

(iv) $c \in L^\infty(R_{n-1} \times (\delta, \infty)) + L^n(R_{n-1} \times (\delta, \infty))$ for all $\delta > 0$
and

$$|c(x)| \leq Bx_n^{-\beta_2} \quad \text{in } R_{n-1} \times (0, b),$$

where C and β_2 are positive constants with $0 \leq \beta_2 < 2$.

$$(C) \quad f \in L^2_{\text{loc}}(R_n^+) \text{ and } \int_{R_n^+} f(x)^2 \Psi(x')^2 [\min(1, x_n)]^\Theta dx < \infty,$$

where $2 \leq \Theta < 3$.

In the sequel we need the following elementary lemmas

LEMMA 1. *If $u \in L^2_\Psi(R_n^+)$ and*

$$\sup_{0 < x_n \leq T} \int_{R_{n-1}} u(x', x_n)^2 \Psi(x')^2 dx' < \infty$$

for certain $T > 0$, then

$$\int_{R_n^+} u(x)^2 \Psi(x')^2 [\min(1, x_n)]^{-\mu} dx < \infty$$

for every $0 \leq \mu < 1$.

LEMMA 2. *Let $u \in W^{1,2}_\Psi(R_n^+)$ and $\int_{R_n^+} |D_n u(x)|^2 \min(1, x_n) \Psi(x')^2 dx < \infty$, then for every T and $0 \leq \mu < 1$*

$$\begin{aligned} \int_{\delta}^T dx_n \int_{R_{n-1}} u(x)^2 \Psi(x')^2 (x_n - \delta)^{-\mu} dx' &\leq \frac{2(T - \delta)^{1-\mu}}{1 - \mu} \int_{R_{n-1}} u(x', T)^2 \Psi(x')^2 dx' + \\ &+ \frac{4T^{1-\mu}}{(1 - \mu)^2} \int_{\delta}^T dx_n \int_{R_{n-1}} |D_n u(x)|^2 (x_n - \delta) \Psi(x')^2 dx \quad \text{for all } \delta \in \left[0, \frac{T}{2}\right]. \end{aligned}$$

The proof is identical to that of Lemma 2 in [3].

In this paper we use the notion of a weak solution of (1) involving the Sobolev spaces $W^{1,2}_{\text{loc}}(R_n^+)$ and $W^{1,2}(R_n^+)$. A function $u \in W^{1,2}_{\text{loc}}(R_n^+)$ is

said to be a weak solution of (1), if it satisfies

$$(3) \quad \int_{R_n^+} \left[\sum_{i,j=1}^n a_{ij}(x) D_i u D_j v + \sum_{i=1}^n b_i(x) D_i u \cdot v + (c(x) + \lambda) uv \right] dx = \\ = \int_{R_n^+} f(x) v dx$$

for every $v \in W^{1,2}(R_n^+)$ with compact support in R_n^+ .

LEMMA 3. *Let $u \in W_{\Psi}^{1,2}(R_n^+)$ be a weak solution of (1) in R_n^+ . Then for every $r > 0$*

$$(4) \quad \int_{2r}^{\infty} \int_{R_{n-1}} |Du(x)|^2 \Psi(x')^2 dx \leq M \left[(|\lambda| + 1) \int_r^{\infty} \int_{R_{n-1}} u(x)^2 \Psi(x')^2 dx + \right. \\ \left. + \int_r^{\infty} \int_{R_{n-1}} f(x)^2 \Psi(x')^2 dx \right],$$

where a positive constant M depends on the norms of the coefficients in $R_{n-1} \times [r, \infty)$, γ and n .

PROOF. Let $v = u \Phi^2$, where $\Phi \in C_0^1(R_n^+)$. Using v as a test function in (2) we obtain

$$\int_{R_n^+} \sum_{i,j=1}^n a_{ij} D_i u D_j u \Phi^2 dx + 2 \int_{R_n^+} \sum_{i,j=1}^n a_{ij} D_i u \cdot u D_j \Phi \cdot \Phi dx + \\ + \int_{R_n^+} \sum_{i=1}^n b_i D_i u \cdot u \Phi^2 dx + \int_{R_n^+} (c + \lambda) u^2 \Phi^2 dx = \int_{R_n^+} f u^2 \Phi^2 dx.$$

It follows from the ellipticity of L and the inequalities of Young and Sobolev that

$$\int_{R_n^+} |Du|^2 \Phi^2 dx \leq M_1 \int_{R_n^+} u^2 (\Phi^2 + |D\Phi|^2) dx + |\lambda| \int_{R_n^+} u^2 \Phi^2 dx + \int_{R_n^+} f^2 \Phi^2 dx,$$

where a positive constant M_1 depends on the norms of the coefficients on $\text{supp } \Phi$, n and γ . Here we have used the fact that $c = c_1 + c_2$, with $c_1 \in L^n(\text{supp } \Phi)$ and $c_2 \in L^\infty(\text{supp } \Phi)$ and applied the Sobolev inequality to the term $\int_{R_n^+} c_1 u^2 \Phi^2 dx$. To complete the proof we put $\Phi(x) = \Psi(x') \Phi_\nu(x)$, where $\{\Phi_\nu\}$ is an increasing sequence of non-negative functions in $C_0^1(R_n^+)$ with the gradient bounded independently of ν and converging to a non-negative function R_n^+ equal to 1 for $x_n \geq 2r$ and vanishing for $x_n < r$.

2. Traces of solutions in $W_\Psi^{1,2}(R_n^+)$.

All constants in the following theorems will be denoted by C_i . The statement « C_i depends on the structure of the operator L » means that C_i depends on $n, \gamma, \beta, \beta_1, \beta_2, \alpha, B, C, b, K, K_1$ and Θ and the norms of the coefficients in the appropriate spaces.

THEOREM 1. *Let $u \in W_\Psi^{1,2}(R_n^+)$ be a weak solution of (1) in R_n^+ . Then the following conditions are equivalent:*

(I) *there exists $T > 0$ such that*

$$\sup_{0 < x_n < T} \int_{R_{n-1}} u(x', x_n)^2 \Psi(x')^2 dx' < \infty,$$

$$(II) \quad \int_{R_n^+} |Du(x)|^2 \Psi(x')^2 \min(1, x_n) dx < \infty.$$

PROOF. The proof is similar to that of Theorem 1 in [3] and therefore we only give an outline.

Let $0 < 3\delta_0 < 1$. We may assume that $3\delta_0 \leq b$. We define a non-negative function $\eta \in C^2([0, \infty))$ such that $\eta(x_n) = x_n$ for $x_n \leq 2\delta_0$, $\eta(x_n) = 1$ for $x_n \geq 3\delta_0$ and $\eta(x_n) \geq \delta$ for all $x_n \geq \delta$ and $0 < \delta < 3\delta_0$. We may also assume that $\lambda = 0$.

Let

$$v(x) = \begin{cases} u(x) (\eta(x_n) - \delta) \Phi(x')^2 & \text{for } x_n > \delta, \\ 0 & \text{elsewhere,} \end{cases}$$

where Φ is a non-negative function in $C_0^1(R_{n-1})$. Since for every $\delta < x_n$,

$v(\cdot, x_n)$ has a compact support in R_{n-1} , it follows from Lemma 3 that v is an admissible test function. Applying the assumption (A) we obtain from (3)

$$\begin{aligned}
 (5) \quad & \gamma \int_{\delta}^{\infty} \int_{R_{n-1}} |Du|^2 (\eta - \delta) \Phi^2 dx \leq \int_{\delta}^{\infty} \int_{R_{n-1}} f u (\eta - \delta) \Phi^2 dx - \\
 & - \int_{\delta}^{\infty} \int_{R_{n-1}} \sum_{i=1}^{n-1} a_{in} D_i u \cdot u D_n \eta \Phi^2 dx - \int_{\delta}^{\infty} \int_{R_{n-1}} a_{nn} D_n u \cdot u D_n \eta \Phi^2 dx - \\
 & - 2 \int_{\delta}^{\infty} \int_{R_{n-1}} \sum_{i,j=1}^n a_{ij} D_i u \cdot u (\eta - \delta) \Phi D_j \Phi dx - \\
 & - \int_{\delta}^{\infty} \int_{R_{n-1}} \sum_{i=1}^n b_i D_i u \cdot u (\eta - \delta) \Phi^2 dx - \int_{\delta}^{\infty} \int_{R_{n-1}} c u^2 (\eta - \delta) \Phi^2 dx .
 \end{aligned}$$

All integrals on the right side can be estimated in the same way as in [3] except the integrals involving coefficients c and b_i . We only derive an estimate of the last integral. Let us denote this integral by J .

To estimate J we use decomposition $c = c_1 + c_2$ on $R_{n-1} \times [\delta_0, \infty)$, with $c_1 \in L^\infty(R_{n-1} \times [\delta_0, \infty))$ and $c_2 \in L^n(R_{n-1} \times [\delta_0, \infty))$ and by the assumption (Biv) we get

$$\begin{aligned}
 |J| \leq C \int_{\delta}^{\delta_0} \int_{R_{n-1}} u^2 x_n^{-\beta_2} (\eta - \delta) \Phi^2 dx + \int_{\delta}^{\infty} \int_{R_{n-1}} |c_1| u^2 (\eta - \delta) \Phi^2 dx + \\
 + \int_{\delta}^{\infty} \int_{R_{n-1}} |c_2| u^2 (\eta - \delta) \Phi^2 dx .
 \end{aligned}$$

To evaluate the last integral we set

$$\bar{c}_2(x) = \begin{cases} c_2(x) & \text{on } R_{n-1} \times [\delta_0, \infty), \\ 0 & \text{elsewhere,} \end{cases}$$

then $\bar{c}_2 \in L^n(R_n^+)$. By Hölder's inequality we have

$$\int_{\delta}^{\infty} \int_{R_{n-1}} |\bar{c}_2| u^2 (\eta - \delta) \Phi^2 dx \leq \|\bar{c}_2\|_{L^\infty} \left[\int_{\delta}^{\infty} \int_{R_{n-1}} u^2 \Phi^2 dx \right]^{1/2} \cdot \left\{ \int_{\delta}^{\infty} \int_{R_{n-1}} [u(\eta - \delta) \Phi]^{2^*} dx \right\}^{1/2^*},$$

where $1/2^* = 1/2 - 1/n$. Now by Sobolev's inequality

$$\left\{ \int_{\delta}^{\infty} \int_{R_{n-1}} |u(\eta - \delta) \Phi|^{2^*} dx \right\}^{1/2^*} \leq S \left[\int_{\delta}^{\infty} \int_{R_{n-1}} |Du|^2 (\eta - \delta) \Phi^2 dx + \int_{\delta}^{\infty} \int_{R_{n-1}} u^2 |D\eta|^2 \Phi^2 dx + \int_{\delta}^{\infty} \int_{R_{n-1}} u^2 (\eta - \delta) |D\Phi|^2 dx \right]^{1/2},$$

where S is a positive constant independent of δ . The last three inequalities yield that

$$|J| \leq \frac{\gamma}{4} \int_{\delta}^{\infty} \int_{R_{n-1}} |Du|^2 (\eta - \delta) \Phi^2 dx + C_1 \left[\int_{\delta}^{\delta_0} \int_{R_{n-1}} u^2 (\eta - x_n) x_n^{-\beta_1} \Phi^2 dx + \int_{\delta}^{\infty} \int_{R_{n-1}} u^2 \Phi^2 dx + \int_{\delta}^{\infty} \int_{R_{n-1}} u^2 (\eta - \delta) \Phi^2 dx + \int_{\delta}^{\infty} \int_{R_{n-1}} u^2 (\eta - \delta) |D\Phi|^2 dx \right],$$

where C_1 is a positive constant dependent on $n, \gamma, C, \|D\eta\|_{L^\infty}$ and $\|c_2\|_{L^n}$. Consequently we derive from (5) the following inequality.

$$(6) \quad \int_{\delta}^{\delta_0} \int_{R_{n-1}} |Du|^2 (\eta - \delta) \Phi^2 dx \leq C_1 \left[\int_{R_{n-1}} u(x', \delta)^2 \Phi^2 dx + \int_{\delta}^{\infty} \int_{R_{n-1}} u^2 |D\Phi|^2 dx + \int_{\delta}^{\infty} \int_{R_{n-1}} u^2 \Phi^2 dx + \int_{\delta_0}^{\infty} \int_{R_{n-1}} |Du|^2 \Phi^2 dx + \int_{\delta}^{\delta_0} \int_{R_{n-1}} u^2 x_n^{-\beta_1} (\eta - \delta) \Phi^2 dx + \int_{\delta}^{\delta_0} \int_{R_{n-1}} u^2 (x_n - \delta)^{2\alpha-1} \Phi^2 dx + \right]$$

$$\begin{aligned}
 & + \int_{\delta}^{\delta_0} \int_{R_{n-1}} u^2 x_n^{-\beta_2} (\eta - \delta) \Phi^2 dx + \int_{\delta}^{\delta_0} \int_{R_{n-1}} u^2 x_n^{-} \Phi^2 dx + \\
 & + \int_{\delta}^{\infty} \int_{R_{n-1}} u^2 (\eta - \delta)^{2-\Theta} \Phi^2 dx + \int_{\delta}^{\infty} \int_{R_{n-1}} f^2 (\eta - \delta)^{\Theta} \Phi^2 dx \Big]
 \end{aligned}$$

where a positive C_2 depends on the structure of the operator L . If the condition (I) holds, then by Lemma 1 for every $0 \leq \mu < 1$ the integral

$$\int_{\delta}^{\delta_0} \int_{R_{n-1}} u^2 (x_n - \delta)^{-\mu} dx \text{ is bounded on } \left[0, \frac{\delta_0}{2}\right].$$

Now we put $\Phi = \Phi_\nu \Psi$, where Φ_ν is an increasing sequence of non-negative functions in $C_0^\infty(R_{n-1})$ converging to 1 as $\nu \rightarrow \infty$ with the gradient bounded independently of ν . Letting $\nu \rightarrow \infty$ in (11) it follows from Lemma 3 and the condition (Ψ) that

$$\begin{aligned}
 & \int_{\delta}^{\delta_0} \int_{R_{n-1}} |Du|^2 (x_n - \delta) \Psi^2 dx \leq \tilde{C}_2 \left[\int_{R_{n-1}} u(x', \delta)^2 \Psi(x'^2) dx + \right. \\
 & + \int_{\delta}^{\infty} \int_{R_{n-1}} u^2 \Psi^2 dx + \int_{\delta_{0/2}}^{\infty} \int_{R_{n-1}} u^2 \Psi^2 dx + \int_{\delta}^{\delta_0} \int_{R_{n-1}} u^2 x_n^{-\beta_1} (\eta - \delta) \Psi^2 dx + \\
 & + \int_{\delta}^{\delta_0} \int_{R_{n-1}} u^2 x_n^{-\beta_2} (\eta - \delta) \Psi^2 dx + \int_{\delta}^{\delta_0} \int_{R_{n-1}} u^2 x_n^{-\beta} \Psi^2 dx + \\
 & + \int_{\delta}^{\delta_0} \int_{R_{n-1}} u^2 (x_n - \delta)^{2\alpha-1} \Psi^2 dx + \int_{\delta}^{\infty} \int_{R_{n-1}} u^2 (\eta - \delta)^{2-\Theta} \Psi^2 dx + \\
 & \left. + \int_{\delta}^{\infty} \int_{R_{n-1}} f^2 (\eta - \delta)^{\Theta} \Psi^2 dx + \int_{\delta_{0/2}}^{\infty} \int_{R_{n-1}} f^2 \Psi^2 dx \right]
 \end{aligned}$$

and consequently the implication $I \Rightarrow II$ follows the Lebesgue Monotone Convergence Theorem.

To show $\text{II} \Rightarrow \text{I}$ we note that

$$\begin{aligned}
& \frac{1}{2} \int_{E_{n-1}} a_{nn}(x', \delta) u(x', \delta)^2 \Phi(x')^2 dx' = \\
& = -\frac{1}{2} \int_{\delta}^{\infty} \int_{R_{n-1}} a_{nn}(x', \delta) u(x)^2 \Phi(x')^2 D_n^2 \eta(x_n) dx + \\
& + \int_{\delta}^{\infty} \int_{R_{n-1}} [a_{nn}(x) - a_{nn}(x', \delta)] D_n u(x) \cdot u(x) D_n \eta(x_n) \Phi(x')^2 dx + \\
& + \int_{\delta}^{\infty} \int_{R_{n-1}} \sum_{i,j=1}^n a_{ij} D_i u D_j u (\eta - \delta) \Phi^2 dx + \int_{\delta}^{\infty} \int_{R_{n-1}} \sum_{i=1}^{n-1} a_{in} D_i u \cdot u D_n \eta \Phi^2 dx + \\
& + 2 \int_{\delta}^{\infty} \int_{R_{n-1}} \sum_{i,j=1}^n a_{ij} D_i u \cdot u (\eta - \delta) \Phi D_i \Phi dx + \\
& + \int_{\delta}^{\infty} \int_{R_{n-1}} \sum_{i=1}^n b_i D_i u \cdot u (\eta - \delta) \Phi^2 dx + \\
& + \int_{\delta}^{\infty} \int_{R_{n-1}} c u^2 (\eta - \delta) \Phi^2 dx - \int_{\delta}^{\infty} \int_{R_{n-1}} f u (\eta - \delta) \Phi^2 dx.
\end{aligned}$$

According to Lemma 2 the condition (II) implies that for every $0 \leq \mu < 1$ the integral

$$\int_{\delta}^{\infty} \int_{R_{n-1}} u^2(x_n - \delta)^{-\mu} \Psi^2 dx$$

is bounded independently of δ . Repeating the argument from the step $\text{I} \Rightarrow \text{II}$ the result follows.

REMARK 1. It follows from the proof of Theorem 1 that the condition (II) implies:

$$\text{for each } T > 0 \sup_{0 < x_n < T} \int_{R_{n-1}} u(x', x_n)^2 \Psi(x')^2 dx' < \infty.$$

THEOREM 2. *Let $u \in W_{\Psi}^{1,2}(R_n^+)$ be a solution of (1) in R_n^+ . Suppose that one of the conditions (I) or (II) holds. Then there exists a function $\varphi \in L_{\Psi}^2(R_{n-1})$ such that*

$$\lim_{\delta \rightarrow 0} \int_{R_{n-1}} u(x', \delta) g(x') \Psi(x')^2 dx' = \int_{R_{n-1}} \varphi(x') g(x') \Psi(x')^2 dx'$$

for every $g \in L_{\Psi}^2(R_{n-1})$.

The proof of this theorem is an obvious modification of the proof of Theorem 2 in [3].

Our next objective is to establish the L_{Ψ}^2 -convergence of $u(\cdot, \delta)$ to φ as $\delta \rightarrow 0$. To do this we first show the norm of $u(\cdot, \delta)$ converges to the norm of φ . The result then follows by the uniform convexity of the space L_{Ψ}^2 .

THEOREM 3. *Let $u \in W_{\Psi}^{1,2}(R_n^+)$ be a solution of (1) in R_n^+ . Suppose that one of the conditions (I) or (II) of Theorem 1 holds. Then there exists a function $\varphi \in L_{\Psi}^2(R_{n-1})$ such that*

$$\lim_{\delta \rightarrow 0} \int_{R_{n-1}} [u(x', \delta) - \varphi(x')]^2 \Psi(x')^2 dx' = 0.$$

The proof is similar to that of theorem 3 in [3] and therefore is omitted.

3. The energy estimate.

The result of Section 2 suggest the following definition of the Dirichlet problem (1), (2).

A weak solution $u \in W_{\Psi}^{1,2}(R_n^+)$ of (1) is a solution of the Dirichlet problem (1), (2) if

$$(7) \quad \lim_{\delta \rightarrow 0} \int_{R_{n-1}} [u(x', \delta) - \varphi(x')]^2 \Psi(x')^2 dx' = 0.$$

To solve the Dirichlet problem (1), (2) we first consider the problem

$$(1_{\delta}) \quad Lu + \lambda u = f \quad \text{in } R_{n-1} \times (\delta, \infty),$$

$$(2_{\delta}) \quad u(x', \delta) = 0 \quad \text{on } R_{n-1}.$$

Here the boundary condition (2_δ) is understood in the sense that $u \in \overset{\circ}{W}^{1,2}(R_{n-1} \times (\delta, \infty))$.

We begin by establishing an energy estimate for a solution u_δ of (1_δ) , (2_δ) (see Theorem 5 below). Thus a solution of (1) , (2) (with $\varphi \equiv 0$) is obtained as a limit of u_δ . The existence of a solution of the problem (1_δ) , (2_δ) will be proved in Section 4.

THEOREM 4. *Let u_δ be a solution in $W^{1,2}(R_{n-1} \times (\delta, \infty))$ of the problem (1_δ) , (2_δ) . Then there exist positive constants λ_0 , δ_1 , τ and C_1 such that*

$$(8) \quad \int_{R_n^+} |Du_\delta(x)|^2 (\eta(x_n) - \delta) \Psi(x')^2 dx + \lambda \int_{R_n^+} u_\delta(x)^2 (\eta(x_n) - \delta) \Psi(x')^2 dx + \\ + \sup_{\delta < s < \delta + \tau} \int_{R_{n-1}} u_\delta(x', s)^2 \Psi(x')^2 dx' \leq C_1 \int_{R_n^+} f(x)^2 [\min(1, x_n)]^\theta \Psi(x')^2 dx$$

for all $\lambda \geq \lambda_0$ and $0 < \delta < \delta_1$.

PROOF. Let $0 < \delta < \delta_0/2$ and put $v(x) = u_\delta(x) (\eta(x_n) - \delta) \Phi(x)^2$ for $x \in R_{n-1} \times (\delta, \infty)$, where Φ is a non-negative function in $C_0^\infty(R_{n-1})$. Taking v as a test function in the integral identity defining a solution u_δ we easily arrive at the estimate

$$(9) \quad \int_{\delta}^{\delta_0} \int_{R_{n-1}} |Du_\delta|^2 (\eta - \delta) \Psi^2 dx + \lambda \int_{\delta}^{\infty} \int_{R_{n-1}} u_\delta^2 (\eta - \delta) \Psi^2 dx \leq \\ \leq C_2 \left[\int_{\delta}^{\infty} \int_{R_{n-1}} u_\delta^2 \Psi^2 dx + \int_{\delta_0}^{\infty} \int_{R_{n-1}} |Du|^2 \Psi^2 dx + \int_{\delta}^{\delta_0} \int_{R_{n-1}} u_\delta^2 (\eta - \delta) x_n^{-\beta_1} \Psi^2 dx + \right. \\ \left. + \int_{\delta}^{\delta_0} \int_{R_{n-1}} u_\delta^2 (\eta - \delta) x_n^{-\beta_2} \Psi^2 dx + \int_{\delta}^{\delta_0} \int_{R_{n-1}} u_\delta^2 (x_n - \delta)^{2\alpha-1} \Psi^2 dx + \right. \\ \left. + \int_{\delta}^{\delta_0} \int_{R_{n-1}} u_\delta^2 x_n^{-\beta} \Psi^2 dx + \int_{\delta}^{\infty} \int_{R_{n-1}} u_\delta^2 (\eta - \delta)^{2-\theta} \Psi^2 dx + \right. \\ \left. + \int_{R_n^+} f^2 [\min(1, x)]^\theta \Psi^2 dx \right],$$

for all $\delta \leq \delta_0/2$, provided λ is sufficiently large, say $\lambda \geq \lambda_1$, where a positive constant C_2 depends on the structure of L . By Lemma 2 we can write the estimate (9) in the form

$$(10) \quad \int_{\delta}^{\delta_0} \int_{R_{n-1}} |Du_{\delta}|^2 (\eta - \delta) \Psi^2 dx + \lambda \int_{\delta}^{\infty} \int_{R_{n-1}} u_{\delta}^2 (\eta - \delta) \Psi^2 dx \leq \\ \leq C_3 \left[\int_{R_n^+} f^2 [\min(1, x_n)]^{\theta} \Psi^2 dx + \int_{R_{n-1}} u(x', \delta_0)^2 \Psi(x')^2 dx' + \right. \\ \left. + \int_{\delta_0}^{\infty} \int_{R_{n-1}} |Du_{\delta}|^2 \Psi^2 dx + \int_{\delta_0}^{\infty} \int_{R_{n-1}} u_{\delta}^2 \Psi^2 dx \right]$$

for all $\delta \leq \delta_0/2$ and $\lambda \geq \lambda_2$, where λ_2 is a sufficiently large and a positive constant C_3 depends on the structure of L . Now note that by an obvious modification of inequality (2) in [10] (p. 179) we have

$$(11) \quad \int_{R_{n-1}} u_{\delta}(x', \delta_0)^2 \Psi(x')^2 dx' \leq S \int_{\delta_0}^{\infty} \int_{R_{n-1}} |Du_{\delta}(x)|^2 \Psi(x')^2 dx,$$

where a positive S is independent of u . Using (11) and Lemma 3 with $r = \delta_0/2$ we can write the estimate (10) in the form

$$(12) \quad \int_{\delta}^{\infty} \int_{R_{n-1}} |Du_{\delta}|^2 (\eta - \delta) \Psi^2 dx + \lambda \int_{\delta}^{\infty} \int_{R_{n-1}} u_{\delta}^2 (\eta - \delta) \Psi^2 dx \leq \\ \leq C_4 \left[\int_{R_n^+} f^2 [\min(1, x_n)]^{\theta} \Psi^2 dx + \int_{\delta_0/2}^{\infty} \int_{R_{n-1}} u_{\delta}^2 \Psi^2 dx \right]$$

for all $\delta \leq \delta_0/4$ and $\lambda \geq \lambda_2$, with a modified λ_2 if necessary. On the other hand note that

$$\eta(x_n) \geq \frac{\delta_0}{8} + \delta \text{ for all } \delta \leq \frac{\delta_0}{8} \text{ and } x_n \geq \frac{\delta_0}{4}$$

and consequently

$$\int_{\delta_0/2}^{\infty} \int_{R_{n-1}} u_{\delta}^2 \Psi^2 dx \leq \frac{8}{\delta_0} \int_{\delta}^{\infty} \int_{R_{n-1}} u_{\delta}^2 \Psi^2 (\eta - \delta) dx.$$

Therefore there exist positive constants C_5 and λ_2 , both depending on the structure of L , such that

$$(13) \quad \int_{\delta}^{\infty} \int_{R_{n-1}} |Du_{\delta}|^2 (\eta - \delta) \Psi^2 dx + \lambda \int_{\delta}^{\infty} \int_{R_{n-1}} u_{\delta}^2 (\eta - \delta) \Psi^2 dx \leq \\ \leq C_4 \int_{R_n^+} f^2 [\min(1, x_n)]^{\Theta} \Psi^2 dx$$

for all $\delta \leq \delta_0/8$ and $\lambda \geq \lambda_2$.

Let $0 < \tau_0 \leq \delta_0/2$, as in Theorem 1 we derive the following inequality

$$(14) \quad \int_{R_{n-1}} u_{\delta}(x', \delta + \tau)^2 \Psi(x')^2 dx' \leq C_6 \left[\int_{R_n^+} f[\min(1, x_n)]^{\Theta} \Psi^2 dx + \right. \\ + \int_{\delta+\tau}^{\infty} \int_{R_{n-1}} |Du|^2 (\eta - \delta - \tau) \Psi^2 dx + \lambda \int_{\delta+\tau}^{\infty} \int_{R_{n-1}} u_{\delta}^2 (\eta - \delta - \tau) \Psi^2 dx + \\ + \int_{\tau+\delta}^{\infty} \int_{R_{n-1}} u_{\delta}^2 \Psi^2 dx + \int_{\tau+\delta_0}^{\infty} \int_{R_{n-1}} |Du|^2 \Psi^2 dx + \\ + \int_{\tau+\delta}^{\tau+\delta_0} \int_{R_{n-1}} u_{\delta}^2 (x_n - \delta)^{2x-1} \Psi^2 dx + \int_{\delta+\tau}^{\tau+\delta_0} \int_{R_{n-1}} u_{\delta}^2 x_n^{-\beta_2} (\eta - \delta - \tau) \Psi^2 dx + \\ + \int_{\tau+\delta}^{\tau+\delta_0} \int_{R_{n-1}} u_{\delta}^2 x_n^{-\beta} \Psi^2 dx + \int_{\tau+\delta}^{\tau+\delta_0} \int_{R_{n-1}} u_{\delta}^2 (\eta - \delta - \tau)^{2-\Theta} \Psi^2 dx + \\ \left. + \int_{\tau+\delta}^{\tau+\delta_0} \int_{R_{n-1}} u_{\delta}^2 x_n^{-\beta_1} (\eta - \delta - \tau) \Psi^2 dx \right]$$

for all $\delta \leq \delta_0/2$ and $0 < \tau \leq \tau_0/2$. Note that if $0 < \tau \leq \delta_0/2$ and $0 < \tau < \delta_0/2$, then for every $0 \leq \mu < 1$ we have

$$(15) \quad \int_{\tau+\delta}^{\tau+\delta_0} \int_{R_{n-1}} u_{\delta}^2 (\eta - \delta - \tau)^{-\mu} \Psi^2 dx \leq \int_{\tau+\delta}^{\delta+\tau_0} \int_{R_{n-1}} u_{\delta}^2 (\eta - \delta - \tau)^{-\mu} \Psi^2 dx +$$

$$\begin{aligned}
 + \int_{\delta + \tau_0}^{\infty} \int_{R_{n-1}} u_{\delta}^2 (\eta - \delta - \tau)^{-\mu} \Psi^2 dx &\leq \frac{(\tau_0)^{1-\mu}}{1-\mu} \sup_{\delta < s < \delta + \tau_0} \int_{R_{n-1}} u(x', s)^2 \Psi(x')^2 dx' + \\
 &+ \left(\frac{2}{\tau_0}\right)^{\mu} \int_{\delta + \tau_0}^{\infty} \int_{R_{n-1}} u_{\delta}^2 \Psi^2 dx.
 \end{aligned}$$

Since $\eta(x_n) \geq \tau_0/2 + \delta$ for all $\delta \leq \tau_0/2$ and $x_n \geq \tau_0$ we get

$$(16) \quad \int_{\delta + \tau_0}^{\infty} \int_{R_{n-1}} u_{\delta}^2 \Psi^2 dx \leq \frac{2}{\tau_0} \int_{\delta}^{\infty} \int_{R_{n-1}} u_{\delta}^2 (\eta - \delta) \Psi^2 dx.$$

On the other hand (11) we have

$$\begin{aligned}
 (17) \quad \sup_{\delta < s < \delta + \tau_0} \int_{R_{n-1}} u_{\delta}(x', s)^2 \Psi(x')^2 dx' &\leq \sup_{\delta < s < \delta + \tau_0/2} \int_{R_{n-1}} u_{\delta}(x', s)^2 \Psi(x')^2 dx' + \\
 + \sup_{\delta + \tau_0/2 < s < \delta + \tau_0} \int_{R_{n-1}} u_{\delta}(x', s)^2 \Psi(x')^2 dx' &\leq \sup_{\delta < s < \delta + \tau_0/2} \int_{R_{n-1}} u_{\delta}(x', s)^2 \Psi(x')^2 dx' + \\
 &+ S \int_{\tau_0/2}^{\infty} \int_{R_{n-1}} |Du_{\delta}|^2 \Psi^2 dx.
 \end{aligned}$$

Consequently combining (13), (14), (15), (16) and (17) we obtain

$$\begin{aligned}
 (18) \quad \sup_{\delta < s < \delta + \tau_0/2} \int_{R_{n-1}} u_{\delta}(x', s)^2 \Psi(x')^2 dx' &\leq \\
 C_7 \left[\int_{R_n^+} f^2 [\min(1, x_n)]^{\theta} \Psi^2 dx + \int_{\tau_0/2}^{\infty} \int_{R_{n-1}} |Du_{\delta}|^2 \Psi^2 dx \right]
 \end{aligned}$$

for all $\delta < \min(\delta_0/8, \tau_0/2)$ and $\lambda \geq \lambda_2$, provided λ_2 is sufficiently large and τ_0 is sufficiently small and C_7 is a positive constant depending on the structure of L . Applying Lemma 3 with $r = \tau_0/4$ we deduce from the last estimate that

$$\begin{aligned}
 (19) \quad \sup_{\delta < s < \delta + \tau_0/2} \int_{R_{n-1}} u_{\delta}(x', s)^2 \Psi(x')^2 dx' &\leq \tilde{C}_7 \left[\int_{R_n^+} f^2 [\min(1, x_n)]^2 \Psi^2 dx + \right. \\
 &+ \int_{\tau_0/4}^{\infty} \int_{R_{n-1}} u_{\delta}^2 \Psi^2 dx + \int_{\tau_0/2}^{\infty} \int_{R_{n-1}} f^2 \Psi^2 dx \left. \right]
 \end{aligned}$$

for all $\delta < \min(\delta_0/8, \tau_0/8)$. Since $\eta(x_n) \geq \tau_0/8 + \delta$ for all $\delta < \tau_0/8$ and $x_n \geq \tau_0/4$ we have

$$(20) \quad \int_{\tau_0/4}^{\infty} \int_{R_{n-1}} u_{\delta}^2 \Psi^2 dx \leq \frac{8}{\tau_0} \int_{R_{n-1}}^{\infty} u_{\delta}^2 \Psi^2 (\eta - \delta) dx$$

for all $\delta \leq \min(\delta_0/8, \tau_0/8)$. Combining the estimates (13), (19) and (20) we easily deduce (8) with $\tau = \tau_0/2$ and $\delta_1 = \min(\delta_0/8, \tau_0/8)$ provided λ_0 is sufficiently large.

4. The existence of a solution to the Dirichlet problem.

We begin by proving the existence of a solution in $W_{\Psi}^{1,2}(R_{n-1} \times (\delta, \infty))$ of the problem (1 _{δ}), (2 _{δ}).

We need the following result due to G. Bottaro and M. E. Marina [1].

THEOREM 5. *Suppose that $f \in L^2(R_n^+)$, $b_i \in L^n(R_n^+) \cap L^\infty(R_n^+)$ ($i = 1, \dots, n$), $c \in L^n(R_n^+) + L^\infty(R_n^+)$ and that $c(x) \geq \text{Const} > 0$ on R_n^+ . Then the Dirichlet problem*

$$Lu = f(x) \quad \text{in } R_n^+; \quad u(x', 0) = \quad \text{on } R_{n-1}$$

has unique solution in $\mathring{W}^{1,2}(R_n^+)$.

THEOREM 6. *Assume that*

$$b_i \in L^n(R_n^+) \cap L^\infty(R_n^+) \quad (i = 1, \dots, n) \quad c \in L^n(R_n^+) + L^\infty(R_n^+)$$

and that $c(x) \geq 0$ on R_n^+ . Then there exists $\lambda_0 > 0$ such that for every $\varphi \in L_{\Psi}^2(R_{n-1})$ and all $\lambda \geq \lambda_0$ there exists a unique solution $u \in W_{\Psi}^{1,2}(R_n^+)$ of the Dirichlet problem (1), (2).

PROOF. The proof is based on the following energy estimate: there exist positive constants d , λ_0 and C_1 depending on the structure of L such that if $u \in W_{\Psi}^{1,2}(R_n^+)$ is a solution of (1), (2) for $\lambda \geq \lambda_0$, then

$$(21) \quad \int_{R_n^+} |Du|^2 \min(1, x_n) \Psi^2 dx + \lambda \int_{R_n^+} u^2 \min(1, x_n) \Psi^2 dx +$$

$$\begin{aligned}
 + \sup_{0 < \delta < d} \int_{R_{n-1}^+} u(x')^2 \Psi(x')^2 dx' \leq C_1 \left[\int_{R_n^+} f^2 [\min(1, x_n)]^\Theta \Psi^2 dx + \right. \\
 \left. + \int_{R_{n-1}} \varphi(x')^2 \Psi(x')^2 dx' \right],
 \end{aligned}$$

the proof of which is now a routine. Let $\{\varphi_m\}$ be sequence of functions in $C_0^1(R_{n-1})$ converging in $L_\Psi^2(R_{n-1})$ to φ . Put

$$f_m(x) = \begin{cases} f(x) & \text{in } (|x'| < m) \times \left(\frac{1}{m}, \infty\right), \\ 0 & \text{elsewhere,} \end{cases}$$

$m = 1, 2, \dots$. It follows from Theorem 6 that for every m and $\lambda > 0$ there exists a unique solution u_m in $W^{1,2}(R_n^+)$ of the Dirichlet problem

$$Lu + \lambda u = f_m \text{ in } R_n^+,$$

$$u(x', 0) = \varphi_m(x') \text{ on } R_{n-1}.$$

According to (21)

$$\begin{aligned}
 \int_{R_n^+} |Du_q - Du_p|^2 \min(1, x_n) \Psi^2 dx + \lambda \int_{R_n^+} (u_p - u_q)^2 \min(1, x_n) \Psi^2 dx + \\
 + \sup_{0 < \delta < d} \int_{R_{n-1}^+} [u_p(x', \delta) - u_q(x', \delta)]^2 \Psi(x')^2 dx' \leq \\
 \leq C_1 \left[\int_{R_n^+} (f_p - f_q)^2 \min(1, x_n) \Psi^2 dx + \int_{R_{n-1}} (\varphi_p - \varphi_q)^2 \Psi^2 dx' \right]
 \end{aligned}$$

for all p and q , provided $\lambda \geq \lambda_0$. Hence $\{u_m\}$ is the Cauchy sequence in the norm

$$\begin{aligned}
 \left[\int_{R_n^+} |Du|^2 \min(1, x_n) \Psi^2 dx + \sup_{0 < \delta < d} \int_{R_{n-1}^+} u(x', \delta)^2 \Psi^2 dx + \right. \\
 \left. + \int_{R_n^+} u^2 \min(1, x_n) \Psi^2 dx \right]^{1/2}
 \end{aligned}$$

and the result follows.

In the sequel we adopt the assumptions of Section 1 with (B iii) replace by

(B' iii) for every $\delta > 0$

$$b_i \in L^n(R_{n-1} \times (\delta, \infty)) \cap L^\infty(R_{n-1} \times (\delta, \infty)) \quad (i = 1, \dots, n)$$

and

$$c \in L^n(R_{n-1} \times (\delta, \infty)) + L^\infty(R_{n-1} \times (\delta, \infty))$$

and moreover there exist positive constants b , C , B , β_1 and β_2 with $0 < \beta_1 < 1$, $0 < \beta_2 < 2$ such that

$$0 < c(x) < Cx_n^{-\beta_2} \quad \text{on } R_{n-1} \times (0, b)$$

and

$$|b_i(x)| < Bx_n^{-\beta_1} \quad (i = 1, \dots, n) \quad \text{on } R_{n-1} \times (0, b).$$

THEOREM 7. *Let $\varphi \in L^2_\Psi(R_{n-1})$. Then there exists $\lambda_1 > 0$ such that for every $\lambda \geq \lambda_1$ the Dirichlet problem (1), (2) admits a unique solution in $W^{1,2}(R_n^+)$.*

PROOF. [We first assume that $\varphi = 0$ on R_{n-1} . Since for every $\delta > 0$ the coefficients b_i ($i = 1, \dots, n$) and c satisfy the assumptions of Theorem 6 in $R_{n-1} \times (\delta, \infty)$, there exists $\lambda_\delta > 0$ such that for every $\lambda \geq \lambda_\delta$ the problem (1 $_\delta$), (2 $_\delta$) admits a unique solution u_δ in $W^{1,2}_\Psi(R_{n-1} \times (\delta, \infty))$. In view of Theorem 7 we may assume that λ_δ can be chosen independently of δ in a small interval $(0, \delta_1]$. Consequently by Theorem 5 there exist positive constants τ , δ_1 , λ_1 and C_1 such that

$$(22) \quad \int_{R_n^+} |Du_\delta|^2 (\eta - \delta) \Psi^2 dx + \lambda \int_{R_n^+} u_\delta^2 (\eta - \delta) \Psi^2 dx + \\ + \sup_{\delta < s < \delta + \tau} \int_{R_n^+} u_\delta(x', s)^2 \Psi(x')^2 dx' \leq C_1 \int_{R_n^+} f^2 [\min(1, x_n)]^\theta \Psi^2 dx$$

for all $\lambda \geq \lambda_1$ and $0 < \delta \leq \delta_1$. The estimate (29) yields the existence of a sequence δ_n and a function $u \in W^{1,2}_{\text{loc}}(R_n^+)$ such that:

for every compact set $K \subset R_n^+$

$$(23) \quad \lim_{n \rightarrow \infty} u_{\delta_n} = u \text{ weakly in } W^{1,2}(K).$$

It is obvious that u satisfies (1). We extend u_δ and $D_i u \sqrt{\eta - \delta}$ ($i = 1, \dots, n$) by 0 outside $R_{n-1} \times (\delta, \infty)$. By a weak compactness of bounded sets in $L^2_\Psi(R_{n-1})$ we may assume that there exist functions v_i ($i = 1, \dots, n$) and v in $L^2_\Psi(R_n^+)$ such that

$$(24) \quad \lim_{n \rightarrow \infty} D_i u_{\delta_n} \sqrt{\eta - \delta_n} = v_i (i = 1, \dots, n) \quad \text{and} \quad \lim_{n \rightarrow \infty} u_{\delta_n} = v$$

weakly in $L^2_\Psi(R_n^+)$. We claim that

$$(25) \quad v_i = D_i u \sqrt{\eta} \quad (i = 1, \dots, n) \quad \text{and} \quad v = u$$

a.e. in R_n^+ . We only show the first relation (32).

Let $g \in L^2_\Psi(R_n^+)$ be an arbitrary function with a compact support. Then

$$\begin{aligned} \left| \int_{R_n^+} (v_i - D_i u \sqrt{\eta}) g \Psi^2 dx \right| &\leq \left| \int_{R_n^+} (v_i - D_i u_{\delta_n} \sqrt{\eta - \delta_n}) g \Psi^2 dx \right| + \\ &+ \left| \int_{\text{supp } g} (D_i u_{\delta_n} \sqrt{\eta - \delta_n} - D_i u_{\delta_n} \sqrt{\eta}) g \Psi^2 dx \right| + \\ &+ \left| \int_{\text{supp } g} (D_i u_{\delta_n} - D_i u) \sqrt{\eta} g \Psi^2 dx \right| + R_1 + R_2 + R_3. \end{aligned}$$

By (23) $\lim_{n \rightarrow \infty} R_3 = 0$. From (24) we obtain $\lim_{n \rightarrow \infty} R_1 = 0$. The Hölder inequality and boundedness of $\int_{\text{supp } g} |Du_{\delta_n}|^2 \Psi^2 dx$ imply that $\lim_{n \rightarrow \infty} R_2 = 0$. Therefore

$$\int_{R_n^+} D_i u \sqrt{\eta} g \Psi^2 dx = \int_{R_n^+} v_i g \Psi^2 dx$$

for every $g \in L^2_\Psi(R_n^+)$ with compact support and the first relation easily follows. It is also clear that

$$\int_{R_n^+} |Du|^2 \min(1, x_n) \Psi^2 dx + \int_{R_n^+} u^2 \Psi^2 dx \leq \int_{R_n^+} f^2 [\min(1, x_n)]^\theta \Psi^2 dx.$$

Theorem 4 implies the existence of a function $h \in L^2_\Psi(R_{n-1})$ such that

$$\lim_{\delta \rightarrow 0} \int_{R_{n-1}} [u(x', \delta) - h(x')]^2 \Psi(x')^2 dx = 0.$$

Hence we have to show that $h = 0$ a.e. on R_{n-1} . Let $\Phi \in C^1_0(R_{n-1})$. Then

$$\begin{aligned} \int_{R_{n-1}} a_{nn}(x', 0) \Phi(x') h(x') \Psi(x')^2 dx' = \\ = \int_{R_n^+} [a_{nn}(x) - a_{nn}(x', 0)] D_n u(x) \Phi(x') D_n \eta(x_n) \Psi(x')^2 dx - \\ - \int_{R_n^+} D_n(a_{nn}(x', 0) D_n \eta(x_n)) \Phi(x') \Psi(x')^2 dx + \\ + \int_{R_n^+} \sum_{i=1}^{n-1} a_{in} D_i u D_n \eta \Phi \Psi^2 dx + \int_{R_n^+} \sum_{i,j=1}^n a_{ij} D_i u D_j (\Phi \Psi^2) \eta dx + \\ + \int_{R_n^+} \sum_{i=1}^n b_i D_i u \Phi \Psi^2 \eta dx + \int_{R_n^+} (c + \lambda) \Phi \Psi^2 \eta dx - \int_{R_n^+} f \Phi \Psi^2 \eta dx = K, \end{aligned}$$

and

$$\begin{aligned} 0 = \int_{\delta}^{\infty} \int_{R_{n-1}} [a_{nn}(x) - a_{nn}(x', \delta)] D_n u_{\delta} D_n \eta \Phi \Psi^2 dx - \\ - \int_{\delta}^{\infty} \int_{R_{n-1}} D_n(a_{nn}(x', \delta) D_n \eta) u_{\delta} \Phi \Psi^2 dx + \int_{\delta}^{\infty} \int_{R_{n-1}} \sum_{i=1}^{n-1} a_{in} D_i u_{\delta} D_n \eta \Phi \Psi^2 dx + \\ + \int_{\delta}^{\infty} \int_{R_{n-1}} \sum_{i,j=1}^n a_{ij} D_i u_{\delta} D_j (\Phi \Psi^2) (\eta - \delta) dx + \int_{\delta}^{\infty} \int_{R_{n-1}} \sum_{i=1}^n b_i D_i u_{\delta} \Phi \Psi^2 (\eta - \delta) dx + \\ + \int_{\delta}^{\infty} \int_{R_{n-1}} (c + \lambda) u_{\delta} \Phi \Psi^2 (\eta - \delta) dx - \int_{\delta}^{\infty} \int_{R_{n-1}} f \Phi \Psi^2 (\eta - \delta) dx = K_{\delta}. \end{aligned}$$

By (24) and (25) it is clear that $\lim_{n \rightarrow \infty} (K - K_{\delta_n}) = 0$ and consequently

$$\int_{R_{n-1}} h \Phi \Psi^2 dx' = 0 \quad \text{for every } \Phi \in C_0^1(R_{n-1}),$$

therefore $h = 0$ a.e. on R_{n-1} .

Now consider the general situation $\varphi \in L_{\Psi}^2(R_{n-1})$ and $\varphi \not\equiv 0$. As in Theorem 4 we derive an energy estimate for solutions in $W_{\Psi}^{1,2}$ of the problem (1), (2), namely, let $u \in W_{\Psi}^{1,2}(R_n^+)$ be a solution of the problem (1), (2), then there exist positive constants λ_0 , τ and C depending on the structure of the operator L such that

$$(26) \quad \int_{R_n^+} |Du|^2 \min(1, x_n) \Psi^2 dx + \lambda \int_{R_n^+} u^2 \min(1, x_n) \Psi^2 dx + \\ + \sup_{0 < s < \tau} \int_{R_{n-1}} u(x', s)^2 \Psi(x')^2 dx' \leq C_1 \left[\int_{R_n^+} f^2 [\min(1, x_n)]^{\theta} \Psi^2 dx + \right. \\ \left. + \int_{R_n} \varphi^2 \Psi^2 dx' \right]$$

for all $\lambda \geq \lambda_0$. To solve the problem (1), (2) we take a sequence $\{\varphi_m\}$ in $C_0^1(R_{n-1})$ such that $\lim_{m \rightarrow \infty} \varphi_m = \varphi$ in $L_{\Psi}^2(R_{n-1})$ and consider the Dirichlet problem (1), (2) with $\varphi = \Phi_m$. Let Φ_m be a function in $C^1(\overline{R_n^+})$ such that $\Phi_m(x', 0) = \varphi(x')$ on R_{n-1} and $\Phi(x', x_n) = 0$ for $x_n \geq \delta_0$. By means of the transformation $v = u - \varphi_m$ this problem can be reduced to the Dirichlet problem with zero boundary data. Therefore by the previous step for every m there exist a solution u_m in $W_{\Psi}^{1,2}$ provided λ is sufficiently large. Now it is obvious that u_m converges to a solution of (1), (2) in the norm defined by the left hand side of the inequality (26) and this completes the proof.

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