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f-radical extensions of rings

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f-Radical Extensions of Rings.

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SUMMARY - If R is a ring with subring A and if $f(x_1, \ldots, x_d)$ is a multilinear, homogeneous polynomial in d non-commutating variables, we say that R is an f-radical extension of A if for every $r_1, \ldots, r_d \in R$ there is an integer $n = n(r_1, \ldots, r_d) \geqslant 1$ such that $f(r_1, \ldots, r_d)^n \in A$. With an addition technical hypothesis added we prove that if R is prime with no non-zero nil left ideals and if R is an f-radical extension of A then: (1) if A is a subdivision ring either $R = A$ or f is power-central valued, (2) if A has no non-zero nilpotent elements either R is a domain or f is power-central valued, and (3) if ψ is an automorphism of R which is the identity on A then either ψ is the identity on R or f is power-central valued.

Let R be a ring and $f = f(x_1, \ldots, x_d)$ a multilinear, homogeneous polynomial in d non-commuting variables. In [4] Herstein, Procesi, and Schacher examine the situation where f is power-central valued, that is, for every $r_1, \ldots, r_d \in R$ there is an integer $n = n(r_1, \ldots, r_d) \geqslant 1$ such that $f(r_1, \ldots, r_d)^n$ is central. It follows from the work in [4] that if R has no non-zero nil left ideals then R must satisfy S_{d+2}, the standard identity in $d + 2$ variables, provided an additional technical hypothesis also holds.

In general, we say that R is an f-radical extension of a subring A if, for every $r_1, \ldots, r_d \in R$ there is an integer $n = n(r_1, \ldots, r_d) \geqslant 1$ such

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that $f(r_1, \ldots, r_d)^n \in A$. The results in [4] can be viewed as the case where R is f-radical over its center. In this paper, we will consider the situation where R is f-radical over an arbitrary subring A and we will be concerned with contrasting the structure of R and A. More precisely, we are interested in seeing when certain properties of A must also hold for R.

At this point, we introduce the following notation, which we will use throughout this paper:

(1) R will be an associative ring with center Z.

(2) $f(x_1, \ldots, x_d)$ will denote a multilinear, homogeneous polynomial in d non-commuting variables. Therefore we will assume that f is of the form $f = \alpha x_1 \ldots x_d + \sum_{\pi \neq 1} \alpha_\pi x_{\pi(1)} \ldots x_{\pi(d)}$, where $\pi \in S_d$, the symmetric group on d letters and $\alpha, \alpha_\pi \in C$ where C is some commutative ring with 1 such that R is an algebra over C and $\alpha R \neq 0$.

(3) $f(x_1, \ldots, x_d)$ will often be abbreviated as f or $f(x_i)$. Similarly, integers $n(r_1, \ldots, r_d)$ may be abbreviated as n or $n(r_j)$.

(4) $T(R) = \{a \in R\colon af(r_i)^n = f(r_i)^n a,\ n = n(a, r_1, \ldots, r_d) \geqslant 1$ for all $r_1, \ldots, r_d \in R\}$.

(5) If S is a subset of R, then $r(S) = \{x \in R\colon Sx = 0\}$.

In our study as in [4], one problem does arise. Suppose R is a division ring of characteristic $p > 0$ and f is a power-central valued, multilinear, homogeneous polynomial of degree d. The proof in [4] that R satisfies S_{d+2} uses the assumption that f is not an identity for the $p \times p$ matrices of characteristic p. It is still an open question as to whether this hypothesis is necessary. However to apply the results in [4] to prime rings with no non-zero nil left ideals, one must use the hypothesis that if the characteristic of R is $p > 0$ then f is not an identity for the $p \times p$ matrices of characteristic p. As a result, we will also need this extra hypothesis in order to prove our main result.

At this point we can now state the main result of this paper.

THEOREM. Let R be a prime ring with no non-zero nil left ideals and let $f(x_1, \ldots, x_d)$ be a multilinear, homogeneous polynomial. Suppose R is an f-radical extension of a subring A where if $\operatorname{char} R = p > 0$

assume f is not an identity for the $p \times p$ matrices of characteristic p. Then:

(1) If A is a subdivision ring either $R = A$ or f is power-central valued.

(2) If A has no non-zero nilpotent elements either R is a domain or f is power-central valued.

(3) If ψ is an automorphism of R which is the identity on A either ψ is the identity on R or f is power-central valued.

We note that if f is power-central valued then R is f-radical over every subring A which contains Z, however in this case R and A need not have much in common. Therefore, in all three parts of our theorem, the general flavor of the results is that if f is not power-central valued then R and A are similar.

In all that follows, unless stated otherwise, *we will assume* that R is a prime ring with no non-zero nil left ideals and R is f-radical over a subring A. Furthermore, assume that f is not power-central valued and if $\operatorname{char} R = p > 0$, assume that f is not an identity for the $p \times p$ matrices of characteristic p.

We now begin the work necessary to prove the first part of our theorem.

LEMMA 1. If A is a subdivision ring of R then R is simple.

PROOF. Let e be the unit element of A; therefore $e \in T(R)$. By a result of Felzenszwalb and Giambruno [1], $T(R) = Z$, thus e is a central idempotent of R. Since R is prime, e is the unit element of R, hence every non-zero element of A is invertible in R.

As a result, if $r_1, \ldots, r_d \in R$ either $f(r_i)$ is nilpotent or invertible. Now, let $I \neq 0$ be a proper ideal of R; if $s_1, \ldots, s_d \in I$ it follows that $f(s_i)$ must be nilpotent. Since I is also a prime ring with no non-zero nil left ideals, by another result of Felzenszwalb and Giambruno [2], f is a polynomial identity for I. Thus f is also an identity for R hence, in particular, f is power-central valued, a contradiction. Therefore, R is simple.

We proceed with

LEMMA 2. If A is a subdivision ring then either R is a division ring or R satisfies a polynomial identity.

PROOF. Suppose R is not a division ring and let L be a proper left ideal; if $s_1, \ldots, s_d \in L$ then $f(s_i)$ is nilpotent. The ring $\bar{L} = L/L \cap r(L)$ is also prime with no non-zero nil left ideals and all the values of f on $\bar{L}$ are nilpotent. By the result in [2], f is an identity for $\bar{L}$, hence $x_{d+1} f(x_1, \ldots, x_d)$ is an identity for L.

Now let M be a left ideal of R maximal with respect to satisfying $x_{d+1} f(x_i)$. If $s \in R$, Ms also satisfies a polynomial identity and, by a result of Rowen [6], $M + Ms$ also satisfies a polynomial identity. If $M + Ms$ is a proper left ideal for every $s \in S$, then $M + Ms$ will satisfy $x_{d+1} f(x_i)$. By the maximality of M, $M + Ms \subset M$, thus $Ms \subset M$ and so, $M = R$. On the other hand, if $M + Ms = R$, for some $s \in R$, then once again, R satisfies a polynomial identity.

We can now state and prove the first part of our main theorem.

THEOREM 3. If A is a subdivision ring of R then $R = A$.

PROOF. By Theorem 1 of [3], if R is a division ring then $R = A$. Therefore, by Lemmas 1 and 2, if $R \neq A$ then R satisfies a polynomial identity and R is the $n \times n$ matrices over a division ring where $n > 1$.

If $Z(A)$, the center of A, were finite than A would be a field since A is finite dimensional over its center. However, in this case, A would then lie in the center of R, contradicting the assumption that f is not power-central valued. Thus $Z(A)$ is infinite and, in particular, there exists a $0 \neq \alpha \in Z(A) \subset Z(R)$ such that $\alpha + 1 \neq 0$. Let $y = \alpha + e_{1n}$; then both y and $1 + y$ are invertible, non-central elements of R.

Since $y \notin T(R)$ there exist $r_1, \ldots, r_d \in R$ such that $yf(r_i)^m \neq f(r_i)^m y$, for every positive integer m. Now let $m > 1$ be such that

$$f(r_i)^m , \qquad f(yr_i y^{-1})^m = yf(r_i)^m y^{-1} ,$$

and

$$f((1+y)r_i(1+y)^{-1})^m = (1+y)f(r_i)^m(1+y)^{-1}$$

all belong to A. Hence

(1) $yf(r_i)^m = ay$ and

(2) $(1+y)f(r_i)^m = b(1+y)$ for some $a, b \in A$.

Subtracting (1) from (2) yields $f(r_i)^m = b + (b - a)y$. If $b = a$ then

$yf(r_i)^m = f(r_i)^m y$, a contradiction. However, if $b \neq a$ then

$$y = (b-a)^{-1}(f(r_i)^m - b)\,, \qquad \text{hence } y \in A\,.$$

As a result, $e_{1n} = y - a \in A$, which is a contradiction, since e_{1n} is nilpotent.

In light of Theorem 3, it is natural to wonder what can be said if we merely assume that A is a domain. Clearly it no longer need be that $R = A$, however one might hope to prove that R is also a domain. In fact, in order to prove that R is a domain, we only need to assume that A has no non-zero nilpotent elements. This is part two of our main theorem which we now record as

THEOREM 4. If A has no non-zero nilpotent elements then R is a domain.

PROOF. Suppose R is not a domain; since R is prime there is some $0 \neq t \in R$ such that $t^2 = 0$. If $r_1, \ldots, r_d \in R$ there is a positive integer m such that $f(r_i t)^m$ and $f((1+t)r_i t(1-t))^m = (1+t)f(r_i t)^m(1-t)$ both belong to A. Therefore $tf(r_i t)^m \in A$ and, since A has no non-zero nilpotent elements, $tf(r_i t)^m = 0$. As in the proof of Lemma 2, f has only nilpotent values on $Rt/Rt \cap r(Rt)$, thus f is an identity for $Rt/Rt \cap r(Rt)$. Thus $tf(x_1 t, \ldots, x_d t)$ is a generalized polynomial identity for R

Now, if R satisfies a polynomial identity then A is a semiprime P.I. ring and so, every non-zero ideal of A intersects $Z(A)$, the center of A, non-trivially. However, $Z(A) \subset T(R) = Z(R)$ and $Z(R)$ is a domain, thus A is prime and therefore is a domain. We can now localize R and A at the non-zero elements of $Z(A)$ to obtain rings R_1 and A_1 respectively. R_1 is an f-radical extension of the division ring A_1 and, by Theorem 2, $R_1 = A_1$. Hence R is a domain.

Therefore, we may now assume that R satisfies a G.P.I. but not a P.I. By a theorem of Martindale [5], if C is the extended center of R then $S = RC$ is a primitive ring with minimal right ideal eS; moreover the commuting division ring $D = eSe$ is finite dimensional over C. Since R does not satisfy a P.I., by a theorem of M. Smith [7], for any $n \geq 1$, R contains a prime P.I. subring $R^{(n)}$ such that $R^{(n)}$ is isomorphic to the ring of $n \times n$ matrices over a subring E of D and $R^{(n)}$ satisfies no P.I. of degree less than n. Let $n > \frac{1}{2}(d+2)$; then by the work in [4], f cannot be power-central valued on $R^{(n)}$. However,

$R^{(n)}$ is an f-radical extension of $R^{(n)} \cap A$, therefore $R^{(n)}$ is a domain contradicting the fact that $n > 1$.

We can now begin the series of reduction necessary to prove the third part of our main theorem. *In all that follows, we will assume* that ψ is an automorphism of R which is the identity on A.

LEMMA 5. If $a \in R$ is invertible or formally invertible then $\psi(a) = \beta a$, for some $\beta \in Z(R)$.

PROOF. If $r_1, \ldots, r_d \in R$, let $m \geqslant 1$ be such that $\psi(f(r_i)^m) = f(r_i)^m$ and $\psi(f(ar_i a^{-1})^m) = \psi(af(r_i)^m a^{-1}) = af(r_i)^m a^{-1}$. Thus

$$\psi(a) f(r_i)^m \psi(a)^{-1} = af(r_i)^m a^{-1}$$

and so,

$$a^{-1}\psi(a) f(r_i)^m = f(r_i)^m a^{-1}\psi(a) .$$

Therefore $a^{-1}\psi(a) = \beta \in T(R) = Z(R)$, hence $\psi(a) = \beta a$.

We continue with

LEMMA 6. If $J(R)$, the Jacobson radical of R, is non-zero then $\psi = 1$.

PROOF. If $r \in J(R)$ then $1 + r$ is formally invertible. By Lemma 5, $\psi(1 + r) = \beta(1 + r)$, for some $\beta \in Z(R)$, which yields $\psi(r) = (\beta - 1) + \beta r$ and, finally $r\psi(r) = \psi(r)r$. It is well-known and easy to prove that if $r\psi(r) = \psi(r)r$, for all r in a non-zero ideal of a prime ring R, then either R is commutative or $\psi = 1$. Since f is not power-central valued, $\psi = 1$.

We now proceed to the hardest part of the theorem, the case where R is primitive.

LEMMA 7. If R is primitive then $\psi = 1$.

PROOF. Let V be a faithful, irreducible right R module with commuting ring D. If $t \in R$, $t^2 = 0$ then, by Lemma 5, $\psi(1 + t) = \beta(1 + t)$, resulting in $\psi(t) = (\beta - 1) + \beta t$. Squaring both sides of this equation yields $0 = (\beta - 1)^2 + 2\beta(\beta - 1)t$ and multiplication by t gives us $0 = (\beta - 1)^2 t$, hence $\beta = 1$. Thus, $\psi(t) = t$.

Now, suppose V is finite dimensional over D, therefore $R = D_n$, for some integer $n \geqslant 1$. If $n = 1$ then by Theorem 1 of [3] or by Theorem 3, $R = A$ and clearly $\psi = 1$. On the other hand, if $n > 1$ then the subring of R generated by all the elements of square zero is all of R. However, by the argument in the previous paragraph, ψ fixes all elements of sqare zero, so again $\psi = 1$. Therefore, we may now assume that V is infinite dimensional over D. Now, since R is primitive, we can write $V = vR$ for some $v \neq 0 \in V$ and, so, we can define an action of C on V as follows: if $w = vr \in V$ and $c \in C$, $wc = vcr$. In this way it is easy to check that C can be identified with a subring of the center of the commuting ring D.

Suppose v_0, v_1 are linearly indepenent elements of V; let $v_2, \ldots, v_d \in V$, be such that $v_0, \ldots, v_d$ are all linearly independent. Now by Jacobson density, let $r_1, \ldots, r_d \in R$ such that $v_0 r_1 = v_2$, $v_i r_i = v_{i+1}$ for $1 \leqslant i \leqslant d-1$, $v_d r_d = v_0$, and $v_i r_j = 0$ otherwise. Since

$$f(x_i) = \alpha x_1 \ldots x_d + \sum_{\pi \neq 1} \alpha_\pi x_{\pi(1)} \ldots x_{\pi(d)} ,$$

we have $v_1 f(r_i) = \alpha v_0$ and $v_0 f(r_i) = \alpha v_0$, thus $v_1 f(r_i)^m = \alpha^m v_0$, for any integer $m \geqslant 1$. If we let m be such that $f(r_i)^m = a \in A$, we have $v_1 a = \alpha^m v_0$ and therefore A acts both faithfully and irreducibly on V. As a result, we may now assume that both R and A act densely on V.

Let $0 \neq u \in V$ and suppose uz and $u\psi(z)$ are linearly dependent, for all $z \in R$. Now, let $x, y \in R$ such that ux and uy are linearly independent then

$$u\psi(x) = \lambda_x ux , \quad u\psi(y) = \lambda_y uy , \quad \text{and} \quad u\psi(x+y) = \lambda_{x+y} u(x+y) ,$$

where $\lambda_x, \lambda_y, \lambda_{x+y} \in D$. Therefore $\lambda_{x+y} ux + \lambda_{x+y} uy = \lambda_x ux + \lambda_y uy$, thus $\lambda_x = \lambda_y$. As a result, $u\psi(z) = \lambda uz$, where λ does not depend on z. However, if we let $z \in A$ such that $uz \neq 0$ we obtain $uz = u\psi(z) = \lambda uz$, hence $\lambda = 1$. By this argument, if uz and $u\psi(z)$ are linearly dependent for all $u \in V$ and $z \in R$, then $\psi = 1$. We may therefore assume that there exists a $u \in V$ and $z \in R$ such that uz and $u\psi(z)$ are linearly independent.

Let $v_1, \ldots, v_{d-1} \in V$ be such that $u, v_1, \ldots, v_{d-1}$ are linearly independent; by the density of A, let $r_2, \ldots, r_d \in A$ be such that $v_i r_{i+1} = v_{i+1}$ for $1 \leqslant i \leqslant d-2$, $v_{d-1} r_d = u$, $ur_j = 0$ for $2 \leqslant j \leqslant d$, and $v_i r_j = 0$ otherwise. In addition, let $a \in A$ such that $u\psi(z)a = v_1$, $uza = 0$ and let $r_1 = za$. Therefore, $uf(r_1, \ldots, r_d) = 0$ and $uf(\psi(r_1), r_2, \ldots, r_d) = \alpha u$. Hence, for any integer $m \geqslant 1$, $uf(r_i)^m = 0$ whereas $u\psi(f(r_i))^m = \alpha^m u$. However, there exists some $m \geqslant 1$ such that $\psi(f(r_i))^m = f(r_i)^m$, a contradiction, thereby proving the lemma.

We now have all the pieces necessary to prove the third part of our main theorem which we record as

THEOREM 8. If ψ is an automorphism of R which is the identity on A, then ψ is the identity on R.

PROOF. By Lemma 6, if $J(R) \neq 0$ then $\psi = 1$, thus it is enough to handle the case where R is semisimple. We can extend the action of ψ to C and then can let $\bar{f}$ denote the polynomial

$$\bar{f}(x_1) = x_i \dots x_d + \sum_{\pi \neq 1} \psi(\alpha_\pi) x_{\pi(1)} \dots x_{\pi(d)} .$$

If P is a primitive ideal of R and $s_1, \dots, s_d \in P$ then $\bar{f}(\psi(s_1)) = \psi(f(s_i))$, hence $\bar{f}(\psi(s_i))^m = \psi(f(s_i))^m = f(s_i)^m \in P$, for some $m \geqslant 1$. Since $\bar{f}$ is nil valued on the ring $\psi(P) + P/P$, the the result in [2], $\bar{f}$ is an identity for $\Psi(P) + P/P$. Thus if $\psi(P) \not\subset P$ then $\psi(P) + P/P$ is a non-zero ideal of the primitive ring R/P, hence R/P also satisfies $\bar{f}$ and clearly $\bar{f}$ is now an identity for R/P.

We now partition the primitive ideals of R into three sets;

$B_1 = \{P\colon \psi(P) \not\subset P\}$,

$B_2 = \{P\colon \psi(P) \subset P$ and f is power-central valued on $R/P\}$,

$B_3 = \{P\colon \psi(P) \subset P$ and f is not power-central valued on $R/P\}$.

In addition, let $I_i = \bigcap_{P \in B_i} P$, for $i = 1, 2, 3$.

Since R is semisimple, $I_1 I_2 I_3 \subset I_1 \cap I_2 \cap I_3 = 0$; however, by the primeness of R, at least one of I_1, I_2, or I_3 is zero. If $I_1 = 0$ then R is a subdirect sum of rings satisfying f, hence f is an identity for R, a contradiction. If $I_2 = 0$ then, by the work in [4], R is a subdirect sum of rings satisfying S_{d+2}, the standard identity in $d + 2$ variables. Since R satisfies a polynomial identity, we can let R_Z denote the localization of R at the non-zero elements of $Z(R)$. In particular, we can extend the action of ψ to R_Z and R_Z is primitive. R does not satisfy f, therefore there exist $r_i \in R$ such that $f(r_i)$ is not nilpotent [2]. If $0 \neq \alpha \in Z(R)$ there is an $m \geqslant 1$ such that $\alpha^m f(r_i)^m = f(\alpha r_1, r_2, \dots, r_d)^m$ and $f(r_i)^m$ belong to A. Hence $\alpha^m f(r_i)^m = \psi(\alpha^m f(r_i)^m) = \psi(\alpha^m) f(r_i)^m$, thus $\psi(\alpha^m) = \alpha^m$. As a result, if $s_i \in R_Z$ there is an $n \geqslant 1$ such that $\psi(f(s_i)^n) = f(s_i)^n$, therefore, by lemma 7, either f is power-central on on R_Z or $\psi = 1$ on R_Z. However, f is not power-central on R, hence $\psi = 1$ on R_Z, so certainly $\psi = 1$ on R.

Finally, if $I_3 = 0$ then R is a subdirect sum of primitive rings on which ψ induces an automorphism $\bar{\psi}$ satisfying all the hypotheses of Lemma 7. Since $\bar{\psi}$ is the identity on R/P, for each $P \in B_3$, ψ is the identity on R. This concludes the proof of the theorem.

By combining Theorems 3, 4, and 8 we obtain our main result which we mentioned at the outset of the paper:

THEOREM. Let R be a prime ring with no non-zero nil left ideals and let $f(x_1, \ldots, x_d)$ be a multilinear, homogeneous polynomial. Suppose R is an f-radical extension of a subring A where if $\operatorname{char} R = p > 0$ assume f is not an identity for the $p \times p$ matrices of characteristic p. Then:

(1) If A is a subdivision ring either $R = A$ or f is power-central valued.

(2) If A has no non-zero nilpotent elements either R is a domain or f is power-central valued.

(3) If ψ is an automorphism of R which is the identity on A either ψ is the identity on R or f is power-central valued.

REFERENCES

[1] B. Felzenswalb - A. Giambruno, *Centralizers and multilinear polynomials in noncommutative rings*, J. London Math. Soc., **19** (1979), pp. 417-428.
[2] B. Felzenszwalb - A. Giambruno, *Periodic and nil polynomials in rings*, Canad. Math. Bull., **23** (1980), pp. 473-476.
[3] A. Giambruno, *Rings radical over P.I. subrings*, Rend. Mat., **13** (1980), pp. 105-113.
[4] I. N. Herstein - C. Procesi - M. Schacher, *Algebraic valued functions on noncommutative rings*, J. Algebra, **36** (1975), pp. 128-150.
[5] W. S. Martindale, *Prime rings satisfying a generalized polynomial identity*, J. Algebra, **12** (1969), pp. 576-584.
[6] L. H. Rowen, *General polynomial identities* II, J. Algebra, **38** (1976), pp. 380-392.
[7] M. Smith, *Rings with an integral element whose centralizer satisfies a polynomial identity*, Duke Math. J., **42** (1975), pp. 137-149.

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