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A Note on Certain Classes of Starlike Functions.

H. M. SRIVASTAVA - SHIGEYOSHI OWA - S. K. CHATTERJEA (*)

SUMMARY - A number of distortion theorems are proved for the classes $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$ of analytic and univalent functions with negative coefficients, studied recently by S. K. Chatterjea [1]. We also present several interesting results for the convolution product of functions belonging to the classes $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$.

1. Introduction.

Let $\mathcal{A}(n)$ denote the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathcal{N} = \{1, 2, 3, \dots\})$$

which are analytic in the unit disk $\mathcal{U} = \{z: |z| < 1\}$. Further, let $\mathcal{S}(n)$ be the subclass of $\mathcal{A}(n)$ consisting of analytic and univalent functions in the unit disk $\mathcal{U}$. Then a function $f(z) \in \mathcal{S}(n)$ is said to be

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in the class $\mathcal{S}_\alpha(n)$ if and only if

$$(1.2) \quad \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha, \quad (z \in \mathcal{U}),$$

for $0 \leq \alpha < 1$. Also a function $f(z) \in \mathcal{S}(n)$ is said to be in the class $\mathcal{K}_\alpha(n)$ if and only if

$$(1.3) \quad \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad (z \in \mathcal{U}),$$

for $0 \leq \alpha < 1$.

We note that $f(z) \in \mathcal{K}_\alpha(n)$ if and only if $zf'(z) \in \mathcal{S}_\alpha(n)$, and that

$$\mathcal{S}_\alpha(n) \subseteq \mathcal{S}_0(n), \quad \mathcal{K}_\alpha(n) \subseteq \mathcal{K}_0(n), \quad \text{and} \quad \mathcal{K}_\alpha(n) \subset \mathcal{S}_\alpha(n)$$

for $0 \leq \alpha < 1$.

The class of functions of the form (1.1) was considered by Chen [2]. The classes $\mathcal{S}_\alpha(1)$ and $\mathcal{K}_\alpha(1)$ were first introduced by Robertson [9], and were studied subsequently by Schild [10], Pinchuk [8], Owa and Srivastava ([6], [7]), and others (see, for example, Duren [3] and Goodman [4]).

Let $\mathcal{T}(n)$ denote the subclass of $\mathcal{S}(n)$ consisting of functions of the form

$$(1.4) \quad f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0).$$

Denote by $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$ the classes obtained by taking intersections, respectively, of the classes $\mathcal{S}_\alpha(n)$ and $\mathcal{K}_\alpha(n)$ with $\mathcal{T}(n)$, that is,

$$(1.5) \quad \mathcal{T}_\alpha(n) = \mathcal{S}_\alpha(n) \cap \mathcal{T}(n) \quad (0 \leq \alpha < 1; n \in \mathcal{N})$$

and

$$(1.6) \quad \mathcal{C}_\alpha(n) = \mathcal{K}_\alpha(n) \cap \mathcal{T}(n) \quad (0 \leq \alpha < 1; n \in \mathcal{N}).$$

The classes $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$ were studied recently by Chatterjea [1]. In particular, $\mathcal{T}_\alpha(1)$ and $\mathcal{C}_\alpha(1)$ are the classes $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$, respectively, which were introduced by Silverman [12].

In order to prove our results for functions belonging to the general

classes $\mathfrak{S}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$, we shall require the following lemmas given by Chatterjea [1]:

LEMMA 1. *Let the function $f(z)$ be defined by (1.4). Then $f(z)$ is in the class $\mathfrak{S}_\alpha(n)$ if and only if*

$$(1.7) \quad \sum_{k=n+1}^{\infty} \left(\frac{k-\alpha}{1-\alpha} \right) a_k \leq 1, \quad (n \geq 1).$$

LEMMA 2. *Let the function $f(z)$ be defined by (1.4). Then $f(z)$ is in the class $\mathcal{C}_\alpha(n)$ if and only if*

$$(1.8) \quad \sum_{k=n+1}^{\infty} \left(\frac{k(k-\alpha)}{1-\alpha} \right) a_k \leq 1, \quad (n \geq 1).$$

REMARK. Lemma 1 follows immediately from a result due to Silverman [12, p. 110, Theorem 2] upon setting

$$(1.9) \quad a_k = 0, \quad k = 2, 3, \dots, n.$$

Lemma 2, on the other hand, is a similar consequence of another result due to Silverman [12, p. 111, Corollary 2].

2. Distortion theorems.

We first prove

THEOREM 1. *Let the function $f(z)$ defined by (1.4) be in the class $\mathfrak{S}_\alpha(n)$. Then*

$$(2.1) \quad |z| - \left(\frac{1-\alpha}{n+1-\alpha} \right) |z|^{n+1} \leq |f(z)| \leq |z| + \left(\frac{1-\alpha}{n+1-\alpha} \right) |z|^{n+1}$$

and

$$(2.2) \quad 1 - \frac{(n+1)(1-\alpha)}{n+1-\alpha} |z|^n \leq |f'(z)| \leq 1 + \frac{(n+1)(1-\alpha)}{n+1-\alpha} |z|^n$$

for $z \in \mathcal{U}$. The results (2.1) and (2.2) are sharp.

PROOF. By virtue of Lemma 1, we have

$$(2.3) \quad \left(\frac{n+1-\alpha}{1-\alpha} \right) \sum_{k=n+1}^{\infty} a_k \leq \sum_{k=n+1}^{\infty} \left(\frac{k-\alpha}{1-\alpha} \right) a_k \leq 1,$$

or

$$(2.4) \quad \sum_{k=n+1}^{\infty} a_k \leq \frac{1-\alpha}{n+1-\alpha}.$$

It follows from (2.4) that

$$(2.5) \quad |f(z)| \geq |z| - |z|^{n+1} \sum_{k=n+1}^{\infty} a_k \geq |z| - \left(\frac{1-\alpha}{n+1-\alpha} \right) |z|^{n+1}$$

and

$$(2.6) \quad |f(z)| \leq |z| + |z|^{n+1} \sum_{k=n+1}^{\infty} a_k \leq |z| + \left(\frac{1-\alpha}{n+1-\alpha} \right) |z|^{n+1}.$$

Note that

$$(2.7) \quad \sum_{k=n+1}^{\infty} k a_k \leq 1-\alpha + \frac{\alpha(1-\alpha)}{n+1-\alpha} = \frac{(n+1)(1-\alpha)}{n+1-\alpha}.$$

Hence

$$(2.8) \quad |f'(z)| \geq 1 - |z|^n \sum_{k=n+1}^{\infty} k a_k \geq 1 - \frac{(n+1)(1-\alpha)}{n+1-\alpha} |z|^n$$

and

$$(2.9) \quad |f'(z)| \leq 1 + |z|^n \sum_{k=n+1}^{\infty} k a_k \leq 1 + \frac{(n+1)(1-\alpha)}{n+1-\alpha} |z|^n.$$

Further, by taking the function given by

$$(2.10) \quad f(z) = z - \frac{1-\alpha}{n+1-\alpha} z^{n+1},$$

we can show that the results (2.1) and (2.2) are sharp.

COROLLARY 1. *Let the function $f(z)$ defined by (1.4) be in the class $\mathcal{G}_\alpha(n)$. Then the unit disk $\mathcal{U}$ is mapped onto a domain that contains the disk $|w| < n/(n+1-\alpha)$.*

Next we state

THEOREM 2. *Let the function $f(z)$ defined by (1.4) be in the class $\mathcal{C}_\alpha(n)$. Then*

$$(2.11) \quad |z| - \frac{1-\alpha}{(n+1)(n+1-\alpha)} |z|^{n+1} \leq |f(z)| \leq \\ \leq |z| + \frac{1-\alpha}{(n+1)(n+1-\alpha)} |z|^{n+1}$$

and

$$(2.12) \quad 1 - \left(\frac{1-\alpha}{n+1-\alpha} \right) |z|^n \leq |f'(z)| \leq 1 + \left(\frac{1-\alpha}{n+1-\alpha} \right) |z|^n$$

for $z \in \mathcal{U}$. The results (2.11) and (2.12) are sharp.

PROOF. In view of Lemma 2, we have

$$(2.13) \quad \sum_{k=n+1}^{\infty} a_k \leq \frac{1-\alpha}{(n+1)(n+1-\alpha)}$$

and

$$(2.14) \quad \sum_{k=n+1}^{\infty} k a_k \leq \frac{1-\alpha}{n+1-\alpha},$$

and the assertions (2.11) and (2.12) of Theorem 2 follow from (2.13) and (2.14), respectively.

Noting that the results (2.11) and (2.12) are sharp for the function given by

$$(2.15) \quad f(z) = z - \frac{1-\alpha}{(n+1)(n+1-\alpha)} z^{n+1},$$

the proof of Theorem 2 is completed.

COROLLARY 2. *Let the function $f(z)$ defined by (1.4) be in the class $\mathcal{C}_\alpha(n)$. Then the unit disk $\mathcal{U}$ is mapped onto a domain that contains the disk $|w| < r_0$, where r_0 is given by*

$$(2.16) \quad r_0 = 1 - \frac{1-\alpha}{(n+1)(n+1-\alpha)}.$$

3. Convolution product.

Let $f_i(z)$ ($i = 1, 2$) be defined by

$$(3.1) \quad f_i(z) = z - \sum_{k=n+1}^{\infty} a_{i,k} z^k \quad (a_{i,k} \geq 0).$$

We denote by $f_1 * f_2(z)$ the convolution product of the functions $f_1(z)$ and $f_2(z)$ defined by

$$(3.2) \quad f_1 * f_2(z) = z - \sum_{k=n+1}^{\infty} a_{1,k} a_{2,k} z^k,$$

and we prove

THEOREM 3. *Let the functions $f_i(z)$ ($i = 1, 2$) be in the class $\mathfrak{C}_\alpha(n)$. Then $f_1 * f_2(z)$ belongs to the class $\mathfrak{C}_\beta(n)$, where*

$$(3.3) \quad \beta = \frac{n+1-\alpha^2}{n+2-2\alpha}.$$

The result is sharp.

PROOF. Employing the technique used earlier by Schild and Silverman [11], we need to find the largest β such that

$$(3.4) \quad \sum_{k=n+1}^{\infty} \left(\frac{k-\beta}{1-\beta} \right) a_{1,k} a_{2,k} \leq 1.$$

Since

$$(3.5) \quad \sum_{k=n+1}^{\infty} \left(\frac{k-\alpha}{1-\alpha} \right) a_{1,k} \leq 1$$

and

$$(3.6) \quad \sum_{k=n+1}^{\infty} \left(\frac{k-\alpha}{1-\alpha} \right) a_{2,k} \leq 1,$$

by the Cauchy-Schwarz inequality we have

$$(3.7) \quad \sum_{k=n+1}^{\infty} \left(\frac{k-\alpha}{1-\alpha} \right) \sqrt{a_{1,k} a_{2,k}} \leq 1.$$

Thus it is sufficient to show that

$$(3.8) \quad \left(\frac{k-\beta}{1-\beta} \right) a_{1,k} a_{2,k} \leq \left(\frac{k-\alpha}{1-\alpha} \right) \sqrt{a_{1,k} a_{2,k}}, \quad (k \geq n+1),$$

that is, that

$$(3.9) \quad \sqrt{a_{1,k} a_{2,k}} \leq \frac{(k-\alpha)(1-\beta)}{(1-\alpha)(k-\beta)}.$$

Note that

$$(3.10) \quad \sqrt{a_{1,k} a_{2,k}} \leq \frac{1-\alpha}{k-\alpha}, \quad (k \geq n+1).$$

Consequently, we need only to prove that

$$(3.11) \quad \frac{1-\alpha}{k-\alpha} \leq \frac{(k-\alpha)(1-\beta)}{(1-\alpha)(k-\beta)}, \quad (k \geq n+1),$$

or, equivalently, that

$$(3.12) \quad \beta \leq \frac{(k-\alpha)^2 - k(1-\alpha)^2}{(k-\alpha)^2 - (1-\alpha)^2}, \quad (k \geq n+1).$$

Since

$$(3.13) \quad \Phi(k) = \frac{(k-\alpha)^2 - k(1-\alpha)^2}{(k-\alpha)^2 - (1-\alpha)^2}$$

is an increasing function of k , letting $k = n+1$ in (3.12) we obtain

$$(3.14) \quad \beta \leq \Phi(n+1) = \frac{n+1-\alpha^2}{n+2-2\alpha},$$

which completes the proof of the theorem.

Finally, by taking the functions given by

$$(3.15) \quad f_i(z) = z - \frac{1-\alpha}{n+1-\alpha} z^{n+1}, \quad (i = 1, 2),$$

we can see that the result is sharp.

Similarly, we have

THEOREM 4. *Let the functions $f_i(z)$ ($i = 1, 2$) defined by (3.1) be in the class $\mathcal{C}_\alpha(n)$. Then $f_1 * f_2(z)$ belongs to the class $\mathcal{C}_\beta(n)$, where*

$$(3.16) \quad \beta = \frac{n(n+1)(n+2-2\alpha)}{(n+1)^3 - 2n(n+2)\alpha + n\alpha^2 - 1}.$$

The result is sharp for the functions given by

$$(3.17) \quad f_i(z) = z - \frac{1-\alpha}{(n+1)(n+1-\alpha)} z^{n+1}, \quad (i = 1, 2).$$

Finally, we prove

THEOREM 5. *Let the functions $f_i(z)$ ($i = 1, 2$) defined by (3.1) be in the class $\mathcal{G}_\alpha(n)$. Then the function*

$$(3.18) \quad h(z) = z - \sum_{k=n+1}^{\infty} (a_{1,k}^2 + a_{2,k}^2) z^k$$

belongs to the class $\mathcal{G}_\beta(n)$, where

$$(3.19) \quad \beta = \frac{n^2 + 2n\alpha(1-\alpha) - (1-\alpha)^2}{n^2 + 2n(1-\alpha) - (1-\alpha)^2}.$$

The result is sharp for the functions $f_i(z)$ ($i = 1, 2$) defined by (3.15).

PROOF. By virtue of Lemma 1, we obtain

$$(3.20) \quad \sum_{k=n+1}^{\infty} \left(\frac{k-\alpha}{1-\alpha} \right)^2 a_{1,k}^2 \leq \left[\sum_{k=n+1}^{\infty} \left(\frac{k-\alpha}{1-\alpha} \right) a_{1,k} \right]^2 \leq 1$$

and

$$(3.21) \quad \sum_{k=n+1}^{\infty} \left(\frac{k-\alpha}{1-\alpha} \right)^2 a_{2,k}^2 \leq \left[\sum_{k=n+1}^{\infty} \left(\frac{k-\alpha}{1-\alpha} \right) a_{2,k} \right]^2 \leq 1.$$

It follows from (3.20) and (3.21) that

$$(3.22) \quad \sum_{k=n+1}^{\infty} \left(\frac{k-\alpha}{1-\alpha} \right)^2 (a_{1,k}^2 + a_{2,k}^2) \leq 2.$$

Therefore, we need to find the largest β such that

$$(3.23) \quad \frac{k-\beta}{1-\beta} \leq \frac{1}{2} \left(\frac{k-\alpha}{1-\alpha} \right)^2, \quad (k \geq n+1),$$

that is,

$$(3.24) \quad \beta \leq \frac{(k-\alpha)^2 - 2k(1-\alpha)^2}{(k-\alpha)^2 - 2(1-\alpha)^2}, \quad (k \geq n+1).$$

Since

$$(3.25) \quad \Psi(k) = \frac{(k-\alpha)^2 - 2k(1-\alpha)^2}{(k-\alpha)^2 - 2(1-\alpha)^2}$$

is an increasing function of k , we readily have

$$(3.26) \quad \beta \leq \Psi(n+1) = \frac{n^2 + 2n\alpha(1-\alpha) - (1-\alpha)^2}{n^2 + 2n(1-\alpha) - (1-\alpha)^2},$$

and Theorem 5 follows at once.

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