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Direct Products of Linearly Compact Primary Rings.

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SUMMARY - Numakura [4], [5] gave criterions for compact rings to be direct products of primary rings. In this note we extend these results to linearly compact rings. As a consequence we get a characterization of those rings which are direct products of division rings, of uni-serial rings (i.e. such artinian rings in which the P^k ($k = 1, 2, \dots$) are all the one-sided ideals where P denotes the unique maximal ideal) and of rings of complete discrete valuations on division rings.

1. All rings considered will be associative rings with identity. In what follows R denotes always a ring and J is its Jacobson radical (briefly: radical). A ring R is said to be *topologically artinian* or *topologically noetherian* if it is the inverse limit of artinian or noetherian left R -modules endowed with the inverse limit topology. A topological ring R is called *linearly compact* if open left ideals form a base for neighbourhoods of zero and every finitely solvable system of congruences $x \equiv x_k \pmod{L_k}$ where the L_k are closed left ideals, is solvable. Remark that every topologically artinian ring is linearly compact (see [2] satz 4) and every compact ring is topologically artinian (see [4] Lemma 5).

By a classical result of Artin every artinian primary ring is a matrix ring over a local ring. In [3] Leptin proved that every topologically artinian primary ring is a matrix ring (not necessarily of finite size) over a local ring. These results can be considered as a generalization

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of Artin-Wedderburn Theorem on simple artinian rings. Throughout this note a primary ring is understood in a more general sense than what is usual. We call a ring *primary* if the factor by its radical is the endomorphism ring of a vector space over a division ring. Now our purpose is to characterize direct products of linearly compact rings of some kind.

Before turning to the main results we need some preparations.

For every subset A of a topological space we denote by $\bar{A}$ its topological closure. Define for every ideal I of a topological ring the following ideals

$${}_1I = \bar{I}, \quad {}_{\mu+1}I = \overline{I \cdot {}_\mu I}, \quad {}_\lambda I = \bigcap_{\mu < \lambda} {}_\mu I \quad \text{if } \lambda \text{ is a limit ordinal,}$$

$$I = \bar{I}, \quad I = \overline{I \cdot I}, \quad I = \bigcap_{\mu < \lambda} I \quad \text{if } \lambda \text{ is a limit ordinal,}$$

$$I_1 = \bar{I}, \quad I_{\mu+1} = \overline{I_\mu \cdot I}, \quad I_\lambda = \bigcap_{\mu < \lambda} I_\mu \quad \text{if } \lambda \text{ is a limit ordinal.}$$

Then there is an ordinal η for which ${}_xi = {}_\eta I$, $I = I_\xi$, $I_\xi = I_\eta$ for all $\xi \geq \eta$. These ${}_xi$, I , I_η will be denoted by ${}_*I$, $\bar{I}$, I_* and $\overline{I \cdot {}_*I} = {}_*I$, $\bar{I}^2 = I$, $\overline{I_* \cdot I} = I_*$ hold clearly. If ${}_*I = 0$ ($I_* = 0$), then I is called *l(r-)transfinitely nilpotent*. If $I_* = 0$, then I is said to be *transfinitely nilpotent*.

PROPOSITION 1 ([2] (3)). *If the set $\{L_\alpha\}$ of closed left ideals in a linearly compact ring has the finite intersection property, then*

$$A + \bigcap L_\alpha = \bigcap (A + L_\alpha)$$

holds for every closed left ideal A .

PROPOSITION 2 ([2] Lemma). *For any two (one-sided) ideals A and B of a topological ring we have*

$$\overline{AB} = \overline{AB} = \overline{AB} = \overline{AB}.$$

PROPOSITION 3 ([2] Satz 9). *The radical of a topologically artinian ring is r-transfinitely nilpotent.*

PROPOSITION 4 ([2] (4)). *If A and B are closed (left) ideals in a linearly compact ring, then $A + B$ is closed.*

PROPOSITION 5 ([2] Satz 4). *Every continuous isomorphism between topologically artinian rings is a topological isomorphism.*

PROPOSITION 6 ([3] Satz 1). *Every linearly compact module over a semisimple linearly compact ring is a direct product of simple modules.*

PROPOSITION 7. *Let A and B be closed (left) ideals in a linearly compact R . If $A + B = R$ holds, then we have $A_\xi + B_\xi = R$ for every ordinal $\xi \geq 1$.*

PROOF. We prove the assertion by transfinite induction. By the assumption we have $A_1 + B_1 = A + B = R$. Suppose that $A_\xi + B_\xi = R$, then there are elements $a \in A_\xi$, $b \in B_\xi$ such that $a + b = 1$ and so $a^2 = 1 - 2b + b^2$. By $2b - b^2 \in B_\xi$ we have $A_{\xi+1} + B_\xi = R$. This implies the existence of elements $a^* \in A_{\xi+1}$ and $b^* \in B_\xi$ such that $a^* + b^* = 1$. This shows that $(b^*)^2 = 1 - 2a^* + (a^*)^2 = 1 - c$ with $c = 2a^* - (a^*)^2 \in A_{\xi+1}$ from which we obtain $A_{\xi+1} + B_{\xi+1} = R$.

If we have now the equality $A_\xi + B_\xi = R$ for every ordinal $\xi < \lambda$ where λ is a limit ordinal, then for each $\mu < \lambda$ it is true that

$$A_\xi + B_\mu = A_\xi + B_\xi + B_\mu = R \quad \text{if } \lambda > \xi \geq \mu$$

and

$$A_\xi + B_\mu = A_\xi + A_\mu + B_\mu = R \quad \text{if } \lambda > \mu > \xi$$

that is, $A_\xi + B_\mu = R$ holds for every $\xi, \mu < \lambda$.

Now we have by Proposition 1 for every fixed $\mu < \lambda$

$$R = \bigcap_{\xi < \lambda} (A_\xi + B_\mu) = \bigcap_{\xi < \lambda} A_\xi + B_\mu = A_\lambda + B_\mu$$

which implies

$$R = \bigcap_{\mu < \lambda} (A_\lambda + B_\mu) = A_\lambda + \bigcap_{\mu < \lambda} B_\mu = A_\lambda + B_\lambda.$$

This completes the proof.

PROPOSITION 8. *For any closed (left) ideals A and B satisfying $A + B = R$ in a linearly compact ring R . $AB = BA$ implies $\overline{AB} = \overline{A \cap B} = \overline{AB}$ and $\overline{AB} = \overline{BA}$ implies $\overline{AB} = A \cap B$.*

PROOF.

$$A \cap B = (A \cap B)R = (A \cap B)(A + B) \subseteq \overline{AB}$$

implying $AB = A \cap B = \overline{AB}$ in the case $AB = BA$.

$$A \cap B = (A \cap B)R = (A \cap B)(A + B) \subseteq \overline{AB + BA} \subseteq AB + BA = \overline{AB}$$

implying $A \cap B = \overline{AB}$ in the case $\overline{AB} = \overline{BA}$.

2. A topological ring R is called of *finite type* if the set of all isomorphism classes of simple submodules in all factors of any finitely generated discrete R -module is finite. Consider now a topologically artinian ring R . Let $\{P_i | i \in I\}$ be the set of all maximal closed ideals of R where I is the index set and $P_i \neq P_j$ for $i \neq j$, $i, j \in I$.

In case $i \neq j$ by Proposition 4 the ideal $P_i + P_j$ is closed in R and hence we get $P_i + P_j = R$. Therefore we obtain $(P_i)_\xi + (P_j)_\xi = R$ for every ordinal ξ by Proposition 7. Thus $(P_i)_* + (P_j)_* = R$ holds. Henceforth for any finite set $\{(P_1)_*, \dots, (P_n)_*\}$ we have by the Chinese Remainder Theorem

$$(P_i)_* + \bigcap_{k \neq i} (P_k)_* = R, \quad k = 1, \dots, n.$$

PROPOSITION 9. *Let R be a topologically artinian ring of finite type. If $\overline{P_i P_j} = \overline{P_j P_i}$, $\overline{P_i(P_j)_\lambda} = \overline{(P_j)_\lambda P_i}$, $\overline{(P_j)_\lambda (P_i)_\xi} = \overline{(P_i)_\xi (P_j)_\lambda}$ hold for any $i \neq j$, $i, j \in I$ and any limit ordinals ξ, λ then $\bigcap_i (P_i)_* = 0$.*

PROOF. First we obtain $\overline{(P_i)_\xi (P_j)_\eta} = \overline{(P_j)_\eta (P_i)_\xi}$ for any $i \neq j$, $i, j \in I$ and any ordinals ξ and η . We proceed by double transfinite induction. Suppose that for some ξ and some fixed η , we have $\overline{(P_i)_\chi (P_j)_\eta} = \overline{(P_j)_\eta (P_i)_\chi}$ for every $\chi \leq \xi$. Then we have by Proposition 2 and the associativity of the multiplication of ideals

$$\begin{aligned} \overline{(P_i)_{\xi+1} (P_j)_\eta} &= \overline{((P_i)_\xi P_i) (P_j)_\eta} = \overline{(P_i)_\xi \overline{(P_i(P_j)_\eta)}} = \\ &= \overline{(P_i)_\xi ((P_j)_\eta P_i)} = \overline{((P_i)_\xi (P_j)_\eta) P_i} = \overline{((P_j)_\eta (P_i)_\xi) P_i} = \\ &= \overline{(P_j)_\eta \overline{(P_i)_\xi P_i}} = \overline{(P_j)_\eta (P_i)_{\xi+1}} \end{aligned}$$

if now $(P_i)_\xi(P_j)_\eta = (P_j)_\eta(P_i)_\xi$ holds for some fixed η and every $\xi < \lambda$ where λ is a limit ordinal, then there exists a limit ordinal τ such that $\eta = \tau + n$ for some natural number n , and hence we have again by Proposition 2 and the associativity of the multiplication of ideals

$$\begin{aligned} \overline{(P_i)_\lambda(P)_\eta} &= \overline{((P_i)_\lambda(P_j)_\tau)P_j^n} = \overline{((P_j)_\tau(P_i)_\lambda)P_j^n} = \overline{(P_j)_\tau((P_i)_\lambda P_j^n)} = \\ &= \overline{(P_j)_\tau(P_j^n(P_i)_\lambda)} = \overline{((P_j)_\tau P_j^n)(P_i)_\lambda} = \overline{(P_j)_\eta(P_i)_\lambda}. \end{aligned}$$

Finally we let η vary and complete the proof in the same way.

Assume now indirectly that $\bigcap_i (P_i)_* \neq 0$. Then there is an element $c \in R$ and an open left ideal L with $c \in \bigcap_i (P_i)_*$ and $c \notin L$. Since R is of finite type, there are finitely many maximal closed ideals P_i , say $P_1, \dots, P_n$ such that every simple submodule of each factor of R/L is annihilated by some one of them. Consider now the submodule M of R/L consisting of those elements which are annihilated by $(P_1)_* \dots (P_n)_*$. We claim $M = R/L$. In fact, if $M \neq R/L$, then there exists an element $m \in R/L$ such that $(Rm + M)/M$ is simple, because R/L is artinian. Therefore $(Rm + M)/M$ is annihilated by some P_i , say P_1 . We obtain now

$$\begin{aligned} [(P_1)_* \dots (P_n)_*]m &= [(P_1)_* P_1 \dots (P_n)_*]m = \\ &= [(P_1)_* \dots (P_n)_*](P_1 m) \subseteq [(P_1)_* \dots (P_n)_*]M = 0 \end{aligned}$$

and consequently $m \in M$, which is a contradiction. Thus $M = R/L$ holds. This implies by $c \in \bigcap_i (P_i)_* \subseteq (P_1)_* \dots (P_n)_*$ that $c \cdot R/L = 0$, i.e. $c \in L$, a contradiction. Therefore $\bigcap_i (P_i)_* = 0$ and the proof is complete.

As $(P_i)_*$ is a closed ideal of the topologically artinian ring R , the factor $R_i = R/(P_i)_*$ is a topologically artinian ring, too. We denote by $\tilde{R}$ the direct product of the R_i , $i \in I$. Then $\tilde{R}$ endowed with the product topology is a topologically artinian ring. R is topologically isomorphic to $\tilde{R}$ by the following proposition.

PROPOSITION 10. *If A_i ($i \in I$) are closed ideals in a topologically artinian ring R such that $\bigcap A_i = 0$ and $A_i + A_j = 0$ hold for all $i \neq j$; $i, j \in I$, then R is topologically isomorphic to the direct product $\prod R/A_i$.*

PROOF. Taking the mapping $\varphi: R \rightarrow \prod R/A_i$ define by $\varphi(x) = (\dots, x_i, \dots)$ with $x_i = x + (A_i) \in R/A_i$ for every $i \in I$, we have a homomorphism from R into $\prod R/A_i$. Further $\varphi(x) = 0$ implies $x_i = 0$ for each $i \in I$, i.e. $x \in \bigcap_{i \in I} A_i = 0$, hence $x = 0$. Thus φ is an injective homomorphism, and it is clear that φ is continuous.

Finally, let $(\dots, \tilde{x}_i, \dots)$ be any element of $\prod R/A_i$, then $M_i = \varphi_i^{-1}(\tilde{x}_i)$ is a coset of the ideal A_i in R where $\varphi_i: R \rightarrow R/A_i$ is the natural projection of R onto R/A_i . Hence we can express M_i as $x_i + A_i$, $x_i \in M_i$. Taking any finite number of M_i —s, say $M_1, \dots, M_n$, we have $\bigcap_{k=1}^n M_k \neq \emptyset$ by the Chinese Remainder Theorem, hence $\bigcap_{i \in I} M_i \neq \emptyset$, since R is topologically artinian. Choosing an element $x \in \bigcap_{i \in I} M_i$, it is obvious that $\varphi(x) = (\dots, \tilde{x}_i, \dots)$. This implies that φ is a continuous isomorphism. Since R is topologically artinian, by Proposition 5 φ is a homeomorphism. This completes the proof of the assertion that R is topologically isomorphic to $\prod R/A_i$.

PROPOSITION 11. *The rings R_i , $i \in I$ are primary rings.*

PROOF. Let $\tilde{P}_i = P_i/(P_i)_*$. Then $R_i/\tilde{P}_i \simeq R/P_i$ which is the endomorphism ring of a vector space over a division ring. Since the radical of R_i is the intersection of all maximal closed ideals which are exactly the images of maximal closed ideals of R by the natural homomorphism $R \rightarrow R_i$, $\tilde{P}_i$ is clearly the radical of R_i . This means that the rings R_i ($i \in I$) are primary rings.

The above considerations yield the following

THEOREM 12. *Let R be a topologically artinian ring. The following conditions are equivalent*

1) *R is a direct product of topologically artinian primary rings R_i , $i \in I$.*

2) *Let $\{P_i | i \in I\}$ be the set of all maximal closed ideals in R then the equalities $P_i P_j = P_j P_i$, $P_i (P_j)_\lambda = (P_j)_\lambda P_i$, $(P_i)_\mu ((P_j)_\lambda) = (P_j)_\lambda (P_i)_\mu$ hold for all $i \neq j$, $i, j \in I$ and limit ordinals μ, λ and R is of finite type.*

3) *The equalities $\overline{P_i P_j} = \overline{P_j P_i}$, $\overline{P_i (P_j)_\lambda} = \overline{(P_j)_\lambda P_i}$, $\overline{(P_i)_\mu (P_j)_\lambda} = \overline{(P_j)_\lambda (P_i)_\mu}$ hold for all $i \neq j$, $i, j \in I$ and limit ordinals μ, λ , and R is of finite type.*

4) *Any ideal K of R such that $K = \overline{K^2}$, is a unital ring.*

PROOF. $1 \Rightarrow 2$. This implication is trivial, since maximal closed ideals in R are the products of a maximal closed ideal in one component with the other components. For any discrete, finitely generated module, let $\{x_1, \dots, x_n\}$ be a generator set of M . The annihilator $\text{ann}_R x_i$ of x_i is an open left ideal of R , and hence it contains almost all R_i . Therefore $\text{ann}_R(x_1, \dots, x_n)$ contains almost all R_i . This shows that almost all R_i is contained in the annihilator $\text{ann}_R M$ of M and henceforth M can be considered as a discrete module over a finite direct sum of primary rings R_i , say $R_1 \times \dots \times R_n$: Since the set of isomorphism classes of discrete simple factors of $R_1 \times \dots \times R_n$ is clearly finite, M is obviously of finite type, i.e. R is of finite type.

$2 \Rightarrow 3$. This implication is also trivial, since $A = B$ implies $\bar{A} = \bar{B}$.

$3 \Rightarrow 1$. This implication is the consequence of the fact that R and $\tilde{R}$ are topologically isomorphic.

$1 \Rightarrow 4$. Let φ_i denote the natural projection of R onto R_i for each $i \in I$. For $K_i = \varphi_i(K)$ it follows by [6] Lemma 6.1 that K is the direct product of the K_i , $i \in I$. Since $K = \bar{K}^2$, we have $K_i = \bar{K}_i^2$, and hence either $K_i = 0$ or $K_i = R_i$, because R_i is a primary ring and its radical is r -transfinitely nilpotent. This shows that K is a unital ring.

$4 \Rightarrow 1$. From the assumption it follows by $\dot{P}_i^2 = \dot{P}_i$ that $\dot{P}_i$ has an identity e_i . It is easy to see that $R_i = R/\dot{P}_i$ is a primary ring. To come to the end of the proof of the implication $4 \Rightarrow 1$ we show $\bigcap_{i \in I} \dot{P}_i = 0$. For this aim let x be an arbitrary element in $\bigcap_{i \in I} \dot{P}_i$. Then we have $x = xe_i$ for each $i \in I$ and $x \in \bigcap_{i \in I} \dot{P}_i = J$. Assume that x is an element of J_λ . By $(J_\lambda/J_{\lambda+1})J = 0$ we can consider $J_\lambda/J_{\lambda+1}$ as a right R/J -module. Since $x = xe_i$, we have $\bar{x}(\bar{1} - \bar{e}_i) = 0$ and hence $\bar{x} \cdot 1 = 0$ where $\bar{a}$ denotes the image of $a \in J_\lambda$ in $J_\lambda/J_{\lambda+1}$. This implies that x belongs to $J_{\lambda+1}$. On the other hand the radical J is r -transfinitely nilpotent so we have $x = 0$. Similarly, to proposition 7 it is routine to verify that $\dot{P}_\xi + \dot{P}_j = R$ for every ordinal ξ and $i \neq j$. Therefore we have $\dot{P}_i + \dot{P}_j = R$ for all $i \neq j$ and thus by Proposition 10 R is topologically isomorphic to the product of the primary rings R_i and this completes the proof of Theorem 12.

If $\bigcap \bar{J}^n = 0$ holds, we have the following.

COROLLARY 13. *Let R be a topologically artinian ring satisfying $\bigcap J^n = 0$. The following assertions are equivalent*

- 1) R is a direct product of topologically artinian primary rings.
- 2) $P_i P_j = P_j P_i$ where $\{P_i: i \in I\}$ is the set of maximal closed ideals of R .
- 3) $Q_i Q_j = Q_j Q_i$ where $Q_i = \bigcap P_i^n$.
- 4) Every ideal K of R with $K = \overline{K^2}$ is a unital ring.
- 5) $\bigcap Q_i = 0$.

First we prove the following.

PROPOSITION 14. *Let R be a topologically artinian ring satisfying $\bigcap J^n = 0$. For any open left ideal L there are (not necessarily different) maximal closed ideals $P_1, \dots, P_n$ with $P_n \dots P_1 \subseteq L$.*

PROOF. Since L is open, the left R -module R/L is artinian. Because $L = L + \left(\bigcap_n J^n\right) = \bigcap_n (L + J^n)$ holds, there is a natural number t with $J^{t+k} + L = J^t + L$ for every non-negative integer k . Hence L contains J^t . The artinian module $R/(J + L)$ can be considered as a left R/J -module, and then by Proposition 6 it is a finite direct sum of simple modules, i.e. it is noetherian. Consider the artinian R -module $(J + L)/(J^2 + L)$. By $J[(J + L)/(J^2 + L)] = 0$ it can be considered as an R/J -module and hence by Proposition 6 it is a finite direct sum of simple modules. Thus it is noetherian. Iterating this process in t steps we get that $(J^{t-1} + L)/(J^t + L) = (J^{t-1} + L)/L$ is noetherian. Thus R/L is noetherian, i.e., it has a composition series $R/L = M_1 \supset M_2 \supset \dots \supset M_n \supset 0$ where M_i/M_{i+1} is simple. Let P_k be the annihilator ideal of M_k/M_{k+1} ($k = 1, \dots, n$), then the P_k are (not necessarily different) maximal closed ideals and $(P_n \dots P_1)R/L = 0$, thus $P_n \dots P_1 \subseteq L$. This completes the proof of Proposition 14.

PROOF OF COROLLARY 13. By Theorem 12 we have $1 \Leftrightarrow 4$ and $1 \Rightarrow 1, 1 \Rightarrow 3$. Next we show $2 \Rightarrow 5$. Suppose that $P_i P_j = P_j P_i$ holds for any $i, j \in I$. If there were an element $0 \neq c \in \bigcap_{i \in I} Q_i$, then we should have an open left ideal L with $c \notin L$. By Proposition 14 there are maximal closed ideals $P_1, \dots, P_n$ with $P_1 \dots P_n \subseteq L$. This

shows by Propositions 7 and 8 that

$$c \in \bigcap_{i \in I} Q_i \subseteq \bigcap_{i=1}^n (P_i^n) \subseteq P_1 \dots P_n \subseteq L$$

which contradicts $c \notin L$.

Similarly we can see that $3 \Rightarrow 5$.

Finally, suppose $\bigcap_{i \in I} Q_i = 0$. Let $R_i = R/Q_i$. From Proposition 7 we get immediately $Q_i + P_j = R$ for all $i \neq j$. This shows clearly that P_i/Q_i is the radical of R_i . By $R/P_i \simeq R_i/(P_i/Q_i)$ the rings R_i ($i \in I$) are primary rings. Consider the direct product $\tilde{R} = \prod \tilde{R}_i$. By Proposition 10 we have $R \simeq \tilde{R}$. This completes the proof of Corollary 13.

PROPOSITION 15. *If in a topologically artinian ring R the products of any two maximal open left ideals commute, then so do those of any two maximal closed ideals, and R/P is a division ring for each maximal closed ideal P .*

PROOF. Let P be a maximal closed ideal. Since P contains the radical, R/P is a primitive ring, consequently it is the endomorphism ring of a vector space over a division ring. To show that R/P is a division ring, we assume indirectly that R/P is the endomorphism ring of a right vector space V over a division ring and the dimension of V is at least 2. Hence there is a basis $\{v_i : i \in I\}$ in V such that the cardinal number of I is greater than 1. Let i, j be any two distinct index in I and we define the endomorphisms e_i and e_j by setting

$$e_i(v_k) = \begin{cases} v_k, & k \neq i \\ 0, & k = i \end{cases}, \quad e_j(v_k) = \begin{cases} v_k, & k \neq j \\ 0, & k = j \end{cases}.$$

It is obvious that $(R/P)e_i$ and $(R/P)e_j$ are two maximal open left ideals in R/P . For the endomorphism e_{ij} define by

$$e_{ij}(v_k) = \begin{cases} v_k, & k \neq j, \quad k \neq i \\ v_i, & k = j \\ v_j, & k = i \end{cases},$$

we obtain $e_i e_{ij} = e_j$ from which it follows that $(R/P)e_i \cdot (R/P)e_j = (R/P)e_j$. Similarly we can see that $(R/P)e_j \cdot (R/P)e_i = (R/P)e_i$.

Since $(R/P)e_i$ and $(R/P)e_j$ are two distinct maximal open left ideals, we have that their products do not commute. This contradicts to the assumption. This implies that P is a maximal open left ideal in R and hence the validity of the proposition is verified.

As an immediate consequence of Proposition 15 and Corollary 13 we have

COROLLARY 16. *A topologically artinian ring satisfying $\bigcap \overline{J^n} = 0$ is a direct product of local rings if and only if the products of any two maximal open left ideals commute.*

In what follows, let R be a topologically artinian local ring satisfying $\bigcap \overline{P^n} = 0$ where P denotes its maximal open ideal. We assume that R satisfies the additional condition:

There exists no one-sided open ideal between P and P^2 .

PROPOSITION 17. *If P^2 is not open in P , then $P = 0$.*

PROOF. For any open left ideal L in P the left ideal $P^2 + L$ is open and it holds $P \supset P^2 + L \supset P^2$. Hence $P^2 + L = P$. This shows that P^2 is everywhere dense in P , so $\overline{P^2} = P$ is true. By induction we get $\overline{P^n} = P$. Therefore $P = \bigcap \overline{P^n} = 0$.

PROPOSITION 18. *$\{P^n, n = 1, 2, \dots\}$ forms a fundamental system of neighbourhoods of zero.*

PROOF. Let L be any open left ideal in R . Since $L = L + (\bigcap \overline{P^n}) = \bigcap (L + P^n)$ and R/L is an artinian module, there is an integer n with $P^n \subseteq L$. This means by [4] Lemma 6 that L is a power of P . (Note that in this case P^k is a union of translates of P^t for every $t \geq k$). Suppose now $P^n = 0$ for some n , and let k be the least integer with this property, i.e. $P^k = 0$, $P^{k-1} \neq 0$. We prove that R is artinian in this case, or in other words, 0 is ipen. In fact, since all non-zero open left ideals are powers of P containing P^{k-1} , they cannot form a fundamental system of neighbourhoods of zero, hence 0 must also be open.

PROPOSITION 19. *Every non-zero one-sided ideal (closed or not) in R coincides with some P^n ($P^0 = R$), $n = 0, 1, 2, \dots$.*

PROOF. Let $0 \neq L \neq R$ be any left ideal in R and $0 \neq c \in L$. The Rc is a closed left ideal in R . This implies $Rc = Rc + (\bigcap \overline{P^n}) =$

$= \bigcap (Rc + P^n)$. Since $Rc + P^n$ is open, $Rc + P^n = P^{k_n}$ for each n . By $\bigcap P^n = 0$, $Rc \neq 0$ there exists an integer t with $Rc = P^t$. Therefore L is open and hence by the proof of Proposition 18 L equals a power of P . By the assumption it is easy to see that R is topologically artinian from the right, too. By symmetry the assertion of Proposition 19 is true.

THEOREM 20. *Let R be a topologically artinian local ring satisfying $\bigcap \overline{P^n} = 0$ and suppose that there exists no one-sided ideal between P and P^2 where P denotes its maximal non-zero open ideal. Then R has no zero-divisors if and only if $P^n \neq 0$ for each n .*

PROOF. The necessity is obvious.

Conversely, by Proposition 19 there is an element a contained in P but not in P^2 , and then $P = Ra = aR$ and by induction $P^n = a^n R = Ra^n$. If x and y are non-zero elements in R , then there are integers k, l such that $x \in P^k - P^{k+1}$, $y \in P^l - P^{l+1}$. Thus it follows that we can express x and y as $x = ua^k$, $y = va^l$ where u, v must be units in R . Therefore $xy = ua^k va^l = uv_1 a^{k+l}$ where v_1 is a unit with the property $v_1 a^k = a^k v$. This implies $xy \neq 0$.

If a and b are any two non-zero elements of R , from the proof of Theorem 20 and by Proposition 19 we have $Ra = P^k$, $Rb = P^l$ for some integers k, l . Hence a, b have a common left multiple $c = b_1 a = a_1 b = 0$. By Ore's well-known result one can now construct a quotient division ring Δ of R whose elements are the quotients $a^{-1}b$. We define the following valuations v on R : $v(0) = +\infty$; if $a \neq 0$ is any element of R , then there is an integer n with $a \in P^n - P^{n+1}$, and let $v(a) = n$. It is routine to check that v defines a discrete valuation on R in the sense of [1] Chap. VI. In the classical way we can extend v to Δ and it is easy to see that v is a discrete valuation on Δ . We show that R is the ring of the valuation v on Δ . In fact, let $x = b^{-1}a$ be any element of Δ which is not contained in R where $a, b \in R$. Then we can write $a = c^k u$, $b = c^l t$ for $c \in P - P^2$, and u, t units in R . Since $x \notin R$ and $x = b^{-1}a = (c^l t)^{-1} c^k u = t^{-1} c^{k-l} u$, $v^{-1}, u \in R$, we have $k < l$. Hence $x^{-1} = a^{-1}b = u^{-1} c^{l-k} t \in R$. Since R is complete, Δ is complete in the topology induced by v .

From the above we have

THEOREM 21. *Let R be a linearly compact local ring satisfying $\bigcap \overline{P^n} = 0$ where P denotes its maximal open ideal. The following conditions are equivalent.*

1) $P^n \neq 0$ for each n and there exists no one-sided ideal between P and P^2 .

2) R is the ring of a complete, discrete valuation on a division ring.

THEOREM 22. *Let R be a linearly compact ring with $\bigcap \overline{J^n} = 0$. R is a direct product of rings of complete, discrete valuations on division rings, of local uni-serial rings, and of division rings if and only if the products of any two maximal open left ideals commute and there exists no one-sided open ideal between P and P^2 for each maximal open ideal P of R .*

In what follows we shall investigate topologically noetherian linearly compact rings. First we prove

THEOREM 23. *A linearly compact ring R is topologically noetherian in the equivalent Leptin-topology if and only if $\star J = 0$, i.e. its radical is l -transfinitely nilpotent.*

PROOF. Assume that R is topologically noetherian. For any open left ideal L of R we have $J[(\star J + L)/L] = (\star J + L)/L$ and hence by Nakayama's Lemma we have $(\star J + L)/L = 0$, consequently $\star J \subseteq L$ and therefore $\star J = 0$. Conversely if $\star J = 0$, then we prove by induction that $R/\mu J$ is topologically noetherian for every ordinal μ from which the statement follows clearly. For $\mu = 1$ the assertion is obvious. If $R/\mu J$ is topologically noetherian, then ${}_{\mu}J/{}_{\mu+1}J$ can be considered as R/J -module and hence it is obviously topologically noetherian in the Leptin-topology. Therefore $R/{}_{\mu+1}J$ is such, too. If λ is a limit ordinal and $R/\mu J$ is topologically noetherian for all $\mu < \lambda$, there R/J is the inverse limit of $R/\mu J$ is trivially topologically noetherian, too. This completes the proof.

A topological ring R is called of *cofinite type* if the set of all isomorphism classes of simple factors of all submodules in any finitely generated discrete R -module is finite. Similarly to Proposition 9 we can prove the next statement.

PROPOSITION 24. *Let R be a topologically noetherian linearly compact ring of cofinite type. If the set $\{P_i : i \in I\}$ of all maximal closed ideals in R satisfies $\overline{P_i P_j} = \overline{P_j P_i}$, $\overline{P_i (P_j)_\lambda} = \overline{(P_i)_\lambda P_i}$ and $\overline{(P_j)_\lambda (P_i)_\xi} = \overline{(P_i)_\xi (P_j)_\lambda}$ for any $i \neq j$, $i, j \in I$ and any limit ordinals ξ, λ then $\bigcap_{i \in I} \star(P_i) = 0$.*

PROOF. Similarly to Proposition 9 we obtain $\overline{(P_i)_\xi(P_j)_\lambda} = \overline{(P_j)_\lambda(P_i)_\xi}$ for any $i \neq j$, $i, j \in I$ and any ordinals ξ and η . Assume now indirectly that $\bigcap_{i \in I} \star(P_i) \neq 0$. Then there is an element $c \in R$ and an open left ideal L with $c \in \bigcap_{i \in I} \star(P_i)$ and $c \notin L$. Since R is of cofinite type, there are finitely many maximal closed ideals P_i , say $P_1, \dots, P_n$ such that every simple factor of each submodule in R/L is annihilated by some one of them. By Zorn's Lemma there exists a minimal submodule M of R/L such that $(R/L)/M$ is annihilated by $\star(P_1) \dots \star(P_n)$. We claim $M = 0$. In fact, if $M \neq 0$, then M as a submodule of the noetherian module R/L is finitely generated. Therefore $JM \neq M$. Since M/JM can be considered as a module over R/J , it is a finite direct sum of simple R -modules by Proposition 6. Thus there is a submodule N of M such that M/N is annihilated by some P_i , say P_1 . Henceforth we obtain now

$$\begin{aligned} [\star(P_1) \dots \star(P_n)][(R/L)/N] &= \\ &= P_1\{[\star(P_1) \dots \star(P_n)][(R/L)/N]\} \subseteq P_1 \cdot M/N = 0, \end{aligned}$$

which contradicts to the minimality of M . Thus $M = 0$ holds. This implies by $c \in \bigcap_{i \in I} \star(P_i) \subseteq \star(P_1) \dots \star(P_n)$ that $c \cdot R/L = 0$, i.e. $c \in L$, which is impossible. Therefore $\bigcap_{i \in I} \star(P_i) = 0$ and we are done.

Consider now a topologically noetherian linearly compact ring R satisfying the condition of Proposition 24. In the case $i \neq j$ by Proposition 4 the ideal $P_i + P_j$ is closed in R and hence we get $P_i + P_j = R$. Similarly to Proposition 7 we obtain $_\xi(P_i) + _\xi(P_j) = R$ for every ordinal ξ . Thus $\star(P_i) + \star(P_j) = R$ holds. Henceforth for any finite set $\{\star(P_1), \dots, \star(P_n)\}$ we have by the Chinese Remainder Theorem

$$\star(P_i) + \bigcap_{k \neq i} \star(P_k) = R, \quad i = 1, \dots, n.$$

As $\star(P_i)$ is a closed ideal of R , the factor ring $R_i = R/\star(P_i)$ is a topologically noetherian linearly compact ring, too. Similarly to Proposition 11, one can see that the rings R_i are primary rings. As it was done in Proposition 10, we can prove that R is isomorphic to the direct product $\prod R_i$ and this isomorphism is continuous, but in general, it is not a topological isomorphism. Furthermore, the topology in $\prod R_i$

induced by this continuous isomorphism is equivalent to the product topology. Therefore this isomorphism is topological if we endow R with the equivalent Leptin-topology. Thus the proof of the following theorem is similar to that of Theorem 12 and hence we omit it.

THEOREM 24. *Let R be a topologically noetherian linearly compact ring endowed with the equivalent Leptin-topology. The following conditions are equivalent.*

1) R is a direct product of topologically noetherian linearly compact primary ring R_i .

2) R is of cofinite type and if $\{P_i: i \in I\}$ is the set of all maximal closed ideals in R , then the equalities $P_i P_j = P_j P_i$, $P_i (P_j)_\lambda = (P_j)_\lambda P_i$, $(P_i)_\mu (P_j)_\lambda = (P_j)_\lambda (P_i)_\mu$ hold for all $i \neq j$, $i, j \in I$ and limit ordinals μ, λ .

3) R is of cofinite type and the equalities $\overline{P_i P_j} = \overline{P_j P_i}$, $\overline{P_i (P_j)_\lambda} = \overline{P_i (P_j)_\lambda} = \overline{(P_j)_\lambda P_i} = \overline{(P_j)_\lambda P_i}$, $\overline{(P_i)_\mu (P_j)_\lambda} = \overline{(P_j)_\lambda (P_i)_\mu}$ hold for all $i \neq j$, $i, j \in I$ and limit ordinals μ, λ .

4) Any ideal K of R such that $K = \overline{K^2}$ is a unital ring.

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