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Integral Functionals Determined by Their Minima.

GIANNI DAL MASO - LUCIANO MODICA (*)

Introduction.

In this paper we study the following problem in Calculus of Variations: determine an integral functional

\[ F(u, A) = \int_A f(x, Du(x)) \, dx \]

by the knowledge of the minima of the Dirichlet's problems for \( F \) with linear boundary values, that is by knowing the numbers

\[ m(p, A) = \min \{ F(u, A) : u(x) = p \cdot x, \quad \forall x \in \partial A \} \]

for every \( p \in \mathbb{R}^n \) and for every bounded open subset \( A \) of \( \mathbb{R}^n \).

Namely, we show that the integrand \( f(x, p) \) can be calculated by a differentiation process of the set function \( A \to m(p, A) \) along a family \((A_\varrho)_{\varrho \to 0} \) of open subsets of \( \mathbb{R}^n \) which shrinks nicely to \( x \) as \( \varrho \to 0^+ \). According to W. Rudin ([13], ch. 8), a family \((A_\varrho)\) is said to shrink to \( x \) nicely as \( \varrho \to 0^+ \) if for every \( \varrho > 0 \)

\[ A_\varrho \subseteq B(x, \varrho) = \{ y \in \mathbb{R}^n : |y - x| < \varrho \}, \quad |A_\varrho| > c|B(x, \varrho)| \]

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where $c > 0$ is a suitable real constant independent of $\varrho$ and $| \cdot |$ denotes the Lebesgue measure in $\mathbb{R}^n$.

The main result we prove is the following.

**Theorem I.** Suppose that the function $f: \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$ satisfies the following hypotheses:

(i) $f(x, \cdot)$ is measurable in $x$ and convex in $\cdot$;

(ii) $\varphi_1(\cdot) \leq f(x, \cdot) \leq \varphi_2(\cdot)$ for all $(x, \cdot) \in \mathbb{R}^n \times \mathbb{R}^s$,

where $\varphi_1$, $\varphi_2: \mathbb{R}^n \to \mathbb{R}$ are convex functions and

\[
\lim_{|\cdot| \to +\infty} \frac{\varphi_1(\cdot)}{|\cdot|} = +\infty.
\]

Then, denoting

\[ m(p, A) = \inf \left\{ \int_A f(y, Du(y)) \, dy : u \in C^\infty(\mathbb{R}^n), u(y) = p \cdot y \quad \forall y \in \partial A \right\}, \]

there exists a measurable subset $N \subseteq \mathbb{R}^n$ with $|N| = 0$ such that

\[
(*) \quad f(x, p) = \lim_{\varrho \to 0^+} \frac{m(p, A_\varrho)}{|A_\varrho|}
\]

for every $p \in \mathbb{R}^n$, $x \in \mathbb{R}^n \setminus N$ and for every family $(A_\varrho)_{\varrho > 0}$ of open subsets of $\mathbb{R}^n$ which shrinks to $x$ nicely as $\varrho \to 0^+$.

Some comments. (a) The superlinearity hypothesis (iii) can be dropped if $f(x, \cdot)$ depends only on $p$ for large $|p|$ (see remark 1.3). (b) In the vector case (when $u(x)$ is a vector in $\mathbb{R}^m$ and $f$ is defined on $\mathbb{R}^s \times \mathbb{R}^{sm}$) the same thesis $(*)$ holds by assuming $f$ quasi-convex but by strengthening (ii) to

\[ (ii)' \quad c_1 |\cdot|^{\alpha} \leq f(x, \cdot) \leq c_2 (1 + |\cdot|^\alpha) \]

with $0 < c_1 < c_2$ and $\alpha > 1$ (see theorem II). The proof in this case relies on a recent approximation result for quasi-convex functions due to P. Marcellini [10]. (c) The case of non-negative integrands $f$ depending not only on $x$ and $Du$ but also on $u$ is more delicate. As an example, we treat here the case of uniform continuity in $u$ and $f(x, u, 0) = 0$ for every $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ (see theorem III).
An application of theorem I is a useful and meaningful characterization of the $I'$-convergence of a sequence of equicoercive functionals: see theorem IV. This theorem is an important step for applying Ergodic Theory in nonlinear stochastic homogenization (see G. Dal Maso - L. Modica [3]).

A particular case of theorem I was obtained by E. De Giorgi and S. Spagnolo [5] when $f$ is a quadratic form, i.e.

$$ f(x, p) = \sum_{i,j=1}^{n} a_{ij}(x) p_i p_j. $$

Their proof relies on Meyers' estimate of the summability exponent for the gradients of the solutions to the Euler equation of the corresponding integral functional $F$. Recently, M. Giaquinta and E. Giusti [9] have found an analogous estimate for the gradients of the minima of integral functionals (even non-differentiable). Nevertheless, we have preferred to employ an elementary and direct method for proving theorem I.

One may also consider the problem of determining an integral functional $F$ by the knowledge of the values of other variational problems for $F$, for instance by knowing the numbers

(1) $$ m(\lambda, w, A) = \inf \left\{ F(u, A) + \lambda \int_A |u - w|^2 dx : u \in C^0(A) \right\} $$

for every bounded open subset $A$ of $\mathbb{R}^n$, $\lambda > 0$, $w \in L^2(A)$ or the numbers

(2) $$ m(\varphi, A) = \inf \left\{ F(u, A) + \int_A \varphi u dx : u \in C^0(A) \right\} $$

for every bounded open subset $A$ of $\mathbb{R}^n$ and $\varphi \in L^2(A)$.

In both cases suitable reformulations of theorem I continue to hold. The first case (1) has been studied in many papers about $I'$-convergence (see, for example, E. De Giorgi - T. Franzoni [4], L. Carbone - C. Sbordone [1], G. Dal Maso - L. Modica [2]), the second case (2) is related to Fenchel's duality for convex functions (see, for example, I. Ekeland - R. Teman [6], R. T. Rockafellar [12]).

We thank the referee for some useful advice.
1. Proof of Theorem 1.

Let us begin by a particular case of theorem I.

1.1. Proposition. Let $f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a function such that $f(x, p)$ is measurable in $x$, convex in $p$, and bounded from below. If there exists a real constant $R$ so that $f(x, p)$ does not depend on $x$ for $|p| > R$, then the thesis (*) of theorem I holds.

**Proof.** Let us fix $x \in \mathbb{R}^n$. A straightforward application of Jensen's inequality gives that

$$
\inf_{A_\varepsilon} \left\{ \int_{A_\varepsilon} f(x, Du(y)) \, dy : u \in C^0(\mathbb{R}^n), \, u(y) = p \cdot y \, \forall y \in \partial A_\varepsilon \right\} =
$$

$$
= \int_{A_\varepsilon} f(x, p) \, dy = |A_\varepsilon| f(x, p) \quad \forall p \in \mathbb{R}^n, \, \varepsilon > 0
$$

so we easily obtain

$$
\left| f(x, p) - \frac{m(p, A_\varepsilon)}{|A_\varepsilon|} \right| \leq \frac{1}{|A_\varepsilon|} \sup_{u \in C^0(\mathbb{R}^n)} \left| \int_{A_\varepsilon} [f(x, Du(y)) - f(y, Du(y))] \, dy \right| \leq
$$

$$
\leq \frac{1}{|A_\varepsilon|} \int_{A_\varepsilon} \sup_{q \in \mathbb{R}^n} |f(x, q) - f(y, q)| \, dy.
$$

If we define

$$
\omega(x, y, p) = |f(x, p) - f(y, p)| \quad (x, y, p \in \mathbb{R}^n),
$$

$$
\varphi(x, y) = \sup_{p \in \mathbb{R}^n} \omega(x, y, p) \quad (x, y \in \mathbb{R}^n),
$$

it remains to prove that there exists a measurable subset $N \subset \mathbb{R}^n$ with $|N| = 0$ such that

$$
\lim_{\varepsilon \to 0^+} \frac{1}{|A_\varepsilon|} \int_{A_\varepsilon} \varphi(x, y) \, dy = 0
$$

for every $x \in \mathbb{R}^n \setminus N$ and $(A_\varepsilon)$ which shrinks to $x$ nicely as $\varepsilon \to 0^+$. 
Since $f(x, p)$ depends only on $p$ for $|p| > R$ and is convex in $p$, we have that
\[ f(x, p) \leq \max_{|q| = R+1} f(x, q) = M, \quad \forall x \in \mathbb{R}^n, p \in \mathbb{R}^n : |p| < R + 1, \]
with $M$ independent of $x$. On the other hand $f$ is bounded from below, so it follows that all the functions $f(x, p)$ are Lipschitz continuous in $p$, uniformly with respect to $x \in \mathbb{R}^n$, on the ball $|p| < R$. If we observe that $\omega(x, y, p) = 0$ for every $p \in \mathbb{R}^n$ such that $|p| > R$, we may infer that
\[ |\omega(x, y, p) - \omega(x, y, q)| < K|x - q| \quad \forall x, y, p, q \in \mathbb{R}^n \]
for a suitable real constant $K$.

Now, let us choose a countable dense subset $D$ of $\mathbb{R}^n$ and let us construct, by Lebesgue’s differentiation theorem (see, for instance, [13], th. 8.8) a measurable subset $N$ of $\mathbb{R}^n$ with $|N| = 0$ such that
\[ \lim_{\varrho \to 0^+} \frac{1}{|A_\varrho|} \int_{A_\varrho} \omega(x, y, p) \, dy = 0 \]
for every $x \in \mathbb{R}^n \setminus N$, $p \in D$ and $(A_\varrho)$ which shrinks to $x$ nicely as $\varrho \to 0^+$.

For every $\varepsilon > 0$ there exists a finite number $p_1, \ldots, p_m$ of elements of $D$ such that
\[ \inf_{1 \leq i \leq m} |p - p_i| < \varepsilon, \quad \forall p \in \mathbb{R}^n : |p| < R, \]
so we have that
\[ \varphi(x, y) < \sum_{i=1}^m \omega(x, y, p_i) + K\varepsilon, \quad \forall x, y \in \mathbb{R}^n, \]
and we may conclude that
\[ \limsup_{\varrho \to 0^+} \frac{1}{|A_\varrho|} \int_{A_\varrho} \varphi(x, y) \, dy \leq K\varepsilon. \]
for every $x \in \mathbb{R}^n \setminus N$ and $(A_\varrho)$ which shrinks to $x$ nicely as $\varrho \to 0^+$. By taking $\varepsilon \to 0^+$, proposition 1.1 is proved.
The general case of theorem I will be obtained by the following approximation lemma.

1.2 Lemma. If $f$ satisfies the hypotheses of theorem I, then there exists an increasing sequence $(f_h)$ of functions such that $f = \sup_h f_h$ and each function $f_h$ fulfills the assumptions of proposition 1.1.

Proof. For every $h \in \mathbb{N}$ we define

$$f_h(x, p) = \inf_{z \in \mathbb{R}^n} [f(x, z) + h|z - p|], \quad ((x, p) \in \mathbb{R}^n \times \mathbb{R}^n).$$

The sequence $(f_h)$ is the usual approximation from below of $f$ by Lipschitz continuous functions. In fact $f_h(x, p)$ is Lipschitz continuous in $p$ (with Lipschitz constant $h$), $f_h \leq f_{h+1} \leq f$ for every $h \in \mathbb{N}$ and it is easy to prove, by remarking that $f(x, p)$ is convex (hence continuous) in $p$, that $\sup_h f_h = f$. The same remark proves that

$$\inf_{z \in \mathbb{R}^n} [f(x, z) + h|z - p|] = \inf_{z \in \mathbb{Q}^n} [f(x, z) + h|z - p|],$$

so $f_h(x, p)$ is measurable in $x$. Finally, a direct calculation shows that $f_h(x, p)$ is convex in $p$.

Then, we define for $h \in \mathbb{N}$ and for $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$

$$f_h(x, p) = \max \{q_1(p) \leq f_h(x, p)\}.$$

If is obvious that $f_h(x, p)$ is measurable in $x$ and convex in $p$. Since

$$f_h(x, p) < q_2(0) + h|p| \quad \forall (x, p) \in \mathbb{R}^n \times \mathbb{R}^n$$

the superlinearity hypothesis (iii) gives that there exist $c \in \mathbb{R}$ and $R_h > 0$ such that

$$c < q_1(p) < f_h(x, p) \quad \forall (x, p) \in \mathbb{R}^n \times \mathbb{R}^n$$

$$f_h(x, p) = q_1(p) \quad \forall (x, p) \in \mathbb{R}^n \times \mathbb{R}^n : |p| > R_h.$$

This concludes the proof of lemma 1.2.

Now, let us prove theorem I.
Proof of theorem I. First, we prove that there exists a measurable set $N' \subseteq \mathbb{R}^n$ with $|N'| = 0$ such that

$$\lim_{e \to 0^+} \frac{1}{|A_e|} \int_{A_e} |f(y, p) - f(x, p)| \, dy = 0$$

for every $x \in \mathbb{R}^n \setminus N'$, $p \in \mathbb{R}^n$ and $(A_e)$ which shrinks to $x$ nicely as $e \to 0^+$. Let $D$ be a countable dense subset of $\mathbb{R}^n$. By Lebesgue’s differentiation theorem (see, for instance, [13], th. 8.8), there exists a measurable set $N' \subseteq \mathbb{R}^n$ with $|N'| = 0$ such that (3) holds for every $x \in \mathbb{R}^n \setminus N'$, $p \in D$ and $(A_e)$ which shrinks to $x$ nicely as $e \to 0^+$. Since $f(x, p)$ is locally Lipschitz continuous in $p$ uniformly with respect to $x$ (by convexity and (ii)), it is easy to see that (3) holds for every $p \in \mathbb{R}^n$.

Now, we have at once

$$\frac{1}{|A_e|} \int_{A_e} f(y, p) \, dy \geq \frac{m(p, A_e)}{|A_e|}$$

so by (3)

$$f(x, p) \geq \limsup_{e \to 0^+} \frac{m(p, A_e)}{|A_e|}$$

for every $x \in \mathbb{R}^n \setminus N'$, $p \in \mathbb{R}^n$ and $(A_e)$ which shrinks to $x$ nicely as $e \to 0^+$.

For the converse inequality, let $(f_h)$ be the sequence given by lemma 1.2, $m_h(p, A_e)$ be the corresponding minima and $N_h$ be the measurable subsets of $\mathbb{R}^n$ with $|N_h| = 0$ given by proposition 1.1 for $f_h$. Define $N^* = \bigcup_{h=1}^{+\infty} N_h$. Since $f \geq f_h$ for every $h \in \mathbb{N}$, we have that

$$f_h(x, p) = \lim_{e \to 0^+} \frac{m_h(p, A_e)}{|A_e|} \leq \liminf_{e \to 0^+} \frac{m(p, A_e)}{|A_e|}$$

and, by taking the limit as $h \to +\infty$, we obtain

$$f(x, p) \leq \liminf_{e \to 0^+} \frac{m(p, A_e)}{|A_e|}$$

for every $x \in \mathbb{R}^n \setminus N^*$, $p \in \mathbb{R}^n$ and $(A_e)$ which shrinks to $x$ nicely as $e \to 0^+$. Then, theorem I is proved by choosing $N = N' \cup N^*$. 

1.3 REMARK. The coerciveness hypothesis (iii) in theorem I is crucial for the approximation lemma 1.2. A particular non-coercive case has been studied by N. Fusco and G. Moscariello [8], who consider non-negative quadratic forms

\[ f(x, p) = \sum_{i,j=1}^{n} a_{ij}(x) p_i p_j \]

and obtain the formula (*) with limsup instead of lim. However, if \( f(x, p) \) does not depend on \( x \) for large \( |p| \), theorem 1 holds without any coerciveness hypothesis, as proposition 1.1 shows.

Theorem I can be generalized as follows.

THEOREM II. Suppose that the function \( f: \mathbb{R}^n \times \mathbb{R}^{mn} \rightarrow \mathbb{R} \) satisfies the following hypotheses:

(i) \( f(x, p) \) is measurable in \( x \) and quasi-convex in \( p \) (in the Morrey's [11] sense).

(ii) \( c_1 |p|^\alpha < f(x, p) \leq c_2 (1 + |p|^\alpha) \forall (x, p) \in \mathbb{R}^n \times \mathbb{R}^{mn} \) with \( 0 < c_1 < c_2, \alpha > 1 \) real constants.

Then the thesis (*) of theorem I holds (\( u \) is a \( m \times n \) matrix).

PROOF. The proof of theorem I can be repeated only substituting lemma 1.2 by theorem 1.2 of P. Marcellini [10].

THEOREM III. Suppose that \( f: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \) satisfies the following hypotheses:

(i) \( f(x, s, p) \) is measurable in \( x \), continuous in \( s \), convex in \( p \);

(ii) \( 0 < \varphi_1(p) \leq f(x, s, p) \leq \varphi_2(p) \forall (x, s, p) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \) where \( \varphi_1, \varphi_2: \mathbb{R}^n \rightarrow \mathbb{R} \) are convex functions and

\[
\lim_{|p| \rightarrow +\infty} \frac{\varphi_1(p)}{|p|} = +\infty;
\]

(iii) \( |f(x, s_1, p) - f(x, s_2, p)| \leq (1 + f(x, s_1, p)) \omega(|s_1 - s_2|) \forall x \in \mathbb{R}^n, s_1, s_2 \in \mathbb{R}, p \in \mathbb{R}^n \) where \( \omega: \mathbb{R}_+ \rightarrow \mathbb{R} \) is a function such that \( \lim_{t \rightarrow 0^+} \omega(t) = 0 \);

(iv) \( f(x, s, 0) = 0 \forall (x, s) \in \mathbb{R}^n \times \mathbb{R} \).
Then, letting $W^{1,\infty}(\mathbb{R}^n)$ be the space of the Lipschitz continuous functions on $\mathbb{R}^n$ and denoting

$$m(x, s, p, A) = \inf_{u \in W^{1,\infty}(\mathbb{R}^n)} \left\{ \int_A \left( f(y, u(y), Du(y)) \right) dy : u(y) = s + p \cdot (y - x) \quad \forall y \in \partial A \right\},$$

there exists a measurable subset $N$ of $\mathbb{R}^n$ with $|N| = 0$ such that

$$f(x, s, p) = \lim_{\varepsilon \to 0^+} \frac{m(x, s, p, A_\varepsilon)}{|A_\varepsilon|}$$

for every $x \in \mathbb{R}^n \setminus N$, $s \in \mathbb{R}$, $p \in \mathbb{R}^n$ and for every family $(A_\varepsilon)_{\varepsilon > 0}$ of open subsets of $\mathbb{R}^n$ which shrinks to $x$ nicely as $\varepsilon \to 0^+$.

**Proof.** Let us introduce the auxiliary function

$$m'(s, p, A) = \inf_{u \in W^{1,\infty}(\mathbb{R}^n)} \left\{ \int_A \left( f(y, s, Du(y)) \right) dy : u(y) = p \cdot y \quad \forall y \in \partial A \right\}$$

and note that

$$m'(s, p, A) = \inf_{u \in W^{1,\infty}(\mathbb{R}^n)} \left\{ \int_A \left( f(y, s, Du(y)) \right) dy : u(y) = s + p \cdot (y - x) \quad \forall y \in \partial A \right\}$$

for every $x \in \mathbb{R}^n$. Hypothesis (iv) assures that the functionals

$$u \mapsto \int_A \left( f(y, u(y), Du(y)) \right) dy, \quad u \mapsto \int_A \left( f(y, s, Du(y)) \right) dy$$

decrease by truncating the function $u$, hence the class of competing functions in the infima (4) and (5) can be restricted to the functions
such that
\[ u(y) = s + p \cdot (y - x) \quad \forall y \in \partial A \]
\[ |u(y) - s| < (\text{diam } A')|p| \]
where \( A' = A \cup \{ x \} \) (« maximum principle »).

Let us fix \( x \in \mathbb{R}^n \). Then, by (ii) and (iii), we have that
\[ |m(x, s, p, A_\varepsilon) - m'(s, p, A_\varepsilon)| < \omega(2\varepsilon|p|)(1 + \varphi_\varepsilon(p))|A_\varepsilon| \]
for every \( s \in \mathbb{R} \), \( p \in \mathbb{R}^n \), and for every open set \( A_\varepsilon \subseteq B(x, \varepsilon) \).

Let \( D \) be a countable dense subset of \( \mathbb{R} \). By theorem I there exists a measurable subset \( N \subseteq \mathbb{R}^n \) with \( |N| = 0 \) such that
\[ \lim_{\varepsilon \to 0^+} \frac{m'(s, p, A_\varepsilon)}{|A_\varepsilon|} = f(x, s, p) \]
for every \( x \in \mathbb{R}^n - N \), \( s \in D \), \( p \in \mathbb{R}^n \), and for every family \( (A_\varepsilon) \) which shrinks to \( x \) nicely as \( \varepsilon \to 0^+ \). Since, by (ii) and (iii), we have
\[ |m'(s_1, p, A_\varepsilon) - m'(s_2, p, A_\varepsilon)| < \omega(|s_1 - s_2|)(1 + \varphi_\varepsilon(p))|A_\varepsilon| \]
it is easy to prove that (7) holds for every \( s \in \mathbb{R} \), and the thesis follows from (6).

1.4 Remark. The same « freezing » technique of the previous proof could be extended also to the vector case. Indeed, the use of the maximum principle can be avoided by taking profit of a result by N. Fusco and J. Hutchinson ([7], lemma 4.1), but by assuming more regularity on \( f \).

2. A characterization of \( \Gamma \)-convergence.

Let us fix \( 0 < c_1 < c_2 \), \( \alpha > 1 \) and let \( \mathcal{F} = \mathcal{F}(x, c_1, c_2) \) be the set of all functionals \( \mathcal{F} : L^\infty_{\text{loc}}(\mathbb{R}^n) \times \mathcal{A}_\varepsilon \to \mathbb{R} \) (\( \mathcal{A}_\varepsilon \) denotes the family of the bounded open subsets of \( \mathbb{R}^n \)) given by
\[ F(u, A) = \begin{cases} \int_A f(x, Du(x)) \, dx & \text{if } u|_A \in W^{1,\alpha}(A), \\ + \infty & \text{otherwise,} \end{cases} \]
where $f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is any function such that $f(x, p)$ is measurable in $x$, convex in $p$ and

$$c_1|p|^a \leq f(x, p) \leq c_2(1 + |p|^a) \quad \forall (x, p) \in \mathbb{R}^n \times \mathbb{R}^n.$$ 

Of course, $W^{1, a}(\mathcal{A})$ denotes the usual first order Sobolev space with summability exponent $a$.

A notion of convergence for sequences of real-extended functions defined on a topological space, the $\Gamma$-convergence (see E. De Giorgi - T. Franzoni [4]), is particularly useful when applied to the sequences in $\mathcal{F}$. We refer to G. Dal Maso - L. Modica [2] for a systematic and self-contained study of the $\Gamma$-convergence on $\mathcal{F}$.

The crucial property of $\Gamma$-convergence is a general theorem on convergence of minima. In particular, we are interested here in the following proposition (see [2], prop. 1.18).

**Proposition 2.1.** Suppose that $(F_h)$ is a sequence in $\mathcal{F}$ which $\Gamma$-converges as $h \to +\infty$ to $F_\infty \in \mathcal{F}$. Then, for every $A \in \mathcal{A}_0$ and $u_0 \in W^{1, a}(A)$, we have that

$$\lim_{h \to +\infty} \min_u \{F_h(u, A): u - u_0 \in W^{1, a}_0(A)\} = \min_u \{F_\infty(u, A): u - u_0 \in W^{1, a}_0(A)\}.$$

In this section, our aim is to prove a converse of the previous proposition and so to obtain a characterization of $\Gamma$-convergence in $\mathcal{F}$ by the convergence of the minima of Dirichlet problems.

**Theorem IV.** Let $(F_n)$ be a sequence in $\mathcal{F}$, let $D$ be a dense subset of $\mathbb{R}^n$ and let $\mathcal{B}$ be a family of bounded open subset of $\mathbb{R}^n$ which contains, for any $x \in \mathbb{R}^n$, a subfamily which shrinks to $x$ nicely. Suppose that

$$\lim_{h \to +\infty} \min_u \{F_h(u, B): u - l_\xi \in W^{1, a}_0(B)\},$$

where $l_\xi(x) = \xi \cdot x$, exists for every $\xi \in D$ and $B \in \mathcal{B}$.

Then, there exists a functional $F_\infty \in \mathcal{F}$ such that $(F_h)$ $\Gamma$-converges to $F_\infty$ and

$$\lim_{h \to +\infty} \min_u \{F_h(u, A): u - l_p \in W^{1, a}_0(A)\} = \min_u \{F_\infty(u, A): u - l_p \in W^{1, a}_0(A)\}$$

for every $p \in \mathbb{R}^n$ and $A \in \mathcal{A}_0$. 
PROOF. By proposition 2.1 it is enough to prove that $(F_h) \Gamma$-converges to a functional $F_\infty \in \mathcal{F}$. It is possible (see [2], prop. 1.21 and cor. 1.22) to define a metric on $\mathcal{F}$ in such a way that $(\mathcal{F}, d)$ is a compact metric space and the convergence of a sequence in $(\mathcal{F}, d)$ is equivalent to $\Gamma$-convergence. By taking profit of this result, it will suffice to prove that, if $(F_{h(h)}^{(1)})$ and $(F_{h(h)}^{(2)})$ are two subsequences of $(F_h)$ which $\Gamma$-converge respectively to $F_\infty^1 \in \mathcal{F}$ and $F_\infty^2 \in \mathcal{F}$, then $F_\infty = F_\infty^1 = F_\infty^2$.

Indeed, by proposition 2.1 and by hypothesis

$$\min_{u} \left\{ F_{\alpha}(u, B) : u - l \in W^{1,\alpha}_0(B) \right\} = \min_{u} \left\{ F_{\alpha}^u(u, B) : u - l \in W^{1,\alpha}_0(B) \right\}$$

for every $\xi \in D$ and $B \in \mathcal{B}$, hence theorem 1 yields that there exists $N \subset \mathbb{R}^n$ with $|N| = 0$ such that

$$f_{\infty}'(x, \xi) = f_{\infty}''(x, \xi) \quad \forall x \in \mathbb{R}^n \setminus N, \forall \xi \in D$$

where $f_{\infty}'$ and $f_{\infty}''$ denote respectively the integrand of $F_{\infty}'$ and $F_{\infty}''$.

Finally, $f_{\infty}'(x, p)$ and $f_{\infty}''(x, p)$ are convex, hence continuous, in $p$ so $f_{\infty}'(x, p) = f_{\infty}''(x, p)$ for every $x \in \mathbb{R}^n \setminus N$ and $p \in \mathbb{R}^n$ and the thesis follows.

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REFERENCES

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