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On the Lattice Automorphisms of $SL(n, q)$ and $PSL(n, q)$.

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Introduction.

In a previous paper [5], a study of the lattice automorphisms (= automorphisms of the lattice of subgroups) of the finite Chevalley groups G has been begun. Starting-point was the fact that if rank $(G) \geqq 2$ (and if some exceptional cases are excluded), then for the group $A(G)$ of lattice automorphisms of G we have

$$(+) \qquad A(G) \cong \operatorname{Aut}(G) \ltimes \Phi$$

where Φ is the kernel of the action of $A(G)$ on the Tits building of G. For a large class of simple Chevalley groups G (essentially those whose Weyl group has a non-trivial center), it was shown that Φ is trivial, i.e. every lattice automorphism is induced by a group automorphism. However, the groups $PSL(n, q)$ do not belong to this class, and in fact it was shown that Φ is not even solvable for many of the groups $PSL(3, q)$. This phenomenon motivated the present paper, where we take a closer look at the groups $G = PSL(n, q)$, $n \geqq 3$. (In the case $n = 2$ and $q > 3$, we have $A(G) \cong \operatorname{Aut}(G)$ by Metelli [2]).

We show that if we exclude the case that q is a power of 3 and $n = 2m$ with m odd, then Φ commutes with the inner automorphisms of G and fixes every unipotent subgroup of G. This allows one to

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determine the exact structure of Φ for certain of the groups $PSL(3, q)$. It remains an open problem whether Φ can be non-trivial for any of the groups $PSL(n, q)$, $n > 3$.

Notations.

For elements x, y of a group G we set $y^x := x^{-1}yx$; the same if y is a subgroup of G. If A, B, $C, \ldots$ are elements or subgroups of G, we let $\langle A, B, C, \ldots \rangle$ denote the subgroup of G generated by them. The following notations will be fixed throughout the paper: q is a power of the prime p, $k = GF(q)$ is the field with q elements, D is a central subgroup of $SL(n, q)$, $n \geqq 3$, and $G = SL(n, q)/D$.

We regard $SL(n, q)$ as a matrix group which acts on the k-vector space k^n in the canonical way; then G acts naturally on the $(n-1)$-dimensional projective space $\mathbf{P}^{n-1}$ over k.

Preliminaries.

Since G is perfect, every lattice automorphism of G preserves the orders of the subgroups of G (see [4, Ch. II, Th. 8]); this will be used constantly in the following (without further reference). In particular, it implies that the group $A(G)$ of lattice automorphisms of G acts on the set of p-Sylow subgroups of G and on the Tits building of G (see [5, sect. 1]); it is easy to see that the kernels of these two actions coincide, and this common kernel will be denoted by Φ. To state it explicitly, Φ is the (normal) subgroup of $A(G)$ consisting of those lattice automorphisms of G that fix every p-Sylow subgroup of G; the elements of Φ will be called *exceptional lattice automorphisms* of G.

In our situation, the Tits building of G is isomorphic to the flag complex of $\mathbf{P}^{n-1}$ and thus $(+)$ follows from classical projective geometry. But nothing of this will be needed in the following, since we exclusively study the group Φ. This is done in a completely elementary fashion; therefore we avoid using the language of algebraic groups, although it would be helpful at some points.

LEMMA 1. (i) If $n \geqq 5$, then Φ fixes all subgroups of G that have n linearly independent fixed points in $\mathbf{P}^{n-1}$.

(ii) If $p \neq 2$, then Φ fixes every subgroup J of G which is the image of a subgroup of order 2 of $SL(n, q)$.

PROOF. Let $P_1, \ldots, P_n$ be linearly independent points of $\mathbf{P}^{n-1}$, let r_i be an element of G interchanging P_i, P_{i+1} and fixing the other P_j's (for $i = 1, \ldots, n-1$), and let U (resp. U^-) be the group of those p-elements of G that fix all the spaces $P_1 + \ldots + P_i$ (resp. $P_{i+1} + \ldots + P_n$) for $i = 1, \ldots, n-1$. Since U and U^- are p-Sylow subgroups of G, Φ fixes the groups $U_i := U \cap (U^-)^{r_i}$ and $U_i^- := U^- \cap U^{r_i}$, hence also $S_i := \langle U_i, U_i^- \rangle$ $(\cong SL(2, q))$.

(i) The normalizer $N_G(U)$ of U in G is the largest subgroup of G containing U but no other p-Sylow subgroup of G. Hence Φ fixes $N_G(U)$, and analogously $N_G(U^-)$, thus also $T := N_G(U) \cap N_G(U^-)$. It is well-known that T is the group of all elements of G that fix $P_1, \ldots, P_n$. Hence it suffices to show that Φ fixes every subgroup of T.

Now T is generated by the groups $T_i := T \cap S_i$ $(i = 1, \ldots, n-1)$, which are fixed by Φ and are all isomorphic to the multiplicative group of k. Furthermore we have $\langle T_1, \ldots, T_{n-2} \rangle \cong T_1 \times \ldots \times T_{n-2}$, hence if $n \geqq 5$ then the (abelian) group T contains at least three independent elements of each occurring order; from this it follows by a theorem of Baer (see [4, Ch. II, Th. 2]) that if $T(l)$ denotes the l-torsion subgroup of T for a prime l, then every lattice automorphism of $T(l)$ is induced by a group automorphism. This shows that for every $\varphi \in \Phi$ there is an automorphism f of T with $X^f = X^\varphi$ for all subgroups X of T.

Now consider the restriction of f to $Y := \langle T_i, T_{i+1} \rangle$, where $1 \leqq i \leqq n-2$. The three cyclic subgroups T_i, T_{i+1} and $(T_{i+1})^{r_i}$ of Y are fixed by f (Note that $(T_{i+1})^{r_i}$ arises in the same way as the T_j's when we renumber the P_j's appropriately) and Y is the direct product of any two of them; hence f acts equivalently in all three. Thus f acts equivalently in all the T_j's, which means that there is some integer m with $f(t) = t^m$ for all t in T. Hence f, and thus also φ, fixes every subgroup of T. This proves (i).

(ii) By the above, we can assume $n \leqq 4$. Then J is either the center of some S_i (for a suitable choice of $P_1, \ldots, P_n$), or the unique central subgroup of order 2 of G. Hence J is fixed by Φ (see e.g. statement (+) in the proof of Lemma 1 in [5]).

REMARK. Lemma 1 (i) fails drastically in the case $n = 3$, see [5, sect. 3].

1. Unipotent subgroups.

A unipotent transformation u of a finite-dimensional vector space V is called *regular* if the fixed space of u in V is 1-dimensional (equivalently, if the Jordan normal form of u consists of only one block).

LEMMA 2. Let K be a field of characteristic $p > 2$. Then for every regular unipotent element u of $SL(m, K)$, $m \geqq 1$, there exists an involution h in $GL(m, K)$ with $u^h = u^{-1}$; if m is even then $\det(h) = (-1)^{m/2}$.

PROOF. With u also u^{-1} is regular, hence $u^h = u^{-1}$ for some h in $GL(m, K)$. Replacing h by its p^m-th power, we may assume that h is semisimple. Then h^2 is a semisimple element commuting with u, hence is a scalar transformation, *i.e.* there is some $t \in K$ with $h^2(x) = tx$ for all x in K^m. Since h fixes the 1-dimensional fixed space of u, h has an eigenvalue s in K. Then $s^2 = t$, and by replacing h by $s^{-1}h$, we get h to be an involution.

For every $i = 0, \ldots, m$ there is exactly one i-dimensional u-invariant subspace W_i of K^m. Hence these W_i must be fixed by h. For every $i = 2, \ldots, m$, the involution h induces in the 2-dimensional space W_i/W_{i-2} an involution h_i which inverts the (non-trivial) transformation induced by u; since $p \neq 2$, it follows that h_i has both 1 and -1 as eigenvalues. This proves the assertion on $\det(h)$. (Note that $W_0 \subset W_1 \subset \ldots \subset W_m$).

LEMMA 3. Let K be a field of characteristic $p > 2$ and suppose that n is either odd or divisible by 4. Then for every unipotent element u of $SL(n, K)$ there exists an involution h in $SL(n, K)$ with $u^h = u^{-1}$.

PROOF. By the Jordan normal form, K^n is the direct sum of a family (E_μ) of u-invariant subspaces, such that the restriction u_μ of u to E_μ is regular for every μ. By Lemma 2 there exist involutions $h_\mu \in GL(E_\mu)$ inverting u_μ. These h_μ combine to yield an involution $h \in GL(n, K)$ inverting u. If some E_μ has odd dimension, then we can force $\det(h) = 1$ by replacing h_μ by $-h_\mu$ (if necessary). If all of the spaces E_μ have even dimension, then n is even, hence divisible by 4 (by assumption) and thus $\det(h) = \prod_\mu \det(h_\mu) = \prod_\mu (-1)^{\dim(E_\mu)/2} = (-1)^{n/2} = 1$ (by Lemma 2).

LEMMA 4. Let K be a field of characteristic 2. Then for every unipotent element u of $SL(m, K)$, $m \geqq 1$, there exists an involution h in $SL(m, K)$ with $u^h = u^{-1}$.

PROOF. By the Jordan normal form we may assume that u is regular. Then u is conjugate to the matrix (u_{ij}) with $u_{ij} = 1$ if $i = j$ or $i = j - 1$ or $i = j - 2 \equiv 1 \pmod 2$, and all other $u_{ij} = 0$. We may assume $u = (u_{ij})$. Setting $h_{ij} = 1$ if $i = j$ or $i = j - 1 \equiv 1 \pmod 2$, and all other $h_{ij} = 0$, the matrix $h = (h_{ij})$ does the job.

LEMMA 5. Let K be a field of characteristic $p > 3$. Then for every unipotent element u of $SL(m, K)$, $m \geqq 1$, there exists a diagonalizable element h of $SL(m, K)$ normalizing $\langle u \rangle$, such that

$$\langle h, h^u \rangle = \langle h, u \rangle .$$

PROOF. First assume that we have already found a $g \in SL(m, K)$ with all eigenvalues in K such that $u^g = u^4$. Then $h := g^{(p^m)}$ is diagonalizable and $u^{4^{(p^m)}-1} = u^{-1}u^h = (h^{-1})^u h \in \langle h, h^u \rangle$. Since $4^{(p^m)} - 1 \equiv$ $\equiv 3^{(p^m)} \not\equiv 0 \pmod p$ and u is of p-power order, it follows that $u \in \langle h, h^u \rangle$, hence the claim.

It remains to prove the existence of g. For this we may assume that u is regular. Then u^4 is also regular (since $p \neq 2$), hence there is some f in $GL(m, K)$ with $u^f = u^4$. Then f fixes the spaces W_i (defined as in the proof of Lemma 2) and we can again consider the action of f and u in W_i/W_{i-2} $(i = 2, \ldots, m)$. Using the matrix identities

$$\begin{pmatrix} c_{i-1} & t \\ 0 & c_i \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_{i-1} & t \\ 0 & c_i \end{pmatrix} = \begin{pmatrix} 1 & c_i c_{i-1}^{-1} \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^4 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix},$$

we conclude that if c_i denotes the eigenvalue of f belonging to the action of f in W_i/W_{i-1} $(i = 1, \ldots, m)$ then we have $c_i = 4c_{i-1}$ (for $i = 2, \ldots, m$). Hence $\det(f) = c_1 \ldots c_m = 4^{m(m-1)/2} c_1^m = (2^{m-1}c_1)^m$, and thus $g := 2^{-m+1} c_1^{-1} f$ does the job. q.e.d.

A finite group is called *dihedral* if it is generated by two elements a, b subject to the relations $a^r = b^2 = 1$, $a^b = a^{-1}$, for some $r \geqq 2$.

LEMMA 6. Let φ be a lattice automorphism of G fixing all sub groups of order 2 of a dihedral subgroup S of G. Then φ fixes every subgroup of S.

PROOF. Easy (see e.g. part (3) in the proof of Lemma 2 in [3]).

PROPOSITION 1. *Suppose that if* $p = 3$ *then* n *is either odd or divisible by* 4. *Then* Φ *fixes every unipotent subgroup* (*i.e.* p-*subgroup*) *of* G.

PROOF. It suffices to show that Φ fixes every cyclic unipotent subgroup $\langle u \rangle$ of G. We first consider the case $p \neq 2$. If furthermore n is either odd or divisible by 4, then the claim follows from Lemma 3, Lemma 1 (ii) and Lemma 6. Now suppose that n is not of this form; then $p \neq 3$ (by assumption) and $n \geqq 6$. Hence it follows from Lemma 5 that there exists some h in G having n linearly independent fixed points in $\mathbf{P}^{n-1}$ and normalizing $\langle u \rangle$, such that $\langle h, u \rangle = \langle h, h^u \rangle$; then Φ fixes $\langle h, u \rangle$ (by Lemma 1 (i)), hence also $\langle u \rangle$, the only p-Sylow subgroup of $\langle h, u \rangle$.

It remains to consider the case $p = 2$. By Lemma 4 and Lemma 6, it suffices to show that Φ fixes $\langle v \rangle$ for every involution v in G. As follows from the Jordan normal form, v can be written as the product of commuting elations $v_1, \ldots, v_s \in G$ (i.e. each v_i is an involution that fixes a hyperplane of $\mathbf{P}^{n-1}$ pointwise). Then $V := \langle v_1, \ldots, v_s \rangle$ carries the structure of a $GF(2)$-vector space, which is generated by its 1-dimensional subspaces $\langle v_i \rangle$; hence if Φ fixes all the $\langle v_i \rangle$, then Φ must act trivially on the lattice of subgroups of V (by the fundamental theorem of projective geometry) and will therefore fix $\langle v \rangle$.

It remains to show that Φ fixes $\langle e \rangle$ for every elation e in G. Now e lies in a subgroup $S \cong SL(2, q)$ of G which can be constructed as the groups S_i in the proof of Lemma 1; hence Φ fixes S and every 2-Sylow subgroup of S (since the 2-Sylow subgroups of S can be constructed as the groups U_i in the proof of Lemma 1). But then Φ fixes every subgroup of S: This is clear if $q = 2$, and if $q > 2$ it follows from Metelli's result [2] that every lattice automorphism of $PSL(2, q)$ ($= SL(2, q)$ in our case) is induced by a group automorphism. In particular, Φ fixes $\langle e \rangle$. q.e.d.

2. The main result.

LEMMA 7. Let μ be a lattice automorphism of G fixing every cyclic subgroup of G that acts reducibly in $\mathbf{P}^{n-1}$. Then $\mu = id$.

PROOF. It suffices to show that μ fixes every maximal cyclic subgroup T of G that acts irreducibly in $\mathbb{P}^{n-1}$. Clearly, T^μ is also maximal cyclic in G and acts irreducibly in $\mathbb{P}^{n-1}$ (Note that μ maps cyclic groups to cyclic groups, see e.g. [4, Ch. I, Th. 2]). Let S_1 (resp. S_2) denote the inverse image of T (resp. T^μ) in $SL(n, q)$, and let $M(n, q)$ denote the ring of $n\times n$-matrices over $k = GF(q)$. It is well-known that the centralizer K_i of S_i in $M(n, q)$ is a subfield of $M(n, q)$ with q^n elements, and $S_i = K_i \cap SL(n, q)$ $(i = 1, 2)$. Choosing $x \neq 0$ in k^n, the bijection $\beta : K_1 \to k^n$ sending h to $h(x)$ endows k^n with the structure of a field F such that β becomes a field isomorphism. Let A_0 be the subgroup of $GL(n, q)$ that acts on F as the Galois group of F over k. By the normal base theorem (see [1, p. 283]) A_0 permutes the elements of a base of k^n (as k-vector space), hence the group $A := \{a^2 : a \in A_0\}$ lies in $SL(n, q)$.

Let s be a generator of S_1 and set $d := [S_1 : D]$, $m :=$ g.c.d. $(d, q^2 - 1)$ (Remember that $G = SL(n, q)/D$). Then the group $\boldsymbol{H} := \langle A, A^s, D\rangle$ contains $S_1^m := \{x^m : x \in S_1\}$ as a normal subgroup and is the semi-direct product of S_1^m and A (namely, let α be the generator of A with $z^\alpha = z^{q^2}$ for all $z \in K_1$ and note that $(\alpha^{-1})^s \alpha = s^{-1} s^\alpha = s^{q^2-1}$).

Since A and A^s act reducibly in $\mathbb{P}^{n-1}$, the image $\bar{H}$ of $\boldsymbol{H}$ in G is fixed by μ. Therefore with $T^m \leqq \bar{H}$ we also have $(T^\mu)^m = (T^m)^\mu \leqq \bar{H}$ implying that $S_2^m \leqq \boldsymbol{H}$. Set $i := [S_1^m S_2^m : S_1^m] = |S_1^m S_2^m \cap A|$. Since $S_1^m \cap S_2^m$ is a subset of K_1 that is centralized by the group $S_1^m S_2^m \cap A$, it follows that $S_1^m \cap S_2^m$ lies in the subfield I of K_1 with $[K_1 : I] = i$. But $S_1^m \cap S_2^m$ also lies in the field K_2, hence in $K_2 \cap I$. Thus for $j := [K_1 : K_2 \cap I]$ we get that $|S_1^m \cap S_2^m|$ divides $q^{n/j} - 1$. But $|S_1^m \cap S_2^m| = i^{-1}|S_2^m|$, hence $|S_2^m|$ divides $i(q^{n/j} - 1)$, thus also $j(q^{n/j} - 1)$. Computing $|S_2^m| = |S_2|/\text{g.c.d.}(m, |S_2|)$ and $|S_2| = (q^n - 1)/(q - 1)$, we finally get

$$(+) \qquad q^n - 1 \text{ divides } j(q^{n/j} - 1)\,(q - 1) \text{ g.c.d.} \left(q^2 - 1, \frac{q^n - 1}{q - 1}\right).$$

Below we are going to show that $(+)$ implies $j = 1$, hence $K_1 = K_2$ and $S_1 = S_2$, which finally means $T = T^\mu$. Then the Lemma is proved. From $(+)$ we deduce $q^{n(j-1)/j} \leqq \sum_{\lambda=0}^{j-1} q^{n\lambda/j} = (q^n - 1)/(q^{n/j} - 1) < jq^3$, hence

$$(++) \qquad q^{-3+n(j-1)/j} < j\,.$$

First we exclude the case that $j = n > 6$: In this case $(++)$

gives $2^{n-4} \leqq q^{n-4} < n$, a contradiction. If $j \neq n$, then $j \leqq n/2$ and $(++)$ gives $2^{-3+n(j-1)/j} < n/2$, hence $2^{-2+n(j-1)/j} < n$; if in addition $n/2 \leqq -2 + n(j-1)/j$, then $2^{n/2} < n$ implying that $n < 4$; if $n/2 > -2 + n(j-1)/j$, then $j \leqq 2$ or $n \leqq 9$.

Now we know that either $j \leqq 2$ or $n \leqq 9$. But if $j = 2$ then $(++)$ gives $2^{-3+n/2} < 2$, hence $n \leqq 6$. Thus we have either $j = 1$ or $n \leqq 9$. From now on we assume $j \neq 1$ and reach a contradiction by considering each $n \leqq 9$ separately.

If $n = 9$, then $j = 3$ or $j = 9$, both of which contradicts $(++)$. Similarly for $n = 8$ and $n = 7$. If $n = 6 = j$ or $n = 6 = 2j$, then $(++)$ gives $q = 2$, contradicting $(+)$. If $n = 6 = 3j$, then $(+)$ gives that $q^6 - 1$ divides

$$2(q^3 - 1)(q - 1)3(q + 1) = 6(q^3 - 1)(q^2 - 1) < 6q^5,$$

hence $q \leqq 5$; checking all $q \leqq 5$, one sees that $(+)$ cannot be fulfilled. If $n = 5$, then $j = 5$ and $(++)$ gives $q \leqq 4$, which again contradicts $(+)$. If $n = 4 = j$, then $(+)$ gives that $q^4 - 1$ divides $4(q-1)\cdot(q-1)2(q+1)$, hence $q^2 + 1$ divides $8(q-1)$, which is impossible. Similarly for $n = 4 = 2j$. Finally if $n = 3$, then $j = 3$ and $(+)$ gives that $q^3 - 1$ divides $3(q-1)(q-1)3 = 9(q-1)^2$, which again is easily seen to be impossible.

THEOREM. *Suppose that if q is a power of 3 then n is either odd or divisible by 4. Then the exceptional lattice automorphisms of $G = SL(n, q)/D$ $(n \geqq 3)$ commute with the inner automorphisms of G and fix every unipotent subgroup of G.*

REMARK. The exceptional lattice automorphisms also fix every « diagonalizable » subgroup of G, if $n \geqq 5$ (see Lemma 1). Furthermore I want to remark that the case $q = 3^r$, $n = 2m$ with odd m, cannot be handled with our methods; I do not know whether the theorem remains valid in this case.

Proof. By Proposition 1 it only remains to show that every $\varphi \in \Phi$ commutes with the inner automorphisms of G. Fix some $g \in G$ and set $X^\mu := \left(((X^\varphi)^g)^{\varphi^{-1}}\right)^{g^{-1}}$ for every subgroup X of G. Then μ is an exceptional lattice automorphism of G and we have to show that $\mu = id$. By Lemma 7 it suffices to show that $X^\mu = X$ for every cyclic subgroup X of G that acts reducibly in $\mathbf{P}^{n-1}$. Noting that $X = X_1 X_2$, where X_1 (resp. X_2) denotes the group of semisimple (resp. unipotent) elements of X, and applying Proposition 1 to X_2 and X_2^φ, we see that we may assume X to consist only of semisimple elements.

By our assumptions on X, there exist non-trivial X-invariant subspaces R and S of k^n with $k^n = R \oplus S$. Let Q (resp. Q^-) be the stabilizer of R (resp. S) in G. Then X lies in $L := Q \cap Q^-$. The (parabolic) subgroups Q and Q^- of G are generated by normalizers of p-Sylow subgroups of G, hence φ fixes Q and Q^-, and thus also L; the same reasoning shows that φ fixes all conjugates of L. For the maximal normal p-subgroup U (resp. U^-) of Q (resp. Q^-) we have $Q = U \rtimes L$ and $Q^- = U^- \rtimes L$.

Claim 1. $(X^\varphi)^u = (X^u)^\varphi$ for every u in $U \cup U^-$.

By symmetry it suffices to consider the case $u \in U$. From $X = L \cap \cap (XU)$ we get $X^\varphi = L^\varphi \cap (XU)^\varphi = L \cap (X^\varphi U)$, hence $(X^\varphi)^u = L^u \cap \cap (X^\varphi U)$. On the other hand, $X^u = L^u \cap (XU)$, hence $(X^u)^\varphi = = (L^u)^\varphi \cap (XU)^\varphi = L^u \cap (X^\varphi U)$. Thus Claim 1 is proved.

Claim 2. $(X^\varphi)^u = (X^u)^\varphi$ for every $u \in \langle U, U^- \rangle$.

Writing $u = u_1 u_2 \dots u_r$ with $u_1, \dots, u_r \in U \cup U^-$, we use induction on r. The case $r = 1$ is just Claim 1. Now assume $r > 1$. Then for $v := u_2 \dots u_r$ we have $(X^\varphi)^u = (X^\varphi)^{v u_1^v} = \big((X^v)^\varphi\big)^{u_1^v}$; the latter equality follows from the induction hypothesis. Since Claim 1 also holds if X, U and U^- are replaced by their v-conjugates, we can continue as follows: $\big((X^v)^\varphi\big)^{u_1^v} = \big((X^v)^{u_1^v}\big)^\varphi = (X^u)^\varphi$. Thus Claim 2 is proved.

It is well-known that $\langle U, U^- \rangle = G$ (namely, it follows from the fact that $\langle U, U^- \rangle$ is normalized by U, U^- and L, hence by G). Thus it follows from Claim 2 that $X^\mu = X$. This was to be shown.

3. The case $n = 3$.

Lemma 8. Let h be a semisimple element of $SL(3, q)$ which is not diagonalizable (over k). Then every exceptional lattice automorphism φ of $G = SL(3, q)/D$ fixes $\langle \bar{h} \rangle$, where $\bar{h}$ denotes the image of h in G.

Proof. The centralizer S of h in $SL(3, q)$ is cyclic, hence it suffices to show that φ fixes the image T of S in G.

Case 1. S does not act irreducibly in k^3.

Then S fixes (exactly) one 2-dimensional subspace E of k^3, and the restriction map $S \to GL(E)$ is injective. Let S_0 denote the subgroup of S consisting of those elements that map to $SL(E)$. Then S_0 acts irreducibly in E (Note that $|S_0| = q+1$ and therefore S_0 cannot embed into $k \setminus \{0\}$). Hence for the image T_0 of S_0 in G we get $C_G(T_0) = T$ (where $C_G(T_0)$ denotes the centralizer of T_0 in G).

There exists an involution f in $SL(3, q)$ with $s^f = s^{-1}$ for all s in S_0. By Lemma 6 and Lemma 1 (ii) (if $p \neq 2$) resp. Proposition 1 (if $p = 2$), it follows that φ fixes T_0. Hence $T_0 \leqq T^\varphi$. But with T also T^φ is cyclic, hence $T^\varphi \leqq C_G(T_0) = T$. Thus $T^\varphi = T$.

CASE 2. S acts irreducibly in k^3.

In this case the proof of Lemma 7 will show that $T^\varphi = T$, provided we know that φ fixes the image in G of the group A (and A^s) occurring in the proof of Lemma 7. But this follows as above from Lemma 6, since there exists an involution in $SL(3, q)$ acting on A by inversion.

COROLLARY. Let T be the image in $G = SL(3, q)/D$ of the group of diagonal matrices in $SL(3, q)$. Then the group Φ of exceptional lattice automorphisms of G fixes T, and the restriction map from Φ to the group of lattice automorphisms of T is injective.

PROOF. In the proof of Lemma 1 it was shown that Φ fixes T. The rest of the claim follows from Lemma 8 and the Theorem.

In [5, sect. 3] we gave conditions for a lattice automorphism of T to have an extension to an exceptional lattice automorphism of G. Combining this with the above Corollary we can completely determine the structure of Φ in certain cases: (A closer analysis would allow one to determine Φ in many more cases.) Letting S_m denote the symmetric group on m letters, we have

PROPOSITION 2. *Suppose P is a prime $\equiv -1 \pmod{12}$ and $p-1$ is square-free. Then the group Φ of exceptional lattice automorphisms of $SL(3, p)$ is isomorphic to*

$$\prod_{i=1}^{r} (S_3)^{n_i} \rtimes S_{n_i},$$

where the n_i are defined from the odd prime divisors $p_1, \ldots, p_r$ of $p-1$ by $n_i := (p_i - 7)/6$ if $p_i \equiv 1 \pmod 3$ and $n_i := (p_i - 5)/6$ if $p_i \equiv -1 \pmod 3$.

PROOF. In view of the above Corollary and [5, Prop. 3 and the discussion following it], it suffices to verify the conditions (i)-(iv) from [5, Prop. 3] for every lattice automorphism λ of T which is the restriction of an exceptional lattice automorphism of $SL(3, p)$. Condition (i) holds by the above Theorem, (iii) follows from Lemma 1 (ii) (since $p - 1 \equiv 10 \pmod{12}$), (iv) follows from (ii) (since $q = p$ is prime) and finally (ii) is easily verified using (i) and the standard arguments involving Lemma 6; we omit the details.

REMARK. (*a*) In the above situation, not only the structure of Φ as abstract group, but also its action on the subgroups of G can be described explicitly, see [5, sect. 3].

(*b*) There is some evidence that the groups $SL(n, q)$ for $n > 3$ will not have such an abundance of exceptional lattice automorphisms; e.g. in the case $G = SL(4, q)$ it can be shown with the above methods that Φ is an elementary abelian 2-group (and is trivial if q is even or $q \equiv 3 \bmod 4$). It remains an open problem whether Φ can be non-trivial for any $n \geqq 4$.

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