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Invertibility of Some Heat Potentials in BMO Norms.

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0. Introduction.

For C^1 -domains D in R^n and L^p boundary data ($1 < p < \infty$), Fabes and Riviere [1] considered the Initial-Dirichlet Problem for the (linear) heat equation

$$(I.D.P.) \quad \left\{ \begin{array}{ll} \Delta_x u - D_t u = 0 & \text{in the cylinder } D \times (0, T), \text{ uniformly} \\ \lim_{t \rightarrow 0} u(X, t) = 0 & \text{on compacts in } D, \\ u(X, t) \rightarrow f(P, s) & \text{a.e. on the surface } \partial D \times (0, T). \end{array} \right.$$

They proved the existence of a unique solution of (I.D.P.) given by the double-layer heat potential of a suitable transform of the boundary data f . Subsequently, in our paper [4], we began to examine a sort of regularity question arising by considering data f in appropriate BMO spaces on $\partial D \times (0, T)$. Due to the more local nature of these norms and to the higher regularity of BMO functions, two modifications

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were needed. The usual BMO norm had to be replaced by a caloric analogue, BMO_C , reflecting the mixed homogeneity of the heat equation. Secondly, a kind of compatibility condition (with the constant initial data) was introduced in the form of restricting ourselves to a subspace B_0MOC of those f in BMO_C having bounded initial behavior at $t = 0$. On this subspace we proved in [4] the continuity of the boundary integral J , where J is the singular integral operator

$$[Jf](P, t) = \lim_{\varepsilon \rightarrow 0} \int_{\partial D} \int_{t-\varepsilon}^{t+\varepsilon} K(P, Q, t-s) f(Q, s) dQ ds$$

and $K(P, Q, t-s)$ as defined below.

In this present paper, the invertibility of the boundary terms $(cI + J)$ in $B_0MOC(\partial D \times (0, T))$ is established. The technique used differs from [1] and elaborates the ideas in [4]. However, the dyadic decomposition of $\partial D \times (0, T)$ and the local analysis on «short time intervals» is finer than the one needed in [4].

Combining the main results here with those in [1] we deduce the unique solvability of (I.D.P.), by means of double-layer heat potentials, with data in the class B_0MOC .

We wish to thank Prof. Eugene Fabes for some helpful conversations concerning the construction in § 3 here.

1. Definitions and preliminaries.

If $D \subset R^n$, $n \geq 2$, is a bounded C^1 domain, we shall consider in the space-time $R^n \times R^+$, the cylinder $D_T = D \times (0, T)$, $0 < T < +\infty$, with lateral boundary $S_T = \partial D \times (0, T)$. Capital letters X, Y will denote points in D , while P, Q will denote points in ∂D . Letters t and s are used for time variables in R^+ . For all $(X, t) \in R^n \times R^+$, we let

$$\Gamma(X, t) = (\pi t)^{-n/2} \exp[-|X|^2/4t]$$

denote the fundamental solution of the heat equation and

$$K(X, t) = \langle N_Q, \nabla_X \Gamma(X, t) \rangle$$

where N_Q is the inner unit normal, denote the kernel of the double

layer heat potential. More explicitly

$$(1.0) \quad K(X, t) = c_n \frac{\langle X - Q, N_Q \rangle}{t^{(n/2+1)}} \exp [-|X - Q|^2/4t] .$$

If $(P, t) \in \mathcal{S}_T$ we call

$$\Delta = \Delta_r(P, t) = \{(Q, s) \in \mathcal{S}_T: |P - Q| < r, |s - t| < r^2\}$$

a caloric surface disc with center (P, t) and radius r , and for any $0 \leq a < T$

$$\Delta^a = \Delta_r^a(P) = \{(Q, s) \in \mathcal{S}_T, |P - Q| < r, a < s < a + r^2\}$$

the initial caloric surface disc, with center P and radius r , with initial point a . Moreover we call

$$\mathcal{S} = \mathcal{S}_r(P) = \{Q \in D: |P - Q| < r\}$$

the spatial surface disc, with center P and radius r . We introduce the spaces $BMO(\mathcal{S}_T)$ and $B_0MO(\mathcal{S}_T)$ [4].

We say that $f \in BMO(\mathcal{S}_T)$ if

$$(1.1) \quad \|f\|_* = \sup_{\Delta} \left\{ |\Delta|^{-1} \int_{\Delta} |f - f_{\Delta}| dQ ds \right\} < +\infty$$

where $f_{\Delta} = |\Delta|^{-1} \int_{\Delta} f$.

With the identification $f_1 \sim f_2$ if $f_1 - f_2 = \text{constant}$, BMO turns out a complete norm space with norm (1.1).

By the anisotropic John-Nirenberg inequality, we have the equivalent norm

$$(1.2) \quad \|f\|_{*,p} = \sup_{\Delta} \left\{ |\Delta|^{-1} \int_{\Delta} |f - f_{\Delta}|^p \right\}^{1/p} .$$

We say that $f \in B_0MO$ if (1.1) is valid and

$$B_0(f) = \sup_{\Delta^0} \left| \left\{ |\Delta^0|^{-1} \int_{\Delta^0} f dQ ds \right\} \right| < +\infty .$$

B_0MOC turns out a complete norm space if we equip it with the norm

$$\|f\|_{0,*} = B_0(f) + \|f\|_*.$$

Set

$$C_p(f) = \sup_{\Delta^0} \left\{ |\Delta^0|^{-1} \int_{\Delta^0} |f|^p dQ ds \right\}^{1/p}.$$

since the finiteness of $B_0(f)$ is equivalent to that of $C_p(f)$ for any $1 < p < \infty$ (see [4]), it follows that we have also the equivalent norm

$$\|f\|_{p,*} = C_p(f) + \|f\|_{*,p}$$

More generally we shall deal with the space $B_aMOC(\partial D \times (a, b))$, $0 \leq a < b \leq T$. We say that $f \in B_aMOC(\partial D \times (a, b))$ if $f \in BMOC(\partial D \times (a, b))$ and

$$B_a(f) = \sup_{\Delta^a} \left\{ |\Delta^a|^{-1} \int_{\Delta^a} f dQ ds \right\} < +\infty.$$

For these spaces, the norms

$$\|f\|_{a,*} = B_a(f) + \|f\|_* \quad \text{and} \quad \|f\|_{a,p,*} = C_{a,p}(f) + \|f\|_{*,p}$$

are equivalent, where $C_{a,p}(f)$ are the corresponding L^p -means relative to initial caloric surface discs Δ^a .

2. Behaviour of the operator J in the strip $\partial D \times (a, b) = \mathcal{S}(a, b)$.

We know that the study of the double layer heat potential, give rise to the singular integral operator

$$(2.0) \quad [Jf](P, t) = \lim_{\varepsilon \rightarrow 0} \int_0^{t-\varepsilon} \int_{\partial D} K(P, Q, t-s) f(Q, s) dQ ds$$

which is a bounded operator on $L^p(\mathcal{S}_T)$, $1 < p < +\infty$, see [1]. In addition J is bounded on $B_0MOC(\mathcal{S}_T)$, see [4].

Moreover as shown in [1], $cI + J$, $c \neq 0$ and I identity operator, is invertible in $L^p(S_T)$. This fact is obtained by showing that the operator J belongs to the class $\mathfrak{J}(S_T)$ of all bounded operators on $L^p(S_T)$ which satisfy the following two conditions

$$\text{i) for all } a, 0 < a \leq T, J\chi_{(a,\infty)} = \chi_{(a,\infty)}J\chi_{(a,\infty)}$$

where $\chi_{(a,b)}$ = characteristic function of (a, b) ,

$$\text{ii) if } (a, b) \subset (0, T), \|J(\chi_{(a,b)}f)\|_{L^p(S(a,b))} \leq \omega_J(b-a)\|f\|_{L^p(S(a,b))}$$

where $\omega_J(\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

The aim of this work is to prove that J also belongs to a corresponding class $\mathfrak{J}(S_T)$ of bounded operators on $B_0MOC(S_T)$ and that $cI + J$ is invertible in this space, for $c \neq 0$.

LEMMA 2.1. If $f \in B_aMOC(\partial D \times (a, b))$, with $(a, b) \subset (0, T)$, then $C_{a,2}(J(\chi_{(a,b)}f)) \leq \gamma(b-a)C_{a,2}(f)$, where $\gamma = \gamma_J$ and $\gamma(r) \rightarrow 0$ as $r \rightarrow 0$.

We observe that, since we work in the strip $\partial D \times (a, b)$, the initial surface disc Δ_r^a are truncated in the time dimension, that is

$$|\Delta_r^a| = cr^{n+1} \quad \text{if } r^2 < b-a; \quad |\Delta_r^a| = cr^{n-1}(b-a) \quad \text{if } r^2 > b-a.$$

Now, recalling theorem 1.3 of [1], we have

$$(2.1) \quad \|J(\chi_{(a,b)}f)\|_{L^2(\partial D \times (a,b))} \leq c\omega_J(b-a)\|f\|_{L^2(\partial D \times (a,b))}$$

In order to show the Lemma it is enough to examine the case $\omega_J(b-a) > b-a$. In fact if $\omega_J(b-a) \leq b-a$, we may take an $\tilde{\omega}_J$ such that $\tilde{\omega}_J(\delta) > \delta$, $\forall \delta$, and observe that the condition ii) of page 5 holds also for $\tilde{\omega}_J(b-a)$.

Let us fix $\Delta_r^a(P)$ and denote it simple by Δ ; set $\omega_J(b-a) = \omega$, and suppose, as we may, $\omega < 1$.

We distinguish three cases:

$$\alpha_1) \quad \omega > b-a \geq r^2, \quad \alpha_2) \quad \omega > r^2 > b-a, \quad \alpha_3) \quad r^2 > \omega > b-a.$$

Let us start with α_1).

Let p be an integer such that $\omega^{p/2(n+1)} \leq r^2 < \omega^{(p-1)/2(n+1)}$.

Set $*\Delta = \Delta^a(2\omega^{p-2/4(n+1)})$, to simplify notation, let now $\delta = \omega^{1/4(n+1)}$

so that $*\Delta = \Delta^a(2\delta^{p-2})$, and consider balls $S_j = S(2^j\delta^{p-2})$ and the discs $\Delta_j = \Delta^a(2^j\delta^{p-2})$. If χ_1 denote the characteristic function of $*\Delta$, then

$$\begin{aligned} \left\{ |\Delta|^{-1} \int_{\Delta} |J(\chi_{(a,b)} f)|^2 \right\}^{1/2} &\leq \left\{ |\Delta|^{-1} \int_{\Delta} |J[\chi_{(a,b)} \chi_1 f]|^2 \right\}^{1/2} + \\ &+ \left\{ |\Delta|^{-1} \int_{\Delta} |J(\chi_{(a,b)} (1 - \chi_1) f)|^2 \right\}^{1/2} = A + B. \end{aligned}$$

Recalling (2.1) we have

$$A < c\omega \left\{ |*\Delta| \cdot |\Delta|^{-1} \cdot |*\Delta|^{-1} \int_{*\Delta} |f|^2 \right\}^{1/2} \leq c\omega^{3/4} C_{a,2}(f)$$

since $*\Delta$ is a initial surface disc and

$$\begin{aligned} |*\Delta| \cdot |\Delta|^{-1} &= c2^{n+1} \delta^{(p-2)(n+1)} r^{-(n+1)} \leq c2^{n+1} \delta^{(p-2)(n+1)} \delta^{-p(n+1)} \\ &\leq c2^{n+1} \omega^{-1/2} \quad \text{if } 2^2 \delta^{2(p-2)} \leq b - a; \\ |*\Delta| \cdot |\Delta|^{-1} &= c2^{n-1} \delta^{(p-2)(n-1)} (b-a) r^{-(n+1)} \\ &\leq c2^{n+1} \delta^{(p-2)(n-1)} (b-a) 2^{-2} \delta^{-p(n+1)} \\ &\leq c2^{n+1} \delta^{-2(n+1)} = c2^{n+1} \omega^{-1/2} \quad \text{if } 2^2 \delta^{2(p-2)} > b - a. \end{aligned}$$

In order to estimate the term B it suffices to show that there is a constant $M > 0$ such that

$$|J[(1 - \chi_1) \chi_{(a,b)} f]| \leq M\omega C_{a,2}(f).$$

If $(Q, s) \in \Delta_j - \Delta_{j-1}$, and $(P, t) \in \Delta$, $a < s < t$, then $2|P - Q| \geq 2^j \delta^{p-2}$. From (1.0) we have the estimate

$$(2.2) \quad |K(P - Q, t - s)| \leq c|P - Q|^{-(n+1)} \exp[-|P - Q|^2/8(t - s)].$$

Hence, we can write

$$\begin{aligned} I \equiv |J((1 - \chi_1) \chi_{(a,b)} f)| &\leq c \sum_{j \geq 2} \left\{ \exp[-2^{2j} \delta^{2(p-2)}/8\omega^{2(p-1)}] / \{2^{j(n+1)} \delta^{(n+1)(p-2)}\} \right\} \cdot \\ &\cdot \int_{\Delta_j} |f| = c \sum_{j \geq 2} R_j \int_{\Delta_j} |f| \end{aligned}$$

and

$$R_j = \begin{cases} c|\Delta_j|^{-1} \exp[-4^j/\delta^2] & \text{if } 2^{2j}\delta^{2(p-2)} \leq b-a \\ c \exp[-4^j/2\delta^2] / \{2^{2j}(2^{j(n-1)}\delta^{(p-2)(n-1)})\delta^{2(p-2)}\} \\ \leq c \exp[-4^j/2\delta^2] |\mathcal{S}_j|^{-1} (b-a)^{-1} = c \exp[\dots] |\Delta_j|^{-1} & \text{if } 2^{2j}\delta^{2(p-2)} > b-a. \end{cases}$$

Then

$$I \leq c \sum_{j \geq 2} \exp[-4^j/8\delta^2] C_{a,1}(f) \leq c \exp[-4^2/8\delta^2] \left\{ \sum_{j \geq 3} \exp[-(4^j - 4^2)/2\delta^2] \right\} C_{a,1}(f).$$

Taking in account that $\delta^2 < 1$ the last series is dominated by a constant independent of δ . Since $C_1(f) \leq C_2(f)$, the estimate on B is complete.

The proof of Case α_2 is similar to the previous case. However since $\delta^2 > b-a$, one should observe that again

$$\begin{aligned} |^*\Delta| \cdot |\Delta|^{-1} &\leq c 2^{n-1} \delta^{(p-2)(n-1)} (b-a) \delta^{-p(n-1)} (b-a)^{-1} = \\ &= c 2^{n-1} \delta^{-2(n-1)} \leq c 2^{n-1} \omega^{-1/2} \end{aligned}$$

while for the R_j we have the second estimate only. For the Case α_3 , let $k \geq 2$ be an integer such that $(k-1)\omega \leq r^2 < k\omega$. We consider the initial surface disc $^*\Delta = \Delta^a(2k^{1/2}\omega^{1/2-1/4n}) = \Delta^a(2\delta)$ with $\delta = k^{1/2}\omega^{1/2-1/4n}$ and for $j \geq 2$ we let $\Delta_j = \Delta^a(2^j\delta)$.

With the same meaning for A and B as above we have

$$A \leq c\omega^{3/8} C_{a,2}(f)$$

since

$$\begin{aligned} |^*\Delta| \cdot |\Delta|^{-1} &= 2^{n-1} \delta^{n-1} (b-a) r^{-(n-1)} (b-a)^{-1} \leq \\ &\leq 2^{n-1} \delta^{n-1} (k-1)^{-(n-1)/2} \omega^{-(n-1)/2} = \\ &= 2^{n-1} (k/k-1)^{(n-1)/2} \omega^{-(n-1)/4n} \leq c 2^{n-1} \omega^{-1/4}. \end{aligned}$$

To estimate the term B , reasoning as in the previous cases, we observe

that: if $(Q, s) \in \Delta_j - \Delta_{j-1}$ (so $a < s < b$, since $r^2 > b - a$) and $(P, t) \in \Delta$, we have $|P - Q| > c 2^j \delta$, for some $c > 0$ independent of j . Hence using estimate (2.2), we obtain

$$\begin{aligned} I &= |J(1 - \chi_1) \chi_{(a,b)} f| \leq c \sum_{j \geq 2} \exp[-c 2^{2j} \delta^2 / 2k\omega] 2^{-j(n+1)} \delta^{-(n+1)} \cdot \int_{\Delta_j} |f| \leq \\ &\leq c(k-1) \exp[\dots] \{k 2^{2j} |S_j|(b-a) \omega^{-1/2n}\}^{-1} \int_{\Delta_j} |f| \leq c \omega^{1/2n} C_{a,1}(f). \end{aligned}$$

We note that the proof of Lemma 2.1 yields a function $\gamma = \omega^{1/s}$ for some $s > 1$. Since $0 < \omega < 1$, we have $\omega < \gamma$.

LEMMA 2.2. If $f \in B_a MOC(\partial D \times (a, b))$, $(a, b) \subset (0, T)$, then

$$\|J(\chi_{(a,b)} f)\|_{*,2} \leq \psi(b-a) \|f\|_{2,*}$$

where $\psi(r) \rightarrow 0$ as $r \rightarrow 0$.

PROOF. For any caloric surface disc $\Delta = S_r(P_0) \times (t_0 - r^2, t_0 + r^2)$ with $a < t_0 < b$, if $r^2 \geq b - a$ then $t_0 < a + r^2$ and Δ is an initial disc Δ^a in the strip $\partial D \times (a, b)$. Therefore

$$|\Delta|^{-1} \int_{\Delta} |Jf|^2 = |\Delta^a|^{-1} \int_{\Delta^a} |Jf|^2 < \{\gamma(b-a) C_{a,2}(f)\}^2$$

by Lemma 2.1.

Next, if $r^2 < b - a$, and γ is the function of Lemma 2.1, as observed at the beginning of this Lemma, we can examine only the case $\gamma(b-a) > b-a$.

Set $\gamma = \gamma(b-a)$ and we may assume that $\gamma < 1$.

We consider $\gamma \geq b-a > r^2$.

Reasoning as in Case α_1) of Lemma 2.1, let p be an integer such that $\delta^p \leq r < \delta^{p-1}$, with $\delta = \gamma^{1/4(n+1)}$ and let now

$$(2.3) \quad {}^*\Delta = \Delta(\delta^{p-2})(P_0, t_0).$$

We distinguish again two cases: $\beta_1) t_0 - a \leq \delta^{2(p-2)}$ and $\beta_2) t_0 - a > \delta^{2(p-2)}$.

Case β): $t_0 - a \leq \delta^{2(p-2)}$.

If we also have $t_0 - a \leq r^2$, we can view Δ as an initial disc so that

$$|\Delta|^{-1} \int_{\Delta} |Jf|^2 \leq \{\gamma C_{a,2}(f)\}^2.$$

When $r^2 < t_0 - a \leq \delta^{2(p-2)}$, letting $h = (t_0 - a + r^2)^{1/2}$, $\Delta \subset \Delta_h^a$ and hence

$$|\Delta|^{-1} \int_{\Delta} |Jf|^2 \leq |\Delta|^{-1} \int_{\Delta_h^a} |Jf|^2 \leq |\Delta|^{-1} |\Delta_h^a| \left\{ |\Delta_h^a|^{-1} \int_{\Delta_h^a} |Jf|^2 \right\}.$$

Now

$$\begin{aligned} |\Delta_h^a| \cdot |\Delta|^{-1} &= (t_0 - a + r^2)^{(n+1)/2} (r^{-2})^{(n+1)/2} = (1 + (t_0 - a)/r^2)^{(n+1)/2} \leq \\ &\leq ((t_0 - a)/r^2 + (t_0 - a)/r^2)^{(n+1)/2} \leq \{2\delta^{-2p+2(p-2)}\}^{(n+1)/2} = c\gamma^{-1/2} \end{aligned}$$

since $\delta = \gamma^{1/4(n+1)}$. Thus, again by Lemma 2.1,

$$|\Delta|^{-1} \int_{\Delta} |Jf|^2 \leq c\gamma^{-1/2} \{\gamma C_{a,2}(f)\}^2 = c\gamma^{3/2} C_{a,2}(f).$$

Case β_2): $t_0 - a > \delta^{2(p-2)}$. Here, $t_0 > \delta^{2(p-2)} > r^2$. Thus, if $J_1 = J_1(P, t) = [J(1)](P, t)$, Lemma 2.1 of [4] shows that there exist a constant $C(\Delta)$ such that, for any $(P, t) \in \Delta$

$$(2.4) \quad |J_1 - C(\Delta)| \leq c\gamma t_0^{-1/2}.$$

Next we let χ_1 be the characteristic function of ${}^*\Delta$, $f_1 = [f - f_{\bullet\Delta}] \chi_1$, $f_2 = |f - f_{\bullet\Delta}|(1 - \chi_1)$ and choose constant $Jf(\Delta) = J(f_2)(P_0, t_0) - f_{\bullet\Delta} \cdot C(\Delta)$ with $f = f_{\bullet\Delta} + f_1 + f_2$, we have

$$\begin{aligned} J(f)(P, t) - Jf(\Delta) &= f_{\bullet\Delta} |J_1(P, t) - C(\Delta)| + J(f_1) + \\ &\quad + [-J(f_2)(P_0, t_0) + J(f_2)]. \end{aligned}$$

We note that, by (2.4) and Holder's inequality

$$\begin{aligned}
 |f \cdot \Delta [J_1 - C(\Delta)]| &\leq c t_0^{-1/2} r |* \Delta|^{-1} \int_{\Delta} |f| \leq c t_0^{-1/2} \delta^{p-1} |* \Delta|^{-1} \int_{\Delta} |f| \leq \\
 &\leq c t_0^{-1/2} \delta^{p-1} |* \Delta|^{-1} |* \Delta|^{n/n+1} \left\{ \int_{\Delta} |f|^{n+1} \right\}^{1/n+1} = \\
 &= c t_0^{-1/2} \delta \left\{ \int_{\Delta} |f|^{n+1} \right\}^{1/n+1} \quad \text{since } |* \Delta| = \delta^{(p-2)(n+1)}.
 \end{aligned}$$

Moreover, since $t_0 > t_0 - a > \delta^{2(p-2)}$, we claim that $*\Delta \subset \Delta^a((2t_0)^{1/2})$. Infact, for $t_0 + \delta^{2(p-2)} \geq b$, we see that $2t_0 - a = t_0 + (t_0 - a) > t_0 + \delta^{2(p-2)} > b$ so that $2t_0 > b - a$. For $t_0 + \delta^{2(p-2)} < b$ the initial surface disc of height $t_0 + \delta^{2(p-2)} - a$ and center P_0 contains $*\Delta$. But, since $\delta^{2(p-2)} < t_0$, this height is $< 2t_0$ as desired. Therefore

$$t_0^{-1/2} \left\{ \int_{\Delta} |f|^{n+1} \right\}^{1/n+1} \leq \left\{ t_0^{-(n+1)/2} \int_{\Delta^a((2t_0)^{1/2})} |f|^{n+1} \right\}^{1/n+1} = c C_{a,q}(f), \quad \text{with } q = n + 1.$$

Consequently,

$$f \cdot \Delta [J_1 - C(\Delta)] \leq c \gamma^{1/4(n+1)} C_{a,q}(f)$$

and hence

$$\left\{ |\Delta|^{-1} \int_{\Delta} |f \cdot \Delta|^2 |J_1 - C(\Delta)|^2 \right\}^{1/2} \leq c \gamma^{1/4(n+1)} C_{a,2}(f),$$

by the equivalence of $C_p(f)$ for various $p \geq 1$.

By (2.3) with $\delta = \gamma^{1/4(n+1)}$, we have $(|* \Delta| \cdot |\Delta|^{-1}) < c_n \gamma^{-1/2}$ as in the proof of Lemma 2.1. Hence, using (2.1), we have

$$\left\{ |\Delta| \int_{\Delta} |J(f_1)|^2 \right\}^{1/2} \leq c \omega \left\{ (|* \Delta| \cdot |\Delta|^{-1}) |* \Delta|^{-1} \int_{\Delta} |f_1|^2 \right\}^{1/2} \leq c \omega \gamma^{-1/4} \|f\|_{*,2}.$$

Since, as noted above, $\omega < \gamma$,

$$\left\{ |\Delta|^{-1} \int_{\Delta} |J(f_1)|^2 \right\}^{1/2} \leq c \gamma^{3/4} \|f\|_{*,2}.$$

In order to estimate the term

$$B = \left\{ |\Delta|^{-1} \int_{\Delta} |J(f_2)(P, t) - J(f_2)(P_0, t_0)| dP dt \right\}$$

let us examine the integrand

$$|J(f_2)(P, t) - J(f_2)(P_0, t_0)|$$

which is majorized by term

$$\int_{S(a,b) \setminus {}^* \Delta} |K(P - Q, t - s) - K(P_0 - Q, t_0 - s)| |f_2(Q, s)| dQ ds$$

where $S(a, b) = \partial D \times (a, b)$. Following [4] we add and subtrat $K(P - Q, t_0 - s)$ and use the Mean-Value Theorem, to see that

$$\begin{aligned} |K(P - Q, t - s) - K(P_0 - Q, t_0 - s)| &\leq |D_t K(P - Q, \tilde{t} - s)| |t - t_0| + \\ &\quad + |\nabla_t K(\tilde{P} - Q, t_0 - s)| \cdot |P - P_0| = B_1 + B_2 \end{aligned}$$

for some $\tilde{t}$ between t and t_0 and $\tilde{P}$ some intermediate point between P_0 and P . Interchanging the order of integration

$$B \leq \int_{S(a,b) \setminus {}^* \Delta} |f - f_{\bullet \Delta}| \left\{ |\Delta|^{-1} \int_{\Delta} (B_1 + B_2) dP dt \right\} dQ ds .$$

Let now $\Delta = \Delta_0$, ${}^* \Delta = \Delta_1$, $S_j = S(2^j \delta^{p-2})$ and $\Delta_j = \Delta(2^j \delta^{p-2}) = S_j^x(t_0 - 2^{2j} \delta^{2(p-2)}, t_0 + 2^{2j} \delta^{2(p-2)})$. If $(P, t) \in \Delta$ and $(Q, s) \in \Delta_j - \Delta_{j-1}$, the same estimates of Theorem 2.3 in [4] yield

$$\begin{aligned} B_1 &\leq c(t - t_0) |\tilde{t} - s|^{-(n+3)/2} \quad \text{for } |P_0 - Q| < 2^{j-1} \delta^{p-2} \\ B_2 &\leq c|t - t_0| |\tilde{P} - Q|^{-(n+3)} \quad \text{for } |P_0 - Q| \geq 2^{j-1} \delta^{p-2} \end{aligned}$$

Moreover, since $r < \delta^{p-1}$, if $|P_0 - Q| < 2^{j-1} \delta^{p-2}$, we have

$$\begin{aligned} |\tilde{t} - s| &\geq \|s - t_0\| - |t_0 - \tilde{t}| \geq 2^{-1}(2^{2(j-1)} \delta^{2(p-2)} - \delta^{2(p-1)}) \geq \\ &\geq 2^{-1} \delta^{2(p-2)} (2^{2(j-1)} - 1) \geq c 2^{2j} \delta^{2(p-2)} . \end{aligned}$$

similary, if $|P_0 - Q| \geq 2^{j-1} \delta^{p-2}$ both $|P - Q|$, $|\tilde{P} - Q| \geq c 2^j \delta^{p-2}$.
Consequently, if $2^j \delta^{2(p-2)} \leq b - a$,

$$B_1 \leq c \delta^{2(p-1)} 2^{-j(n+3)} \delta^{-(p-2)(n+3)} = c \delta^{2(p-1)} 2^{-2j} \delta^{-2(p-2)} |\Delta_j|^{-1} = c \delta^2 2^{-2j} |\Delta_j|^{-1}$$

while if $2^{2j} \delta^{2(p-2)} > b - a$

$$B_1 \leq c \delta^{2(p-1)} 2^{-j(n+3)} \delta^{-(p-2)(n+3)} = c \delta^{2(p-1)} (b - a) [2^{j(n-1)} \delta^{(p-2)(n-1)} 2^{4j} \delta^{4(p-2)}]^{-1} \cdot \\ \cdot (b - a)^{-1} \leq c \delta^{2(p-1)} 2^{2j} \delta^{2(p-2)} |\Delta_j|^{-1} 2^{-4j} \delta^{-4(p-2)} = c \delta^2 |\Delta_j|^{-1} 2^{-2j}.$$

So, in both cases, $B_1 \leq c 2^{-j} |\Delta_j|^{-1} \delta$.

In the same manner, we obtain

$$B_2 \leq c \delta 2^{-j} |\Delta_j|^{-1} \quad \text{when } (P, t) \in \Delta \text{ and } (Q, s) \in \Delta_j - \Delta_{j-1}.$$

Therefore, we have

$$B \leq \sum_{j \geq 2} \int_{\Delta_j - \Delta_{j-1}} |f - f_{\Delta}| \left\{ |\Delta|^{-1} \int_{\Delta} (B_1 + B_2) dP dt \right\} dQ ds \leq \\ \leq c \sum_{j \geq 2} \left\{ \delta 2^{-j} |\Delta_j|^{-1} \int_{\Delta_j} |f - f_{\Delta}| dQ ds \right\} \leq c \sum_{j \geq 2} \delta 2^{-j} \{1 + (j-1)2^{n+1}\} \|f\|_{*,2} \leq \\ \leq c \gamma^{1/4(n+1)} \|f\|_{*,2} \quad \text{with norms on } \partial D \times (a, b).$$

Combining the two Cases, the conclusion follows at once.

COROLLARY 2.3. Let $f \in B_a MOC(\partial D \times (a, b))$ with $(a, b) \subset (0, T)$.
Then

$$(2.5) \quad \|J(\chi_{(a,b)} f)\|_{2,*} \leq \varphi(b-a) \|f\|_{2,*}$$

where $\varphi(r) > 0$ and $\varphi(r) \rightarrow 0$ as $r \rightarrow 0$.

This follows from Lemma 2.1 and 2.2 with $\varphi = \gamma + \psi$.

3. Construction of the global solution of $(J + cI)f = g$ on S_T .

We shall construct the global solution f of $(J + cI)f = g$ for a given $g \in B_0 MOC(S_T)$. First let us verify, given numbers c, d, m such

that $c < d < m < T$, if $h \in BMO C(\partial D \times (c, m))$ and $h(P, t) = 0$ on $\partial D \times (c, d)$, then $h \in B_a MOC(\partial D \times (d, m))$.

In fact, if $'\Delta_r = \Delta_r(P, d) = S_r \times (d - r^2, d + r^2)$ and $h_r = |\Delta_r|^{-1} \int_A h$, we have

$$\begin{aligned} \|h\|_{BMO C(\partial D \times (c, m))} &\geq |\Delta_r|^{-1} \int_{\Delta_r} |h - h_r| \geq |\Delta_r|^{-1} \int_{d-r^2}^d \int_{S_r} |h - h_r| = \\ &= h_r/2 = 2^{-1} \left| |\Delta_r|^{-1} \int_d^{d+r^2} \int_{S_r} h \right| \end{aligned}$$

that is $h \in B_a MOC(\partial D \times (d, m))$.

THEOREM 3.1. The operator $J + cI$ is invertible on

$$B_0 MOC(\partial D \times (0, T)).$$

PROOF. A standard argument, see [1], [2], shows that the operator is one-to-one. Let $g \in B_0 MOC(\partial D \times (0, T))$. We partition $(0, T)$ in $N = N(\varepsilon)$ subinterval of length $\varepsilon > 0$, so small that $cI + J$ is invertible on each of the spaces $B_{k\varepsilon} MOC(\partial D \times (k\varepsilon, (k+1)\varepsilon))$, $k = 0, 1, \dots, N-1$, by Corollary 2.3.

Consider g on $\partial D \times (0, \varepsilon)$ only. Since $g \in B_0 MOC(\partial D \times (0, \varepsilon))$, there exists $f_1 \in B_0 MOC(\partial D \times (0, \varepsilon))$ such that $(J + cI)f_1 = g$. Next let $\tilde{f}_1$ be any $B_0 MOC$ extension of f_1 to $\partial D \times (0, 2\varepsilon)$ for example:

$$\tilde{f}_1(P, t) = \begin{cases} f_1(P, t) & \text{for } t \in (0, \varepsilon) \\ f_1(P, 2\varepsilon - t) & \text{for } t \in (\varepsilon, 2\varepsilon). \end{cases}$$

Since $\tilde{f}_1 \in B_0 MOC(\partial D \times (0, 2\varepsilon))$ so does $(J + cI)\tilde{f}_1$ by [4]. Clearly $g - (J + cI)\tilde{f}_1$ is in $B_0 MOC(\partial D \times (0, 2\varepsilon))$, and is identically zero in $\partial D \times (0, \varepsilon)$. Thus, by the remark preceding the Theorem $g - (J + cI)\tilde{f}_1$ is in $B_\varepsilon MOC(\partial D \times (\varepsilon, 2\varepsilon))$, and hence there exists an f_2 in $B_\varepsilon MOC(\partial D \times (\varepsilon, 2\varepsilon))$ such that $(J + cI)f_2 = g - (J + cI)\tilde{f}_1$. If we extend f_2 to equal zero on $\partial D \times (0, \varepsilon)$, we obtain

$$(J + cI)(\tilde{f}_1 + f_2) = g \quad \text{on } D \times (0, 2\varepsilon)$$

and $\tilde{f}_1 + f_2$ remains in $B_0 MOC(\partial D \times (0, 2\varepsilon))$. Iterating this process we obtain a function $f \in B_0 MOC(\partial D \times (0, T))$ such that $(J + cI)f = g$.

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