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GABRIELLA D'ESTE

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Extended Strings and Admissible Words.

GABRIELLA D'ESTE (*)

Let m be a natural number ≥ 2 , let Γ be the quiver

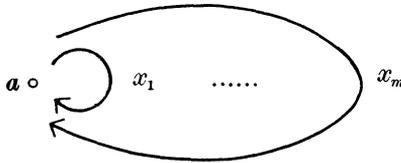


Fig. 1.

and let A be the path algebra of Γ over an algebraically closed field k (see [1] for the definitions). Then, even if $m = 2$, the injective representation $I(a)$ with socle given by the simple representation $S(a)$ of the form

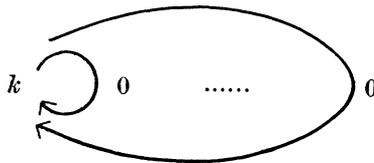


Fig. 2.

seems to be quite complicated. As a first step to investigate its structure, we will examine some special representations of Γ similar to infinite « strings ».

With a terminology suggested by [2], we call any infinite sequence $W = (l_n)_{n \in \mathbf{N}}$, where $l_n \in \{x_1, \dots, x_m\}$ for any $n \in \mathbf{N}$, a *word* in the letters

(*) Indirizzo dell'A.: Gabriella D'Este, Seminario Matematico, Università di Padova, via Belzoni 7, I-35131 Padova, Italy.

x_1, \dots, x_m . For any word $W = (l_n)_{n \in \mathbb{N}}$, we denote by $\bar{M}(W)$ the representation of Γ illustrated, in an obvious way, by the following picture:

$$\circ \xleftarrow{l_0} \circ \xleftarrow{l_1} \circ \xleftarrow{l_2} \circ \leftarrow \circ \leftarrow \circ \dots$$

$v_0 \quad v_1 \quad v_2 \quad v_3$

Finally, we denote by $\bar{M}(W)$ the representation of Γ of the form

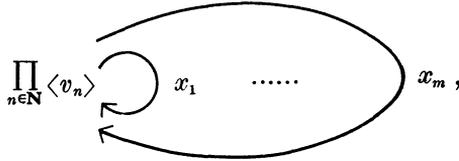


Fig. 3.

with the x_i 's componentwise defined on the vector space $V = \prod_{n \in \mathbb{N}} \langle v_n \rangle$ by means of their action on the underlying vector space $\bigoplus_{n \in \mathbb{N}} \langle v_n \rangle$ of $\bar{M}(W)$. More precisely, if $i = 1, \dots, m$ and $v = (t_n v_n)_{n \in \mathbb{N}}$ with $t_n \in k$ for any n , then $x_i(v) = (t_{i,n} v_n)_{n \in \mathbb{N}}$, where

$$t_{i,n} = \begin{cases} t_{n+1} & \text{if } l_n = x_i \\ 0 & \text{otherwise} \end{cases}$$

If the simple module generated by v_0 is the socle of $\bar{M}(W)$, then regarding $\bar{M}(W)$ as a submodule of $I(a)$, we shall say that $\bar{M}(W)$ is an *extended string contained in $I(a)$* , or, more briefly, that the word W is an *admissible word*.

As we shall see, there exist completely different admissible words. Roughly speaking, we can say that a word is admissible if and only if it is not « locally periodic ». The idea of this characterization arises from the direct check that, if x and y are two distinct letters, then the following facts hold:

- (*) The words $(x, y, y, y, y, y, y, y, y, \dots)$ and $(x, y, x, x, y, x, x, x, y, x, x, x, y, \dots)$ are admissible.
- (**) The word $(x, x, x, x, x, x, x, x, x, \dots)$ is not admissible.

With all notation as above, we prove

THEOREM 1. *Let $W = (l_n)_{n \in \mathbf{N}}$ be a word. The following conditions are equivalent:*

(i) *W is an admissible word.*

(ii) *There exists $d \in \mathbf{N}$ such that $l_0 \dots l_d \neq l_{n-d} \dots l_n$ for any $n > d$.*

Proof (i) \Rightarrow (ii) Assume that W is admissible. For any $i \in \mathbf{N}$, let π_i denote the projection of V onto $\langle v_i \rangle$ with $\text{Ker } \pi_i = \prod_{n \neq i} \langle v_n \rangle$. Next, let v denote the vector $(v_n)_{n \in \mathbf{N}} \in V$, and choose some $f \in \mathcal{A}$ such that $f(v) = v_0$. View f as a polynomial in the (non commutative) variables x_1, \dots, x_m , and let d be the degree of f . We claim that $l_0 \dots l_d \neq l_{n-d} \dots l_n$ for all $n > d$. To see this, fix any $n > d$, and let v' and v'' denote the following vectors:

$$v' = v_0 + \dots + v_d \quad \text{and} \quad v'' = v_{n-d} + \dots + v_n.$$

Evidently $\pi_0 f(v')$ and $\pi_{n-d} f(v'')$ depend on f and on the paths $l_0 \dots l_d$ and $l_{n-d} \dots l_n$. Hence, to show that these two paths are different, it suffices to check that $\pi_0 f(v') \neq 0$, while $\pi_{n-d} f(v'') = 0$. Since the endomorphisms x_1, \dots, x_m of V are componentwise defined, the same holds for f . Combining this observation with the hypothesis that $\text{deg } f = d$, we obtain $\pi_i f\left(\prod_{n > i+d} \langle v_n \rangle\right) = 0$ for any $i \in \mathbf{N}$. Since $f(v) = v_0$, it follows that $\pi_0 f(v') = \pi_0 f(v) = v_0$, while $\pi_{n-d} f(v'') = \pi_{n-d} f(v) = 0$, as we wished to show. Hence

$$l_0 \dots l_d \neq l_{n-d} \dots l_n,$$

and so condition (ii) holds.

(ii) \Rightarrow (i) Let d be as in condition (ii). To prove that W is admissible, take any $0 \neq v = (t_n v_n)_{n \in \mathbf{N}} \in V$. If $t_n = 0$ for any $n > d + 1$, then, clearly, $v_0 \in \mathcal{A}v$. Otherwise, let $m = \min \{n \in \mathbf{N} : n > d + 1, t_n \neq 0\}$, and let $u = l_{d+1} \dots l_{m-1}(v)$. Then the choice of m guarantees that $\pi_{d+1}(u) \neq 0$, while the definition of d implies that

$$0 \neq l_0 \dots l_d(u) \in \langle v_0 \rangle.$$

Consequently, $v_0 \in Au \subseteq Av$; thus W is admissible, as claimed in (i).

If $W = (l_n)_{n \in \mathbb{N}}$ is a word and $i \in \mathbb{N}$, we shall denote by W_i the following word:

$$W_i = (l_i, l_{i+1}, l_{i+2}, \dots).$$

The relation between W and the W_i 's is given by the following corollary.

COROLLARY 2. *Let $W = (l_n)_{n \in \mathbb{N}}$ be a word and let $i > 0$. The following facts hold:*

- (1) *If W_i is admissible, then also W is admissible.*
- (2) *If $W_i = W$, then W is not admissible.*

Proof (1) Assume that W_i is admissible. Then, by Theorem 1, we can find some $d \geq i$ such that $l_i \dots l_d \neq l_{n-(d-i)} \dots l_n$ for any $n > d$. This means that $l_0 \dots l_d \neq l_{n-d} \dots l_n$ for all $n > d$. Hence, applying again Theorem 1, we conclude that W is admissible.

(2) Suppose that $W_i = W$. Then, for any $d \in \mathbb{N}$, we obviously have $l_0 \dots l_d = l_{ir} \dots l_{ir+d}$ for all $r \in \mathbb{N}$. Thus W cannot satisfy condition (ii) of Theorem 1, and so W is not admissible.

Finally, we point out a « translated » version of Theorem 1.

COROLLARY 3. *Let $W = (l_n)_{n \in \mathbb{N}}$ be a word. The following conditions are equivalent:*

- (a) *W is an admissible word.*
- (b) *There exist $r, s \in \mathbb{N}$ with $r \geq s$ such that*

$$l_{r-s} \dots l_r \neq l_{n-s} \dots l_n \quad \text{for all } n > r.$$

Proof (a) \Rightarrow (b) If W is admissible and d satisfies condition (ii) of Theorem 1, then (b) holds for $r = d$ and $s = d$.

(b) \Rightarrow (a) If r and s satisfy (b), then, by Theorem 1, the word W_{r-s} is admissible. Hence, by Corollary 2, also W is admissible.

REFERENCES

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- [2] C. M. RINGEL, *The indecomposable representations of the dihedral 2-groups*, *Math. Ann.*, **214** (1975), pp. 19-34.

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