Rendiconti
del
Seminario Matematico
della
Università di Padova

H. M. Srivastava

A class of finite q-series

Rendiconti del Seminario Matematico della Università di Padova, tome 75 (1986), p. 15-24

<http://www.numdam.org/item?id=RSMUP_1986__75__15_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 75 (1986)

A Class of Finite q-Series.

H. M. SRIVASTAVA (*)

Dedicated to the memory of
Professeur JOSEPH KAMPÉ DE FÉRIET

SUMMARY - Some simple ideas are used here to prove an interesting unification (and generalization) of several finite summation formulas associated with various special hypergeometric functions of one and two variables. Further generalizations involving series with essentially arbitrary terms and their q-extensions are also presented. The main results (2.1), (3.1) and (3.8), as also the special cases (2.7) and (2.14), are believed to be new.

1. Introduction.

Making use of the Pochhammer symbol $(\lambda)_n = \Gamma(\lambda + n)/\Gamma(\lambda)$, let $F^{p:r;u}_{k:s;v}$ denote the generalized (Kampé de Fériet's) double hypergeometric function defined by (*cf.* [4]; see also [1], p. 150, and [9], p. 423)

$$(1.1)\qquad F^{p:r;u}_{k:s;v}\left[\begin{matrix}(a_p): & (c_r); & (\alpha_u); \\ & & & x, y \\ (b_k): & (d_s); & (\beta_v);\end{matrix}\right] = \\ = \sum_{l,m=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{l+m}\prod_{j=1}^{r}(c_j)_l\prod_{j=1}^{u}(\alpha_j)_m}{\prod_{j=1}^{k}(b_j)_{l+m}\prod_{j=1}^{s}(d_j)_l\prod_{j=1}^{v}(\beta_j)_m}\frac{x^l}{l!}\frac{y^m}{m!},$$

(*) Indirizzo dell'A.: Department of Mathematics, University of Victoria, Victoria, British Columbia V8W 2Y2, Canada.
Supported, in part, by NSERC (Canada) Grant A-7353.

where, for convergence of the double hypergeometric series,

(i) $p+r<k+s+1$, $p+u<k+v+1$, $|x|<\infty$, and $|y|<\infty$, or

(ii) $p+r=k+s+1$, $p+u=k+v+1$, and

$$\text{(1.2)}\qquad \begin{cases} |x|^{1/(p-k)}+|y|^{1/(p-k)}<1\,, & \text{if } p>k\,,\\ \max\{|x|,|y|\}<1\,, & \text{if } p\leqq k\,,\end{cases}$$

unless, of course, the series terminates; here, and in what follows, (a_p) abbreviates the array of p parameters $a_1, \dots, a_p$, with similar interpretations for (b_k), *et cetera.*

Recently, Shah [6] extended certain earlier results of Munot [5], involving finite sums of single and double hypergeometric functions, to hold true for some very special Kampé de Fériet functions. We recall here these finite summation formulas of Shah in the following (essentially equivalent) forms (¹) (*cf.* [6], p. 93, Equation (1.1), and p. 94, Equation (2.1)):

$$\text{(1.3)}\qquad \sum_{n=0}^{N}\frac{(-1)^n}{n!\,(N-n)!}F^{2:1;1}_{1:1;1}\left[\begin{matrix}-\varrho+\delta,\delta: & -n; & -N+n; \\ -\varrho+\delta-\sigma: & \alpha+1; & \beta+1;\end{matrix}\; x,y\right]=$$

$$=\frac{(-\varrho+\delta)_N\,(\delta)_N}{(-\varrho+\delta-\sigma)_N}\,\frac{(-1)^N(x+y)^N}{(\alpha+1)_N(\beta+1)_N}\,P_N^{(\alpha,\beta)}\left(\frac{y-x}{y+x}\right),$$

$$\text{(1.4)}\qquad \sum_{n=0}^{N}\frac{(-1)^n}{n!\,(N-n)!}\cdot$$

$$\cdot F^{1:2;2}_{1:1;1}\left[\begin{matrix}-\varrho+\delta: & -n,b; & -N+n,\delta-b; \\ -\varrho+\delta-\sigma: & \alpha+1\,; & \beta+1;\end{matrix}\; x,x\right]=$$

$$=\frac{(-\varrho+\delta)_N(\delta)_N}{(-\varrho+\delta-\sigma)_N(\beta+1)_N}\,\frac{(-x)^N}{N!}\,{}_3F_2\left[\begin{matrix}-N,\alpha+\beta+N+1,b; \\ \alpha+1,\delta;\end{matrix}\;1\right],$$

where (according to Shah [6]) σ and N are *both* non-negative integers,

(¹) Incidentally, the summation formula (1.3) was given earlier by M. A. Pathan [*Proc. Nat. Acad. Sci. India Sect. A*, **47** (1977), pp. 58-60; especially see p. 59, Equation (2.4)].

and $P_N^{(\alpha,\beta)}(z)$ denotes the Jacobi polynomial defined by

$$(1.5)\qquad P_N^{(\alpha,\beta)}(z) = \sum_{n=0}^{N} \binom{N+\alpha}{N-n}\binom{N+\beta}{n}\left(\frac{z-1}{2}\right)^n\left(\frac{z+1}{2}\right)^{N-n} =$$

$$= \binom{N+\alpha}{N}\,{}_2F_1\left[\begin{matrix} -N,\ \alpha+\beta+N+1; \\ \\ \alpha+1; \end{matrix}\ \frac{1-z}{2}\right].$$

Shah's proofs of (1.3) and (1.4), as also Munot's similar proofs of the special cases of (1.3) and (1.4) when $\sigma = 0$, are long and involved. In fact, our simple and direct proofs of Munot's results, presented in our earlier paper (see [8], pp. 94-96), apply *mutatis mutandis* to establish (1.3) and (1.4). The object of the present note is to show that our proofs extend easily to much more general results than (1.3) and (1.4). Our summation formulas (2.1), (2.7), (2.14) and (3.1), and the q-extension (3.8), are believed to be new.

2. Finite series of generalized Kampé de Fériet functions.

In this section we establish the following general result, involving Kampé de Fériet's function, which indeed unifies the summation formulas (1.3) and (1.4):

$$(2.1)\qquad \sum_{n=0}^{N} \frac{(-1)^n}{n!\,(N-n)!}\cdot$$

$$\cdot F^{p:r+1;u+1}_{k:s\ \ ;v}\left[\begin{matrix} (a_p): & -n,\ (c_r); & -N+n,\ (\alpha_u); \\ (b_k): & (d_s); & (\beta_v); \end{matrix}\ x, y\right] =$$

$$= \frac{\prod_{j=1}^{p}(a_j)_N \prod_{j=1}^{r}(c_j)_N}{\prod_{j=1}^{k}(b_j)_N \prod_{j=1}^{s}(d_j)_N}\,\frac{x^N}{N!}\,{}_{u+s+1}F_{v+r}\left[\begin{matrix} -N,\ (\alpha_u),\ 1-(d_s)-N; \\ \\ (\beta_v),\ 1-(c_r)-N; \end{matrix}\ (-1)^{r-s}\frac{y}{x}\right],$$

where N is a non-negative integer, and the various parameters and variables are so constrained that each member of (2.1) exists.

PROOF. For convenience, let Ω denote the left-hand side of (2.1). Also let

$$(2.2)\qquad \lambda_n = \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{k}(b_j)_n}, \quad \mu_n = \frac{\prod_{j=1}^{r}(c_j)_n}{\prod_{j=1}^{s}(d_j)_n}, \quad \nu_n = \frac{\prod_{j=1}^{u}(\alpha_j)_n}{\prod_{j=1}^{v}(\beta_j)_n}, \quad n \geqq 0 .$$

Making use of (2.2) and the definition (1.1), we find from (2.1) that

$$\Omega = \sum_{n=0}^{N} \sum_{l=0}^{n} \sum_{m=0}^{N-n} \lambda_{l+m}\mu_l\nu_m \frac{(-1)^{l+m+n}}{(n-l)!\,(N-m-n)!} \frac{x^l}{l!}\frac{y^m}{m!} =$$

$$= \sum_{l,\,m\geqq 0} \lambda_{l+m}\mu_l\nu_m \frac{(-x)^l}{l!}\frac{(-y)^m}{m!} \sum_{n=l}^{N-m} \frac{(-1)^n}{(n-l)!\,(N-m-n)!} =$$

$$= \sum_{l,m=0}^{l+m\leqq N} \frac{\lambda_{l+m}\mu_l\nu_m}{(N-l-m)!} \frac{x^l}{l!}\frac{(-y)^m}{m!} \sum_{n=0}^{N-l-m} (-1)^n \binom{N-l-m}{n}.$$

Since

$$(2.3)\qquad \sum_{n=0}^{N} (-1)^n \binom{N}{n} = \delta_{N,0}, \qquad N = 0, 1, 2, \ldots,$$

where $\delta_{m,n}$ is the familiar Kronecker delta, it follows at once that

$$(2.4)\qquad \Omega = \sum_{l+m=N} \lambda_{l+m}\mu_l\nu_m \frac{x^l}{l!}\frac{(-y)^m}{m!},$$

or, equivalently, that

$$(2.5)\qquad \Omega = \lambda_N x^N \sum_{m=0}^{N} \mu_{N-m}\nu_m \frac{(-y/x)^m}{(N-m)!\,m!}.$$

Recalling from (2.2) that

$$\mu_{N-m} = \frac{\prod_{j=1}^{r}(c_j)_{N-m}}{\prod_{j=1}^{s}(d_j)_{N-m}} = (-1)^{(r-s)m}\mu_N \frac{\prod_{j=1}^{s}(1-d_j-N)_m}{\prod_{j=1}^{r}(1-c_j-N)_m}, \qquad m \geqq 0,$$

we can rewrite (2.5) in the form:

$$\Omega = \lambda_N \mu_N \frac{x^N}{N!} \sum_{m=0}^{N} \frac{(-N)_m \prod_{j=1}^{u} (\alpha_j)_m \prod_{j=1}^{s} (1-d_j-N)_m}{\prod_{j=1}^{v} (\beta_j)_m \prod_{j=1}^{r} (1-c_j-N)_m} \cdot \frac{\{(-1)^{r-s}(y/x)\}^m}{m!}, \quad N \geqq 0, \tag{2.6}$$

which, in view of (2.2), is precisely the second member of the summation formula (2.1).

Two special cases ([2]) of (2.1) are worthy of mention. First of all, if in (2.1) we set

$$r = s-1 = u = v-1 = 0\,, \quad d_1 = \alpha + 1\,, \quad \beta_1 = \beta + 1\,,$$

and identify the resulting hypergeometric ${}_2F_1$ function as a Jacobi polynomial defined by (1.5), we obtain the summation formula:

$$\sum_{n=0}^{N} \frac{(-1)^n}{n!\,(N-n)!} F^{p:1;1}_{k:1;1}\left[\begin{matrix}(a_p): & -n; & -N+n; \\ (b_k): & \alpha+1; & \beta+1;\end{matrix}\; x, y\right] = \tag{2.7}$$

$$= \frac{\prod_{j=1}^{p} (a_j)_N}{\prod_{j=1}^{k} (b_j)_N} \frac{(-1)^N (x+y)^N}{(\alpha+1)_N (\beta+1)_N} P_N^{(\alpha,\beta)}\left(\frac{y-x}{y+x}\right), \quad N \geqq 0\,.$$

Formula (2.7) corresponds, when $p-1=k=1$, to Shah's result (1.3). For $p=k=1$, the Kampé de Fériet function occurring in (2.7) reduces to Appell's function F_2 (*cf.* [1], p. 14, Equation (12)), and we are led immediately to Munot's result ([5], p. 691, Equation (2.1)).

([2]) A further special case of (2.1) when $p = k = 0$ was proven earlier by H. L. Manocha and B. L. Sharma [*Compositio Math.*, **18** (1967), pp. 229-234; see p. 233, Equation (16)] by repeatedly using certain operators of *fractional* derivative.

Next we consider a special case of (2.1) when

$$x = y, \quad r = s = u = v = 1, \quad c_1 = \gamma, \quad d_1 = \alpha + 1,$$
$$\alpha_1 = \delta, \quad \beta_1 = \beta + 1,$$

and we thus obtain

$$(2.8) \quad \sum_{n=0}^{N} \frac{(-1)^n}{n!(N-n)!} F^{p:2;2}_{k:1;1}\left[\begin{matrix}(a_p): & -n, \gamma; & -N+n, \delta; \\ (b_k): & \alpha+1; & \beta+1;\end{matrix}\; x, x\right] =$$
$$= \frac{\prod_{j=1}^{p}(a_j)_N}{\prod_{j=1}^{k}(b_j)_N} \frac{(\gamma)_N}{(\alpha+1)_N} \frac{x^N}{N!} {}_3F_2\left[\begin{matrix}-N, \delta, -\alpha-N; \\ \beta+1, 1-\gamma-N;\end{matrix}\; 1\right],$$

or, equivalently,

$$(2.9) \quad \sum_{n=0}^{N} \frac{(-1)^n}{n!(N-n)!} F^{p:2;2}_{k:1;1}\left[\begin{matrix}(a_p): & -n, \gamma; & -N+n, \delta; \\ (b_k): & \alpha+1; & \beta+1;\end{matrix}\; x, x\right] =$$
$$= \frac{\prod_{j=1}^{p}(a_j)_N}{\prod_{j=1}^{k}(b_j)_N} \frac{(\delta)_N}{(\beta+1)_N} \frac{(-x)^N}{N!} {}_3F_2\left[\begin{matrix}-N, \gamma, -\beta-N; \\ \alpha+1, 1-\delta-N;\end{matrix}\; 1\right],$$

which would follow readily from (2.8) if, upon reversing the sum on the left-hand side, we interchange α and β, and γ and δ.

Now in (2.9) we appropriately apply the known transformation (*cf.*, *e.g.*, [3], p. 499, Equation (6.1))

$$(2.10) \quad {}_3F_2\left[\begin{matrix}a, b, c; \\ d, e;\end{matrix}\; 1\right] =$$
$$= \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-c)\Gamma(d+e-a-b)} {}_3F_2\left[\begin{matrix}d-a, d-b, c; \\ d, d+e-a-b;\end{matrix}\; 1\right],$$

which holds true, by analytic continuation, when both series terminate or when

$$\min\{\operatorname{Re}(d+e-a-b-c),\ \operatorname{Re}(e-c)\} > 0\,,$$

and we find from (2.9) that

$$(2.11)\quad \sum_{n=0}^{N} \frac{(-1)^n}{n!\,(N-n)!} F^{p:2;2}_{k:1;1}\left[\begin{matrix}(a_p): & -n,\gamma; & -N+n,\delta; \\ (b_k): & \alpha+1\ ; & \beta+1; \end{matrix}\ x, x\right] =$$

$$= \frac{\prod_{j=1}^{p}(a_j)_N}{\prod_{j=1}^{k}(b_j)_N} \frac{(\alpha+\beta-\gamma-\delta+2)_N}{(\beta+1)_N} \frac{x^N}{N!} \cdot$$

$$\cdot {}_3F_2\left[\begin{matrix}-N, \alpha+\beta+N+1, \alpha-\gamma+1; \\ \alpha+1, \alpha+\beta-\gamma-\delta+2; \end{matrix}\ 1\right].$$

Multiplying the hypergeometric series identity (*cf.* [7], p. 31, Equation (1.7.1.3))

$$(2.12)\quad {}_2F_1\left[\begin{matrix}-N, b; \\ c; \end{matrix}\ x\right] = (1-x)^N {}_2F_1\left[\begin{matrix}-N, c-b; \\ c; \end{matrix}\ -\frac{x}{1-x}\right]$$

by $x^{a-1}(1-x)^{d-a-1}$ and integrating the resulting equation from $x=0$ to $x=1$, we obtain

$$(2.13)\quad {}_3F_2\left[\begin{matrix}-N, a, b; \\ c, d; \end{matrix}\ 1\right] = \frac{(d-a)_N}{(d)_N} {}_3F_2\left[\begin{matrix}-N, a, c-b; \\ c, a-d-N+1; \end{matrix}\ 1\right],$$

where N is a non-negative integer.

In view of (2.13), the summation formula (2.11) assumes its *equivalent* form:

$$(2.14)\quad \sum_{n=0}^{N} \frac{(-1)^n}{n!\,(N-n)!} F^{p:2;2}_{k:1;1}\left[\begin{matrix}(a_p): & -n,\gamma; & -N+n,\delta; \\ (b_k): & \alpha+1; & \beta+1; \end{matrix}\ x, x\right] =$$

$$= \frac{\prod_{j=1}^{p}(a_j)_N}{\prod_{j=1}^{k}(b_j)_N} \frac{(\gamma+\delta)_N}{(\beta+1)_N} \frac{(-x)^N}{N!} {}_3F_2\left[\begin{matrix}-N, \alpha+\beta+N+1, \gamma; \\ \alpha+1, \gamma+\delta; \end{matrix}\ 1\right],$$

which, for $p = k = 1$, corresponds to (1.4). Moreover, in its special case when $p = k = 0$ or, alternatively, when

$$p = k, \quad a_j = b_j, \quad j = 1, \ldots, p \ (\text{or } k),$$

our summation formula (2.14) yields Munot's result ([5], p. 693, Equation (2.12)).

3. Further generalizations and q-extensions.

A closer look at our proof of the summation formula (2.1) suggests the existence of an immediate further generalization of (2.1) in the form:

$$(3.1) \qquad \sum_{n=0}^{N} (-1)^n \binom{N}{n} \sum_{l=0}^{n} \sum_{m=0}^{N-n} \lambda_{l+m} \mu_l \nu_m (-n)_l (-N+n)_m \frac{x^l}{l!} \frac{y^m}{m!} = \\ = \lambda_N x^N \sum_{n=0}^{N} \binom{N}{n} \mu_{N-n} \nu_n \left(-\frac{y}{x}\right)^n,$$

where $\{\lambda_n\}$, $\{\mu_n\}$ and $\{\nu_n\}$ are *arbitrary* complex sequences.

Formula (3.1) would evidently reduce to the hypergeometric form (2.1) when the arbitrary coefficients λ_n, μ_n and ν_n $(n = 0, 1, 2, \ldots)$ are chosen as in (2.2).

In order to present the q-extensions of the finite summation formulas considered in the preceding section, we begin by recalling the definition (*cf.* [2]; see also [7], Chapter 3)

$$(3.2) \qquad (\lambda; q)_\mu = \prod_{j=0}^{\infty} \left(\frac{1 - \lambda q^j}{1 - \lambda q^{\mu+j}} \right)$$

for arbitrary q, λ and μ, $|q| < 1$, so that

$$(3.3) \qquad (\lambda; q)_n = \begin{cases} 1, & \text{if } n = 0, \\ (1-\lambda)(1-\lambda q) \ldots (1-\lambda q^{n-1}), & \forall n \in \{1, 2, 3, \ldots\}, \end{cases}$$

and

$$\lim_{q\to 1}\left\{\frac{(q^\lambda;q)_n}{(q^\mu;q)_n}\right\}=\frac{(\lambda)_n}{(\mu)_n},\qquad n=0,1,2,\ldots, \tag{3.4}$$

for arbitrary λ and μ, $\mu\neq 0,-1,-2,\ldots$.

We shall also need the q-binomial coefficient defined, for arbitrary λ, by

$$\begin{bmatrix}\lambda\\ n\end{bmatrix}=(-1)^n q^{\frac{1}{2}n(2\lambda-n+1)}\frac{(q^{-\lambda};q)_n}{(q;q)_n},\qquad n=0,1,2,\ldots, \tag{3.5}$$

so that, if N is an integer,

$$\begin{bmatrix}N\\ n\end{bmatrix}=\frac{(q;q)_N}{(q;q)_n(q;q)_{N-n}}=\begin{bmatrix}N\\ N-n\end{bmatrix},\qquad 0\leqq n\leqq N. \tag{3.6}$$

Furthermore, we have the elementary q-identity

$$\sum_{n=0}^{N}(-1)^n\begin{bmatrix}N\\ n\end{bmatrix}q^{\frac{1}{2}n(n-1)}=\delta_{N,0}, \tag{3.7}$$

which provides an interesting q-analogue of the combinatorial identity (2.3).

Assuming the coefficients λ_n, μ_n and ν_n $(n=0,1,2,\ldots)$ to be arbitrary complex numbers, it is not difficult to prove, using (3.7) along the lines detailed in the preceding section, the following q-extension of the general result (3.1):

$$\sum_{n=0}^{N}(-1)^n\begin{bmatrix}N\\ n\end{bmatrix}\sum_{l=0}^{n}\sum_{m=0}^{N-n}q^{\frac{1}{2}n(n-2m-1)}\lambda_{l+m}\mu_l\nu_m(q^{-n};q)_l(q^{-N+n};q)_m\cdot$$
$$\cdot\frac{x^l}{(q;q)_l}\frac{y^m}{(q;q)_m}=$$
$$=\lambda_N\left(\frac{x}{q}\right)^N\sum_{n=0}^{N}\begin{bmatrix}N\\ n\end{bmatrix}q^{\frac{1}{2}n(n-2N+1)}\mu_{N-n}\nu_n\left(-\frac{y}{x}\right)^n, \tag{3.8}$$

which holds true whenever both sides exist.

By specializing the coefficients λ_n, μ_n and ν_n $(n=0,1,2,\ldots)$ in a manner analogous to (2.2) but using the definition (3.3), we can deduce appropriate q-extensions of the summation formulas (2.1), (2.7) and

(2.14) as particular cases of the q-series (3.8). Moreover, in view of (3.4), the q-summation formula (3.8) would naturally yield (3.1) in the limit when $q \to 1$.

REFERENCES

[1] P. APPELL - J. KAMPÉ DE FÉRIET, *Fonctions Hypergéométriques et Hypersphériques; Polynômes d'Hermite*, Gauthier-Villars, Paris, 1926.

[2] H. EXTON, *q-Hypergeometric Functions and Applications*, Halsted Press (Ellis Horwood Ltd., Chichester), John Wiley & Sons, New York, Brisbane, Chichester and Toronto, 1983.

[3] G. H. HARDY, *A chapter from Ramanujan's note-book*, Proc. Cambridge Philos. Soc., **21** (1923), pp. 492-503.

[4] J. KAMPÉ DE FÉRIET, *Les fonctions hypergéométriques d'order supérieur à deux variables*, C.R. Acad. Sci. Paris, **173** (1921), pp. 401-404.

[5] P. C. MUNOT, *On Jacobi polynomials*, Proc. Cambridge Philos. Soc., **65** (1969), pp. 691-695.

[6] M. SHAH, *On the study of Kampé de Fériet functions*, Mat. Vesnik, **6** (**19**), (**34**) (1982), pp. 93-97.

[7] L. J. SLATER, *Generalized Hypergeometric Functions*, Cambridge Univ. Press, Cambridge, London and New York, 1966.

[8] H. M. SRIVASTAVA, *Certain formulas involving Appell functions*, Comment. Math. Univ. St. Paul., **21** (1972), fasc. 1, pp. 73-99.

[9] H. M. SRIVASTAVA - R. PANDA, *An integral representation for the product of two Jacobi polynomials*, J. London Math. Soc. (2), **12** (1976), pp. 419-425.

Manoscritto pervenuto in redazione il 10 maggio 1984.