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A Property of Multiplication in Sobolev Spaces. Some Applications.

TULLIO VALENT (*)

SUMMARY - Let Ω be an open subset of $\mathbb{R}^n$ having the cone property. In Sect. 1, Theorem 1 concerns the conditions on the numbers p, q, r and m for (the pointwise) multiplication is a continuous function of $W^{m,p}(\Omega) \times W^{m,q}(\Omega)$ into $W^{m,r}(\Omega)$. As a consequence of Theorem 1, multiplication is a continuous function of $W^{m,p}(\Omega) \times W^{m,q}(\Omega)$ into $W^{m,q}(\Omega)$ if the following conditions are satisfied: $q < p$, $mp > n$ and, if $p \neq q$ and the volume of Ω is infinite, $mq < n$. In particular one deduces the well known fact that $W^{m,p}(\Omega)$ is a Banach algebra if $mp > n$. In Sect. 2 we apply Theorem 1 in showing a property of the Nemytsky operator: see Theorem 2. The proof of such a property given in [2] (see Lemma 1) is not completely correct.

1. A property of multiplication in Sobolev spaces.

Let Ω be an open subset of $\mathbb{R}^n (n \geq 1)$, let m be an integer ≥ 1 and let p, q, r be real numbers ≥ 1 . $W^{m,p}(\Omega)$ will denotes the vector space $\{v \in L^p(\Omega): D^\alpha v \in L^p(\Omega), 0 \leq |\alpha| \leq m\}$ with the norm $\|\cdot\|_{m,p}$ defined by

$$\|v\|_{m,p} = \left(\sum_{|\alpha| \leq m} \|D^\alpha v\|_{0,p}^p \right)^{1/p},$$

where $\|\cdot\|_{0,p}$ is the usual norm of $L^p(\Omega)$. We will put $D_i = \partial/\partial x_i$, ($i = 1, \dots, n$).

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THEOREM 1. *Assume that Ω has the cone property, and that $p \geq r$, $q \geq r$ and*

$$\frac{m}{n} > \frac{1}{p} + \frac{1}{q} - \frac{1}{r}.$$

If the volume of Ω is infinite, assume further that $mp \leq n$ when $q \neq r$, that $mq \leq n$ when $p \neq r$ and that

$$\frac{m-1}{n} \leq \frac{1}{p} + \frac{1}{q} - \frac{1}{r} \quad \text{when } p \neq r, q \neq r.$$

Then, if $u \in W^{m,p}(\Omega)$ and $v \in W^{m,q}(\Omega)$, we have $uv \in W^{m,r}(\Omega)$ and there exists a positive number c independent of u and v such that $\|uv\|_{m,r} \leq c \|u\|_{m,p} \|v\|_{m,q}$.

PROOF (by induction on m). As a first step we prove that the statement is true for $m = 1$. Then we suppose that $u \in W^{1,p}(\Omega)$ and $v \in W^{1,q}(\Omega)$ with $p \geq r$, $q \geq r$, $p \leq n$ if $q \neq r$, $q \leq n$ if $p \neq r$,

$$(1.1) \quad \frac{1}{p} + \frac{1}{q} - \frac{1}{r} < \frac{1}{n}$$

and

$$(1.2) \quad \frac{1}{p} + \frac{1}{q} - \frac{1}{r} \geq 0 \quad (\text{if } p \neq r \neq q),$$

and we show that $uv \in W^{1,r}(\Omega)$ and that $\|uv\|_{1,r} \leq c \|u\|_{1,p} \|v\|_{1,q}$, where c is a positive number independent of u and v . Moreover we show that, if Ω has finite volume, the conclusion holds without assumption (1.2) and without the conditions $q \neq r \Rightarrow p \leq n$ and $p \neq r \Rightarrow q \leq n$.

If $q > r$ [resp. $p > r$] let α [resp. β] be the real number such that

$$\frac{1}{\alpha} + \frac{1}{q} = \frac{1}{r}, \quad \left[\text{resp. } \frac{1}{p} + \frac{1}{\beta} = \frac{1}{r} \right],$$

(i.e., $\alpha = qr/(q-r)$ and $\beta = pr/(p-r)$). Holder's inequality yields

the following implications

$$(1.3) \quad \left\{ \begin{array}{ll} w_1 \in L^p(\Omega), & p > r, \quad w_2 \in L^\beta(\Omega) \Rightarrow w_1 w_2 \in L^r(\Omega), \\ & \|w_1 w_2\|_{0,r} \leq c_1 \|w_1\|_{0,p} \|w_2\|_{0,\beta}, \\ w_1 \in L^q(\Omega), & q > r, \quad w_2 \in L^\alpha(\Omega) \Rightarrow w_1 w_2 \in L^r(\Omega), \\ & \|w_1 w_2\|_{0,r} \leq c_1 \|w_1\|_{0,q} \|w_2\|_{0,\alpha}, \end{array} \right.$$

where c_1 is a positive number independent of w_1 and w_2 .

Note that, by virtue of (1.1), $p \leq n$ implies $q > r$ and $q \geq n$ implies $p > r$.

A basic remark is that, if $p < n$ [resp. $q < n$], condition (1.1) is equivalent to the condition

$$\alpha < \frac{np}{n-p} \quad \left[\text{resp. } \beta < \frac{nq}{n-q} \right],$$

while condition (1.2) is equivalent to the condition $p \leq \alpha$ [resp. $q \leq \beta$].

Hence, by the Sobolev imbedding theorem (see e.g. Adams [1], Theorem 5.4) the following continuous imbedding holds if $p \leq n$ [resp. $q \leq n$]:

$$(1.4) \quad W^{1,p}(\Omega) \subseteq L^\alpha(\Omega), \quad [\text{resp. } W^{1,q}(\Omega) \subseteq L^\beta(\Omega)].$$

We are now in a position to easily recognize that

$$(1.5) \quad \left\{ \begin{array}{ll} w_1 \in L^p(\Omega), & w_2 \in W^{1,q}(\Omega) \Rightarrow w_1 w_2 \in L^r(\Omega), \\ & \|w_1 w_2\|_{0,r} \leq c_2 \|w_1\|_{0,p} \|w_2\|_{1,q}, \\ w_1 \in L^q(\Omega), & w_2 \in W^{1,p}(\Omega) \Rightarrow w_1 w_2 \in L^r(\Omega), \\ & \|w_1 w_2\|_{0,r} \leq c_2 \|w_1\|_{0,q} \|w_2\|_{1,p}, \end{array} \right.$$

where c_2 is a positive number independent of w_1 and w_2 .

Indeed, if $p < n$ and $q < n$, then by (1.1) we have $p > r$ and $q > r$; thus (1.5) is an immediate consequence of (1.3) and (1.4). If $q > n$ and $p \leq n$ [resp. $p > n$ and $q \leq n$] then $p = r$ and $W^{1,q}(\Omega) \subseteq L^\beta(\Omega)$ [resp. $q = r$ and $W^{1,p}(\Omega) \subseteq L^\alpha(\Omega)$], and therefore the first [resp. second] of the implications (1.5) follows from Hölder's inequality because $W^{1,q}(\Omega)$ [resp. $W^{1,p}(\Omega)$], by the Sobolev imbedding theorem, can be

continuously imbedded into $L^\infty(\Omega)$, while the second [resp. first] of the implications (1.5) is a consequence of the second [resp. first] of the implications (1.3). Finally, if $p > n$ and $q > n$, then $p = r = q$ and $W^{1,p}(\Omega)$ can be continuously imbedded into $L^\infty(\Omega)$; thus (1.5) follows once more from Hölder's inequality.

Observe that, if Ω has finite volume, then $s_1 \leq s_2 \Rightarrow L^{s_1}(\Omega) \subseteq L^{s_2}(\Omega)$; therefore, in this case, the continuous imbedding (1.4) does not need condition (1.2), and the deduction of (1.5) does not need the implications $q \neq r \Rightarrow p \leq n$ and $p \neq r \Rightarrow q \leq n$.

In view of (1.5) we have

$$(1.6) \quad uv \in L^r(\Omega), \quad vD_i u \in L^r(\Omega), \quad uD_i v \in L^r(\Omega), \quad (i = 1, \dots, n).$$

Let now $(u_k)_{k \in \mathbb{N}}$ and $(v_k)_{k \in \mathbb{N}}$ be sequences in $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ and in $C^\infty(\Omega) \cap W^{1,q}(\Omega)$ respectively such that

$$(1.7) \quad \lim_{k \rightarrow \infty} \|u_k - u\|_{1,p} = 0, \quad \lim_{k \rightarrow \infty} \|v_k - v\|_{1,q} = 0.$$

Since by (1.5) we have

$$\begin{aligned} \|v_k D_i u_k - v D_i u\|_{0,r} &\leq \|v_k (D_i u_k - D_i u)\|_{0,r} + \\ &+ \|(v_k - v) D_i u\|_{0,r} \leq \|D_i u_k - D_i u\|_{0,p} \|v_k\|_{1,q} + \\ &+ \|v_k - v\|_{1,q} \|D_i u\|_{0,p} \leq \|u_k - u\|_{1,p} \|v_k\|_{1,q} + \|v_k - v\|_{1,q} \|u\|_{1,p}, \end{aligned}$$

and

$$\begin{aligned} \|u_k D_i v_k - u D_i v\|_{0,r} &\leq \|u_k (D_i v_k - D_i v)\|_{0,r} + \\ &+ \|(u_k - u) D_i v\|_{0,r} \leq \|D_i v_k - D_i v\|_{0,q} \|u_k\|_{1,p} + \\ &+ \|u_k - u\|_{1,p} \|D_i v\|_{0,q} \|v_k - v\|_{1,q} \|u_k\|_{1,p} + \|u_k - u\|_{1,p} \|v\|_{1,q}, \end{aligned}$$

then from (1.7) it follows that

$$(1.8) \quad \lim_{k \rightarrow \infty} \|(v_k D_i u_k + u_k D_i v_k) - (v D_i u + u D_i v)\|_{0,r} = 0.$$

Using Hölder's inequality we can immediately deduce from (1.8)

that

$$\begin{aligned} & \int_{\Omega} (v D_i u + u D_i v) \varphi \, dx + \int_{\Omega} uv D_i \varphi \, dx = \\ & = \lim_{k \rightarrow \infty} \left[\int_{\Omega} (v_k D_i u_k + u_k D_i v_k) \varphi \, dx + \int_{\Omega} u_k v_k D_i \varphi \, dx \right] \quad \forall \varphi \in \mathfrak{D}(\Omega), \end{aligned}$$

whence

$$(1.9) \quad \int_{\Omega} (v D_i u + u D_i v) \varphi \, dx + \int_{\Omega} uv D_i \varphi \, dx = 0 \quad \forall \varphi \in \mathfrak{D}(\Omega),$$

because, being $D_i(u_k v_k) = v_k D_i u_k + u_k D_i v_k$, we have

$$\int_{\Omega} (v_k D_i u_k + u_k D_i v_k) \varphi \, dx + \int_{\Omega} u_k v_k D_i \varphi \, dx = 0$$

Note that (1.9) means that

$$v D_i u + u D_i v = D_i(uv);$$

hence $D_i(uv) \in L^r(\Omega)$ because of (1.6). Moreover by (1.6) uv belongs to $L^r(\Omega)$. Thus we conclude that $uv \in W^{1,r}(\Omega)$. Finally, from (1.5) we obtain

$$\begin{aligned} \|uv\|_{1,r} & \leq c_3 \left(\|uv\|_{0,r} + \sum_{i=1}^n \|v D_i u + u D_i v\|_{0,r} \right) \leq \\ & \leq c_4 (\|u\|_{1,p} \|v\|_{1,q}) + \sum_{i=1}^n (\|D_i u\|_{0,p} \|v\|_{1,q} + \|u\|_{1,p} \|D_i v\|_{0,q}) \leq c_5 \|u\|_{1,p} \|v\|_{1,q}, \end{aligned}$$

where c_3 , c_4 and c_5 are positive numbers independent of u and v .

As a second step of our induction argument, we now suppose that the statement of Theorem 1 is true for an $m (\geq 1)$ and we will prove that, consequently, it is true even when m is replaced by $m+1$. Accordingly, let p_1 , q_1 , r_1 be real numbers ≥ 1 such that $p_1 \geq r_1$, $q_1 \geq r_1$ and

$$(1.10) \quad \frac{m+1}{n} > \frac{1}{p_1} + \frac{1}{q_1} - \frac{1}{r_1}$$

and let $u_1 \in W^{m+1,p}(\Omega)$ and $v \in W^{m+1,q}(\Omega)$. If the volume of Ω is infinite we also suppose that

$$(1.11) \quad \frac{m}{n} \leq \frac{1}{p_1} + \frac{1}{q_1} - \frac{1}{r_1} \quad \text{in the case } p_1 \neq r_1 \neq q_1,$$

that $(m+1)p_1 \leq n$ in the case $q_1 \neq r_1$ and that $(m+1)q_1 \leq n$ in the case $p_1 \neq r_1$.

We begin by considering the case when $mp_1 \leq n$ and $mq_1 \leq n$, with $m > 1$. We set

$$\tilde{p}_1 = \frac{np_1}{n-p_1} \quad \text{and} \quad \tilde{q}_1 = \frac{nq_1}{n-q_1}.$$

By the Sobolev imbedding theorem, under our hypotheses, we have

$$u \in W^{m,\tilde{p}_1}(\Omega), \quad v \in W^{m,\tilde{q}_1}(\Omega),$$

besides

$$D_i u \in W^{m,p_1}(\Omega), \quad D_i v \in W^{m,q_1}(\Omega), \quad (i = 1, \dots, n).$$

Remark that, since

$$(1.12) \quad \frac{1}{\tilde{p}_1} = \frac{1}{p_1} - \frac{1}{n} \quad \text{and} \quad \frac{1}{\tilde{q}_1} = \frac{1}{q_1} - \frac{1}{n},$$

(1.10) implies

$$(1.13) \quad \frac{m}{n} > \frac{1}{\tilde{p}_1} + \frac{1}{q_1} - \frac{1}{r_1} \quad \text{and} \quad \frac{m}{n} > \frac{1}{p_1} + \frac{1}{\tilde{q}_1} - \frac{1}{r_1}.$$

If the volume of Ω is finite, this suffices to deduces (via the induction hypothesis) that

$$vD_i u \in W^{m,r_1}(\Omega), \quad uD_i v \in W^{m,r_1}(\Omega)$$

and that

$$(1.14) \quad \|vD_i u\|_{m,r_1} \leq c_6 \|v\|_{m,\tilde{q}_1} \|D_i u\|_{m,p_1}, \quad \|uD_i v\|_{m,r_1} \leq c_6 \|u\|_{m,\tilde{p}_1} \|D_i v\|_{m,q_1},$$

where c_8 is a positive number independent of u , v and i ; then, in view of the Sobolev imbedding theorem, there exists a positive number c_7 independent of u , v and i such that

$$(1.15) \quad \|vD_i u\|_{m,r_1} \leq c_7 \|u\|_{m+1,p_1} \|v\|_{m+1,q_1}, \quad \|uD_i v\|_{m,r_1} \leq c_7 \|u\|_{m+1,p_1} \|v\|_{m+1,q_1}.$$

If the volume of Ω is infinite, it is not difficult to realize that our assumptions imply that

$$(1.16) \quad \frac{m-1}{n} \leq \frac{1}{\tilde{p}_1} + \frac{1}{q_1} - \frac{1}{r_1} \quad \text{and} \quad \frac{m-1}{n} \leq \frac{1}{p_1} + \frac{1}{\tilde{q}_1} - \frac{1}{r_1}.$$

Indeed, by (1.12) each of the conditions (1.16) is equivalent to (1.11) and therefore (1.16) holds if $p_1 \neq r_1 \neq q_1$; moreover (1.16) also holds if $p_1 = r_1$ and if $q_1 = r_1$, because (1.11) becomes $mq_1 \leq n$ if $p_1 = r_1$ and becomes $mq_1 \leq n$ if $q_1 = r_1$.

Furthermore, since $m\tilde{p}_1 = (m+1)p_1$ and $m\tilde{q}_1 = (m+1)q_1$, if the volume of Ω is infinite we have $m\tilde{p}_1 \leq n$ in the case $q_1 \neq r_1$ and $m\tilde{q}_1 \leq n$ in the case $p_1 \neq r_1$.

Therefore, by the induction hypothesis, estimates (1.15), and consequently (1.16), are true even when the volume of Ω is infinite.

By an analogous way as we obtained (1.15) we can show that $uv \in W^{m,r_1}(\Omega)$ and that a positive number c_8 independent of u and v exists such that

$$(1.17) \quad \|uv\|_{m,r_1} \leq c_8 \|u\|_{m+1,p_1} \|v\|_{m+1,q_1}.$$

We now prove that estimates (1.15) and (1.17) hold also in the four cases: $mp_1 > n$, $mq_1 > n$, $p_1 = n$ with $m = 1$ and $q_1 = n$ with $m = 1$.

If $mp_1 > n$ or $mq_1 > n$ it is easily seen that all hypotheses of the statement of Theorem 1 are satisfied, so that (by the induction assumption) multiplication is a continuous operator from $W^{m,p_1}(\Omega) \times W^{m,q_1}(\Omega)$ to $W^{m,r_1}(\Omega)$. This is obvious if the volume of Ω is finite; if the volume of Ω is infinite we need only remark that, if $mp_1 > n$ [resp. $mq_1 > n$], then $q_1 = r_1$ [resp. $p_1 = r_1$].

Let now $p_1 = n$ [resp. $q_1 = n$] and $m = 1$. If the volume of Ω is finite it may occur that $q_1 > r_1$ [resp. $p_1 > r_1$]: in this case all hypotheses of the statement of Theorem 1 are again satisfied. If the volume of Ω is infinite we have $q_1 = r_1$ [resp. $p_1 = r_1$]. Note that, in the case when $p_1 = n$, $q_1 = r_1$ [resp. $q_1 = n$, $p_1 = r_1$] and $m = 1$, the hypotheses

of the statement of Theorem 1 are satisfied provided p_1 [resp. q_1] is replaced by $\tilde{p}_1$ [resp. $\tilde{q}_1$], where $\tilde{p}_1$ [resp. $\tilde{q}_1$] is any number $> p_1$ [resp. $> q_1$]. Thus, recalling that (by the Sobolev imbedding theorem) $W^{2,p_1}(\Omega)$ [resp. $W^{2,q_1}(\Omega)$] can be continuously imbedded into $W^{1,\tilde{p}_1}(\Omega)$ [resp. $W^{1,\tilde{q}_1}(\Omega)$], from the induction hypothesis we get that, if $p_1 = n$ [resp. $q_1 = n$], then multiplication is a continuous operator from $W^{2,p_1}(\Omega) \times W^{1,q_1}(\Omega)$ [resp. $W^{1,p_1}(\Omega) \times W^{2,q_1}(\Omega)$] to $W^{1,r_1}(\Omega)$.

This evidently shows what we wanted: that (1.15) and (1.17) are true also in the four cases $mp_1 > n$, $mq_1 > n$, $p_1 = n$ with $m = 1$, and $q_1 = n$ with $m = 1$.

Now, using the density of $C^\infty(\Omega) \cap W^{m,s}(\Omega)$ in $W^{m,s}(\Omega)$, $1 \leq s \in \mathbb{R}$, we can deduce, by a procedure quite analogous to the one developed in the first step, that $D_i(uv) = v D_i u + u D_i v$. Then, in view of (1.15) and (1.17), we can conclude that $uv \in W^{m+1,r_1}(\Omega)$ and that $\|uv\|_{m+1,r_1} \leq c_9 \|u\|_{m+1,p_1} \|v\|_{m+1,q_1}$, where c_9 is a positive number independent of u and v . Thus the induction argument is complete. $\square$

2. A property of the Nemytsky operator.

Let N be an integer ≥ 1 and let $(x, y) \mapsto f(x, y)$ be a real function defined in $\Omega \times \mathbb{R}^N$. For any function $\sigma: \Omega \rightarrow \mathbb{R}^N$ let $F(\sigma): \Omega \rightarrow \mathbb{R}$ be the function defined by setting

$$(2.1) \quad F(\sigma)(x) = f(x, \sigma(x)), \quad x \in \Omega.$$

We will denote by $C^m(\bar{\Omega} \times \mathbb{R}^N)$ the set of real functions defined in $\Omega \times \mathbb{R}^N$ which are restrictions to $\Omega \times \mathbb{R}^N$ of some C^m -function of $\mathbb{R}^n \times \mathbb{R}^N$ into $\mathbb{R}$.

THEOREM 2. *Assume that Ω is bounded and has the cone property, that $f \in C^m(\bar{\Omega} \times \mathbb{R}^N)$ and that $mp > n$. Then $\sigma \mapsto F(\sigma)$ is a continuous operator of $(W^{m,p}(\Omega))^N$ into $W^{m,p}(\Omega)$.*

PROOF (by induction on m). We denote by $F_{x_i}(\sigma)$ and $F_{y_j}(\sigma)$, ($i = 1, \dots, n$; $j = 1, \dots, N$), the real functions defined in Ω by setting

$$F_{x_i}(\sigma)(x) = \frac{\partial f}{\partial x_i}(x, \sigma(x)), \quad F_{y_j}(\sigma)(x) = \frac{\partial f}{\partial y_j}(x, \sigma(x)).$$

We begin with the case $m = 1$. Accordingly, let $f \in C^1(\bar{\Omega} \times \mathbb{R}^N)$ and

$p > n$. By the Sobolev imbedding theorem each $v \in W^{1,p}(\Omega)$ is an equivalence class of functions containing a continuous and bounded function, which we still denote by v , and there exists a positive number $c_{1,p}$ independent of v such that

$$(2.2) \quad \|v\|_{0,\infty} \leq c_{1,p} \|v\|_{1,p} \quad \forall v \in W^{1,p}(\Omega),$$

where $\|\cdot\|_{0,\infty}$ is the norm of $L^\infty(\Omega)$. Then, if $\sigma \in (W^{1,p}(\Omega))^N$, the equivalence classes $F(\sigma)$, $F_{x_i}(\sigma)$ and $F_{y_j}(\sigma)$ can be identified with continuous and bounded functions. Let $\sigma = (\sigma_j)_{j=1, \dots, N} \in (W^{1,p}(\Omega))^N$ and let $(\sigma^k)_{k \in \mathbb{N}}$ be a sequence in $(C^\infty(\Omega) \cap W^{1,p}(\Omega))^N$ which converges to σ in $(W^{1,p}(\Omega))^N$, and therefore by (2.2) in $(L^\infty(\Omega))^N$. We have

$$(2.3) \quad D_i F(\sigma^k) = F_{x_i}(\sigma^k) + \sum_{j=1}^N F_{y_j}(\sigma^k) D_i \sigma_j^k.$$

Since $(\sigma^k)_{k \in \mathbb{N}}$ converges to σ in $(L^\infty(\Omega))^N$, then $(F(\sigma^k))_{k \in \mathbb{N}}$, $(F_{x_i}(\sigma^k))_{k \in \mathbb{N}}$ and $(F_{y_j}(\sigma^k))_{k \in \mathbb{N}}$ converge in $L^\infty(\Omega)$ respectively to $F(\sigma)$, $F_{x_i}(\sigma)$ and $F_{y_j}(\sigma)$, and therefore $\left(F_{x_i}(\sigma^k) + \sum_{j=1}^N F_{y_j}(\sigma^k) D_i \sigma_j^k\right)_{k \in \mathbb{N}}$ converges in $L^p(\Omega)$ to $F_{x_i}(\sigma) + \sum_{j=1}^N F_{y_j}(\sigma) D_i \sigma_j$. Consequently, by Hölder's inequality we have, for any $\varphi \in \mathcal{D}(\Omega)$,

$$(2.4) \quad \int_{\Omega} \left(F_{x_i}(\sigma) + \sum_{j=1}^N F_{y_j}(\sigma) D_i \sigma_j \right) \varphi \, dx + \int_{\Omega} F(\sigma) D_i \varphi \, dx = \\ = \lim_{k \rightarrow \infty} \left[\int_{\Omega} \left(F_{x_i}(\sigma^k) + \sum_{j=1}^N F_{y_j}(\sigma^k) D_i \sigma_j^k \right) \varphi \, dx + \int_{\Omega} F(\sigma^k) D_i \varphi \, dx \right].$$

Because of (2.3) we have for any $k \in \mathbb{N}$ and any $\varphi \in \mathcal{D}(\Omega)$

$$\int_{\Omega} \left(F_{x_i}(\sigma^k) + \sum_{j=1}^N F_{y_j}(\sigma^k) D_i \sigma_j^k \right) \varphi \, dx + \int_{\Omega} F(\sigma^k) D_i \varphi \, dx = 0$$

and therefore, by (2.4), we obtain

$$\int_{\Omega} \left(F_{x_i}(\sigma) + \sum_{j=1}^N F_{y_j}(\sigma) D_i \sigma_j \right) \varphi \, dx + \int_{\Omega} F(\sigma) D_i \varphi \, dx = 0 \quad \forall \varphi \in \mathcal{D}(\Omega),$$

which means

$$(2.5) \quad D_i F(\sigma) = F_{x_i}(\sigma) + \sum_{j=1}^N F_{y_j}(\sigma) D_i \sigma_j.$$

Since the equivalence classes $F(\sigma)$, $F_{x_i}(\sigma)$ and $F_{y_j}(\sigma)$ contain a continuous and bounded function, from (2.5) it follows that $F(\sigma) \in W^{1,p}(\Omega)$.

To prove that $F: (W^{1,p}(\Omega))^N \rightarrow W^{1,p}(\Omega)$ is continuous we need only remark that, if a sequence $(\sigma^k)_{k \in \mathbb{N}}$ converges to σ in $(W^{1,p}(\Omega))^N$, then, by (2.2), $(\sigma^k)_{k \in \mathbb{N}}$ converges to σ in $(L^\infty(\Omega))^N$, and therefore the sequences $(F(\sigma^k))_{k \in \mathbb{N}}$, $(F_{x_i}(\sigma^k))_{k \in \mathbb{N}}$ and $(F_{y_j}(\sigma^k))_{k \in \mathbb{N}}$ converge in $L^\infty(\Omega)$ respectively to $F(\sigma)$, $F_{x_i}(\sigma)$ and $F_{y_j}(\sigma)$: then $(D_i F(\sigma^k))_{k \in \mathbb{N}}$ converges to $D_i F(\sigma)$ in $L^p(\Omega)$ in view of (2.5), and thus $(F(\sigma^k))_{k \in \mathbb{N}}$ converges to $F(\sigma)$ in $W^{1,p}(\Omega)$.

As a next step, we suppose that the statement of the theorem is true for an $m \geq 1$ and we show that, consequently, it holds when m is replaced by $m+1$. In order to do this, we assume that $f \in C^{m+1}(\bar{\Omega} \times \mathbb{R}^N)$, that $(m+1)p > n$ and that $\sigma \in (W^{m+1,p}(\Omega))^N$, and we prove that $F(\sigma) \in W^{m+1,p}(\Omega)$ and that $\sigma \mapsto F(\sigma)$ is a continuous operator from $(W^{m+1,p}(\Omega))^N$ to $W^{m+1,p}(\Omega)$.

Let us recall that (by the Sobolev imbedding theorem) each $v \in W^{m+1,p}(\Omega)$ can be identified with a continuous function and there is a positive number $c_{m+1,p}$ independent of v such that $\|v\|_{0,\infty} \leq c_{m+1,p} \|v\|_{m+1,p} \forall v \in W^{m+1,p}(\Omega)$. Then, by arguments quite similar to the ones given in the case $m=1$, we can show that F is a continuous operator from $(W^{m+1,p}(\Omega))^N$ to $W^{1,p}(\Omega)$ and that (2.5) holds.

It is now convenient to distinguish the cases $p > n$, $p = n$ and $p < n$.

If $p > n$, from the (induction) assumption it follows that F_{x_i} and F_{y_j} are continuous operators of $(W^{m,p}(\Omega))^N$ into $W^{m,p}(\Omega)$; therefore F is a continuous operator of $(W^{m+1,p}(\Omega))^N$ into $W^{m+1,p}(\Omega)$, in view of (2.5), because $W^{m,p}(\Omega)$ is a Banach algebra.

Let now $p = n$, and let $q \in \mathbb{R}$ be such that $n < q$. Thus $mq > n \forall m \geq 1$ and (by the Sobolev imbedding theorem) $W^{m+1,n}(\Omega)$ can be continuously imbedded into $W^{m,q}(\Omega)$; furthermore, by the (induction) assumption, F_{x_i} and F_{y_j} are continuous operators of $(W^{m,q}(\Omega))^N$ into $W^{m,q}(\Omega)$.

Note that, since $mq > n$, from Theorem 1 it follows that the pointwise multiplication is a continuous operator of $W^{m,n}(\Omega) \times W^{m,q}(\Omega)$ into $W^{m,n}(\Omega)$. Then we can deduce by (2.5) that $\sigma \mapsto D_i F(\sigma)$ is a continuous operator of $(W^{m+1,n}(\Omega))^N$ into $W^{m,n}(\Omega)$. Consequently

$\sigma \mapsto F(\sigma)$ is a continuous operator of $(W^{m+1,n}(\Omega))^N$ into $W^{m+1,n}(\Omega)$.

Finally, let us consider the case $p < n$. In this case the condition $(m+1)p > n$ is equivalent to the condition

$$m \frac{np}{n-p} > n.$$

Now: F_{x_i} and F_{ν_j} are continuous operators of $(W^{m,np/(n-p)}(\Omega))^N$ into $W^{m,np/(n-p)}(\Omega)$ (because of the induction hypothesis), $W^{m+1,p}(\Omega)$ can be continuously imbedded into $W^{m,np/(n-p)}(\Omega)$ (by the Sobolev imbedding theorem), and the pointwise multiplication is a continuous operator of $W^{m,p}(\Omega) \times W^{m,np/(n-p)}(\Omega)$ into $W^{m,p}(\Omega)$ (by Theorem 1). This implies, by (2.5), that $\sigma \mapsto D_i F(\sigma)$ is a continuous operator of $(W^{m+1,p}(\Omega))^N$ into $W^{m,p}(\Omega)$. Therefore, also in this case $\sigma \mapsto F(\sigma)$ is a continuous operator of $(W^{m+1,p}(\Omega))^N$ into $W^{m+1,p}(\Omega)$. $\square$

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