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On Dedekind Domains in Infinite Algebraic Extensions.

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SUMMARY - The aim of this note is to prove the following result: Let A be a Dedekind domain with quotient field F, and K an algebraic extension of K. Let W be a set of discrete valuations of rank one on K, and $O = \{x: x \in K$ such that $w(x) \geqslant 0$ for all $w \in W\}$. Then O is a Dedekind domain whose quotient field is K, if and only if for every maximal ideal P of A the set $W(p) = \{w | w \in W$ such that w extend the valuation on F defined by $P\}$, is finite. In such a way we can give various examples of Dedekind domains in infinite algebraic extensions.

1. The theory of Dedekind domains was created as a generalisation of results concerning the rings of integers in finite extensions of the rational field, obtained mainly by Dedekind.

We shall say that a field K has a classical ideal theory relative to a Dedekind domain A, if A is a proper subring of K and K is its quotient field. It is clear that not every field has a classical ideal theory (for example the finite fields and algebraically closed fields).

The aim of this work is to make some remarks on the Dedekind domains in infinite algebraic extensions of a field F which has a classical ideal theory. Since the theorem which we shall prove has a general character, the main applications of this theorem concerne infinite algebraic extensions of rational numbers.

In [3] are defined so-called Stiemke fields. In a Stiemke field,

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there exist a classical ideal theory relative to a smallest Dedekind domain. On the other hand we can define infinite extensions of rational numbers which have a classical ideal theory relative to a smallest Dedekind domain but these fields are not Stiemke fields (Example 3). Moreover (Example 1) we show that there exist infinite extensions of rational numbers which do not contain a smallest Dedekind domain.

Another question is relative to residue class fields of a Dedekind domain. We show (Example 1) that in some infinite extensions of the field of rational number there exist Dedekind domains A with property that A/P is a finite field for every maximal ideal P of A. On the other hand (Example 2) we may define Dedekind domains A whose field of quotients is an infinite algebraic extension of rational numbers and such that A contains infinitely (or finitely) many maximal ideals P such that A/P is finite, and in the same time infinitely (or finitely) many maximal ideals P such that A/P is an infinite field.

2. Let A be a Dedekind domain and F its quotient field. Denote M the set of all maximal ideals of A. For every $P \in M$ denote v_P the valuation on F associated with P.

The set $V = \{v_P\}_{P \in M}$ is called the set of the valuations of F which define A.

Let K be an algebraic extension of F. Assume that for every maximal ideal P of A there exists a non-empty set $W(P)$ of non-equivalent (rank-one and discrete) valuations on K which extend the valuation v_P. Let $W = \bigcup_{P \in M} W(P)$. For every $v \in W$, denote O_v the valuation ring of v in K, and let P_v be the maximal ideal of O_v. Let us denote

$$O = \bigcap_{v \in W} O_v .$$

THEOREM. The hypotheses and notations are as above. The following assertions are equivalent:

a) The ring O is a Dedekind domain whose quotient field is K.

b) For every $P \in M$, the set $W(P)$ is finite.

PROOF. *a*) $\Rightarrow$ *b*). Let $P \in M$. There exists a non-zero element $a \in F$ such that $v_P(a) \neq 0$. Thus one has:

$$W(P) \subseteq \{v \in W,\ v(a) \neq 0\} .$$

The hypothesis O Dedekind domain shows that the set in the right of the above inclusion is finite, i.e. $W(F)$ is finite.

$b) \Rightarrow a)$. We shall show that conditions I-IV in [3], p. 95 are accomplished (see also [1]).

(I). Let $x \in K$ and $O_x = O \cap F(x)$. Let A_x be the integral closure of A in $F(x)$. It is clear that $A_x \subseteq O_x$. Since $F(x)$ is a finite extension of F, one sees that $F(x)$ is the quotient field of A_x hence of O_x. This means that $x = ab^{-1}$, where $a, b \in O_x$. Therefore the quotient field of O is K.

(II). Let x be a non-zero element of K. The inclusion $A_x \subseteq O_x$ shows that O_x is also a Dedekind domain. Hence the set of all $v \in W$ such that $v(x) \neq 0$ must be finite.

(III). Let $v_1, v_2 \in W$, $v_1 \neq v_2$. This means that there exist $x \in K$ such that $v_1(x) \neq v_2(x)$. Since $O_x = F(x) \cap O$ is a Dedekind domain, there exist $a \in O_x$ such that $v_1(a) > 0$ and $v_2(a) = 0$.

(IV). We must show that every prime non-zero ideal of O is maximal. Denote by D the set of all finite subsets of K. It is clear that $K = \bigcup_{S \in D} F(S)$, where $F(S)$ is the smallest subfield of K which contains F and S. For every $S \in D$, denote $O_S = O \cap F(S)$, and A_S the integral closure of A in $F(S)$. Since $A_S \subseteq O_S \subseteq F(S)$ and A_S is obviously a Dedekind domain it follows that O_S is also a Dedekind domain. One has

$$O = O \cap K = O \cap \Big(\bigcup_{S \in D} F(S')\Big) = \bigcup_{S \in D} (O \cap F(S)) = \bigcup_{S \in D} O_S .$$

Now let I be an ideal of O and $I_S = I \cap O_S$. Then I_S is an ideal of O_S and as above $I = \bigcup_{S \in D} I_S$. Furthermore, let P be a non-zero prime ideal of O, and I an ideal of O such that $P \subset I$ and $P \neq I$. Since P is non-zero, there exists an element $S_1 \in D$ so that $P_{S_1} \neq 0$. Also for any $S \in D$ one has $P_S \subset I_S$, and the condition $P \neq I$, implies that there exists $S_2 \in D$, such that $P_{S_2} \neq I_{S_2}$. Let $S = S_1 \cup S_2$. Since $0 \neq P_{S_1} \subset P_S$ one sees that P_S is a non-zero prime ideal of O_S. On the other hand, the condition $P_{S_2} \neq I_{S_2}$, means that $P_S \neq I_S$, hence $I_S = O_S$. Therefore the ideal I_S, hence also I, contains the identity element of O i.e. $I = O$. This proves that P is a maximal ideal of O.

As a consequence of the above Theorem one has the following result.

COROLLARY ([3], Ch. IV, Theorem 5). Let A be a Dedekind domain, F its quotient field and K an algebraic extension of K. Let V be the set of all valuations of F which define A and denote W the set of all prolongations of all elements of V to K. The following assertions are equivalent:

1) The ring $O = \bigcap_{w \in W} O_w$ is a Dedekind domains.

2) Every $v \in V$ has a finite number of prolongations to K, and every element of W is a discrete valuation.

A field K which verifies the equivalent condition of the Corollary is called a Stiemke field with respect to the pair (A, F) (see [3], pag. 110). Also, [3], pag. 111 gives an example of a Stiemke field with respect to the pair $(\mathbb{Z}, Q)$, where $\mathbb{Z}$ is the ring of integers and Q the field of rational numbers. It is easy to see that every Stiemke field with respect to a pair (A, F) can be constructed as in [3], Lemma 31, pag. 111.

On the other hand, using the ideas of [3] (Lemma 31, pag. 111) we can define various examples of Dedekind domains.

In what follows let $p_1, p_2, \ldots, p_n \ldots$ be the increasing sequence of prime numbers; denote v_n the valuation on Q defined by p_n.

EXAMPLE 1. We shall define an infinite algebraic extension K of Q such that the prolongation to K of any valuation v on Q, is a discrete valuation.

For every natural number m we may define an algebraic number field K_n such that:

i) $[K_{n+1} : K_n] = 2$, $K_n \subset K_{n+1}$ for all $n \geqslant 1$.

ii) For every $1 \leqslant i \leqslant n$, the valuation v_i has at least 2^{n-i+1} prolongations to K_m.

Suppose that K_n is already constructed, and let $W_1, \ldots, W_r$ be the set of all prolongations to K_m of the first $n+1$ valuations $v_1, \ldots, v_{n+1}$. Let B be the integral closure of $\mathbb{Z}$ in K_n. There exists for every j, $1 \leqslant j \leqslant r$, a monic polynomial $p_j(X) \in B[X]$ of degree two and such that the image of $p_j(X)$ in the residue class field associated with w_j, has two distinct roots. By the Chinese Remainder Theorem there exists then an irreducible polynomial $p(X)$ of degree two, $p(X) \in B[X]$ such that $p(X) \equiv p_j(X) \pmod{P_j}$, where P_j is the prime ideal of B associated with w_j, $1 \leqslant j \leqslant r$. Let K_{n+1} be generated by a root of $p(X)$.

Each valuation w_j determines in K_{n+1} precisely two prolongations.

We let $K = \bigcup_n K_n$. It is easy to see that every valuation of K is discrete and its residue class field is finite. Also it is plain that every valuation v_n of Q has infinitely many prolongations to K.

Let A be the integral closure of $\mathbb{Z}$ in K. According to the above theorem one sees that A is not a Dedekind domain. On the other hand according to the above theorem we can define infinitely many Dedekind domains A_n, $n \in N$, such that K is the quotient field of A_n for all n.

EXAMPLE 2. We shall define an infinite algebraic extension K of Q, such that there exist on K infinitely many discrete valuations, whose residue class field is infinite, and also infinitely many discrete valuations, whose residue class field is finite. In this way, according to the above theorem we can define examples of Dedekindian domains A such that A contains infinitely (or finitely) many maximal ideals P such that A/P is an infinite field, and also infinitely (or finitely) many maximal ideals P' such that A/P' is a finite field.

We wish to construct K as the join $\bigcup_{n\geqslant 1} K_n$ of finite extensions K_n/Q such that:

1) $K_n \subset K_{n+1}$ and $[K_{n+1}:K_n] = 2$.

2) For every $1 \leqslant k \leqslant n$, the valuation v_{2k} has at least $2(n-k+1)$ prolongations to K_n.

3) For every $1 \leqslant k \leqslant n$ the valuation v_{2k-1} has at the most $2k-1$ prolongations to K_n.

Let us assume that K_n has been defined such that above conditions are accomplished. We shall define K_{n+1}. By the Chinese Remainder Theorem, there exists a monic polynomial $f(X)$ in $B[X]$ (B being the ring of integers in K_n) of degree two such that for every $1 \leqslant k \leqslant n+1$, the image of $f(X)$ is irreducible in the residue class field of all prolongations of v_{2k-1} and that the image of $f(X)$ has two distinct roots in the residue class field of all prolongations of v_{2k}. Let us define $K_{n+1} = K_n(a)$ where a is a root of $f(X)$.

EXAMPLE 3. Now we shall define an infinite algebraic extension K of Q such that every valuation on Q has finitely many prolongations to K which are discrete (of course, generally a valuation on Q has another set, eventually infinite, of prolongations to K which are not discrete).

For any natural number n we shall define an extension K_n of Q such that

1) $K_n \subset K_{n+1}$ and $[K_{n+1}:K_n] = 2$.

2) The valuation v_n has a prolongation w_n to K_n such that w_n has only a prolongation to K_{n+t} for all $t \geqslant 1$ and this prolongation is unramified.

3) If $u_1, \ldots, u_s$ are other prolongations of v_n to K_n, then these valuations are totally ramified in K_{n+t}, for all $t \geqslant 1$.

Let us assume that K_n has been defined such that above conditions are accomplished. We shall define K_{n+1}. For that let w_i, be the extension of v_i $1 \leqslant i \leqslant n$, at K_n which satisfies the above condition 2) and let us choose w_{n+1} an prolongation of v_{n+1} to K_n. Furthermore let $u_1, \ldots, u_h$ be all the prolongations of $v_1, \ldots, v_{n+1}$ to K_n which are distinct of $w_1, \ldots, w_{n+1}$. Denote by B the ring of algebraic integers of K_n. According to the Chinese Remainder theorem we can define a monic polynomial, of degree two $f(X) \in B[X]$ such that the image of $f(X)$ over the residue field of w_i, $1 \leqslant i \leqslant n+1$ is irreducible and its image over the residue field of all u_i, $1 \leqslant i \leqslant h$ is the square of an irreducible polynomial. It is clear that $f(X)$ is irreducible over K_n and thus we define $K_{n+1} = K_n(a)$, where a is a root of $f(X)$. It is plain to show that above conditions 1)-3) are accomplished. Let $K = \bigcup\limits_{n \geqslant 1} K_n$. The field K is an infinite extension of Q and every valuation v_n of Q has only one prolongation $\overline{w}_n$ to K which is discrete. According to the above Theorem the ring $A = \bigcap\limits_{n \geqslant 1} Q_{\overline{w}_n}$ is the smallest Dedekind domain of K, and obviously K is not Stiemke field.

REFERENCES

[1] K. Matusita, *Über ein bewertungstheoretisches Axiomensystem für die Dedekind-Noethersche Idealtheorie*, Japan. J. Math., **19** (1944), pp. 97-110.

[2] W. Narkiewicz, *Elementary and analytic theory of algebraic numbers*, Warszawa, 1974.

[3] O. F. G. Schilling, *The theory of valuations*, A.M.S. Mathematical Surveys No. 4, 1950.

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