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**Group Graded Rings and Smash Products.**

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**Introduction.**

Let $R = \bigoplus_{\sigma \in G} R_{\sigma}$ be a $k$-algebra graded by a finite group $G$, with $R_e$ the component corresponding to the identity element $e$ of $G$ ($k$ is a commutative ring). In the paper [1] M. Cohen and S. Montgomery define the ring $R \neq k[G]^*$, called the «Smash product» associated to the graded ring $R$. This ring may be used to obtain many properties of the graded ring $R$. The main tools are provided by the two Duality Theorems: Duality Theorem for Action and Duality Theorem for Coaction (see Theorem 3.2 and 3.5 of [1]). In this paper we give a new characterization of the Smash Product $R \neq k[G]^*$ (Theorems 1.2 and 1.3) and we deduce directly from it and in a little more general form Cohen and Montgomery Duality Theorems (Theorems 2.2 and 2.3).

1. **The rings $\text{End}_{R,gr}(U)$, $\text{End}_R(U)$ and their structure.**

If $R = \bigoplus_{\sigma \in G} R_{\sigma}$ is a $k$-algebra graded by a finite group $G$, we denote by $R$-gr the category of all unital right graded $R$-modules. If $M = \bigoplus_{\sigma \in G} M_{\sigma}$, $N = \bigoplus_{\sigma \in G} N_{\sigma}$ are two objects of $R$-gr, then the morphisms in $R$-gr are $R$-homomorphisms $f: M \to N$ such that $f(M_{\sigma}) \subset f(N_{\sigma})$ for all $\sigma \in G$. It is well known that $R$-gr is a Grothendieck category (see [2]).

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If $M = \bigoplus_{\lambda \in \mathbb{G}} M_{\lambda}$ is a graded $R$-module and $\sigma \in G$, then $M(\sigma)$ is the graded module obtained from $M$ by putting $M(\sigma)_{\lambda} = M_{\lambda\sigma}$; the graded module $M(\sigma)$ is called the $\sigma$-suspension of $M$ [2]. It is well-known that the mapping $M \to M(\sigma)$ defines a functor from $R$-$\text{gr}$ to $R$-$\text{gr}$ which is an equivalence of categories for all $\sigma \in G$. $M \in R$-$\text{gr}$ is said to be $G$-invariant [2] if for all $\sigma \in G$, $M \simeq M(\sigma)$ in $R$-$\text{gr}$. Consider now the graded modules $M$ and $N$. A $R$-linear mapping $f: M \to N$ is said to be a graded morphism of degree $\tau$, $\tau \in G$, if $f(M_{\sigma}) \subset N_{\sigma\tau}$ for all $\sigma \in G$. Graded morphisms of degree $\tau$ build up an additive subgroup $\text{HOM}_R(M, N)_\tau$ of $\text{Hom}_R(M, N)$. It is clear that $\text{HOM}_R(M, N) = \bigoplus_{\tau \in \mathbb{G}} \text{HOM}_R(M, N)_\tau$ is a graded abelian group of type $G$ and $\text{HOM}_R(M, N)_\sigma = \text{Hom}_{R$-$\text{gr}}(M, N)$. In particular, if $M = N$, then $\text{HOM}_R(M, N) = \text{End}_R(M)$ is a graded ring of type $G$. In the sequel we will denote by $U = \bigoplus \mathbb{R}(\sigma)$. Since $\{\mathbb{R}(\sigma)\}_{\sigma \in \mathbb{G}}$ is a family of generators for $R$-$\text{gr}$ [2], it follows that $U$ is a generator for $R$-$\text{gr}$. When $G$ is a finite group, we also remark that $\text{End}_R(U) = \text{End}_R(U)$ (see [2]).

**Proposition 1.1.** [2] If $G = \{g_1 = e, g_2, \ldots, g_n\}$ is a finite group, then the ring $\text{End}_R(U)$ is isomorphic to the matrix ring $M_n(R)$ equipped with the grading

$$M_n(R) = \bigoplus_{\lambda \in \mathbb{G}} M_n(R)_\lambda,$$

where

$$M_n(R)_\lambda = \begin{pmatrix}
R_{g_1\lambda^{-1}} & R_{g_1g_2^{-1}} & \cdots & R_{g_{n-1}g_{n}^{-1}} \\
R_{g_2\lambda^{-1}} & R_{g_2g_2^{-1}} & \cdots & R_{g_{n-2}g_{n-1}^{-1}} \\
& & \ddots & \vdots \\
& & & R_{g_{n-1}\lambda^{-1}} & R_{g_{n-1}g_{n}^{-1}} & \cdots & R_{g_{n-1}g_{n}^{-1}}
\end{pmatrix}.$$  

In particular, the ring $\text{End}_{R$-$\text{gr}}(U)$ is isomorphic to the matrix ring

$$\begin{pmatrix}
R_e & R_{g_1^{-1}} & \cdots & R_{g_{n-1}^{-1}} \\
R_{g_1g_1^{-1}} & R_e & \cdots & R_{g_{n-1}^{-1}} \\
& & \ddots & \vdots \\
& & & R_e & R_{g_2^{-1}} & \cdots & R_{g_{n-1}^{-1}}
\end{pmatrix}.$$
PROOF. See Corollary 1.5.3 of [2].

By an \textit{action} of a group $G$ on a $k$-algebra $A$ we mean a group morphism $\alpha: G \to \text{Aut}_k(A)$; let $\alpha_g$ denote the image of $g \in G$ in $\text{Aut}_k(A)$. We may define the skew group ring (or \textit{trivial crossed product}) denoted by $A \ast G$, as being the free right and left $A$-module with basis $\{g; g \in G\}$ and with multiplication given by $(ag) \cdot (bh) = ax(b)gh$, where $a, b \in A$, $g, h \in G$. The ring $A \ast G$ is a graded ring: $A \ast G = \bigoplus_{\sigma \in G} (A \ast G)_\sigma$, where $(A \ast G)_\sigma = A\sigma = \{a\sigma; a \in A\}$.

**Theorem 1.2.** If $G = \{e, g_2, \ldots, g_n\}$ is a finite group, then the ring $\text{End}_R(U)$ is isomorphic to the skew group ring $\text{End}_{R, gr}(U) \ast G$.

**Proof.** By Proposition 1.1 we have that $\text{End}_R(U)$ is isomorphic to the matrix ring $M_n(R)$. We consider the set

$$U_\lambda = \begin{pmatrix} 0 & \cdots & 0 & R_{e} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \end{pmatrix}$$

where on the first row $R_{e}$ is on the $k_1$-th position, $k_1$ being such that $g_{k_1} = g_1 \lambda$; on the second row $R_{e}$ is on the $k_2$-th position, where $k_2$ is such that $g_{k_2} = g_2 \lambda$; \ldots; on the $n$-th row $R_{e}$ is on the $k_n$-th position, where $g_{k_n} = g_n \lambda$. Since $G$ is a group it is easy to see that $\{1, 2, \ldots, n\} = = \{k_1, k_2, \ldots, k_n\}$. Moreover one may see that $U_\lambda \subset M_n(R)\lambda$. Let $u_\lambda \in U_\lambda$, $u_\lambda = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \end{pmatrix}$.

From the definition of $U_\lambda$ it follows that $u_\lambda$ has a 1 on each column and all the other entries are 0. We will show now that the system $\{u_\lambda\}_{\lambda \in G}$ has the property that $u_\lambda u_\mu = u_{\lambda \mu}$, for all $\lambda, \mu \in G$. Let

$$u_\mu = \begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \\ 0 & \cdots & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 1 & 0 \\ \end{pmatrix}$$
where 1 in the first row is on the \( \lambda_i \)-th column, where \( g_{i_1} = g_{1_1} \mu \); 1 in the second row is on the \( \lambda_2 \)-th column, where \( g_{i_2} = g_{2_2} \mu \); \ldots ; 1 in the \( n \)-th row is on the \( \lambda_n \)-th column, where \( g_{i_n} = g_{n_n} \mu \).

There exists a unique column, say the \( s \)-th, such as its intersection with the \( \lambda_i \)-th row has a 1 and the rest of its entries are zero. Thus we have that \( g_s = g_{s_1} \mu \). Since \( g_{s_1} = g_{s_1} \lambda \), then \( g_s = g_{s_1} \lambda \mu \) and so in the matrix \( u_1 u_\mu \) we have 1 on the first row in the \( s \)-th position, all the other entries of the first row being zero. Hence the first row of the matrix \( u_1 u_\mu \) is the same as the first row of the matrix \( u_{\lambda_1} \).

Using the same argument for the other rows, we deduce that \( u_{\lambda_1} u_\mu = = u_{\lambda_1} \). In particular, since \( u_\lambda \) is equal to the unit matrix, we obtain that \( u_{\lambda_1}^{-1} = u_{\lambda_1}^{-1} \), for each \( \lambda \in G \). Using now Theorem 5.3.23 of [2], we obtain that \( \text{End}_R(U) \) is isomorphic to the skew group ring \( \text{End}_{R \# G}(U) \). Q.E.D.

Let now \( A = \bigoplus_{g \in G} A_g \) be a graded \( k \)-algebra, where \( k \) is a commutative ring and \( G \) is a finite group. By [1], the construction of the smash product \( A \neq k[G]^* \) is the following: \( A \neq k[G]^* \) is the free left and right \( A \)-module with basis \( \{ p_g : g \in G \} \), a set of orthogonal idempotents whose sum is 1 and with the multiplication given by the rule: \( \cdot (bp_h) = abp_{gh}, \ a, b \in A. \)

**Theorem 1.3.** Let \( R = \bigoplus_{g \in G} R_g \) be a graded \( k \)-algebra, where \( G = \{ e, g_1, g_2, \ldots, g_n \} \) is a finite group. Then the ring \( \text{End}_{R \# G}(U) \) is isomorphic to the smash product

\[ \text{End}_{R \# G}(U) \cong k[G]^*. \]

**Proof.** We have seen that the ring \( \text{End}_{R \# G}(U) \) is isomorphic to the matrix ring

\[ T = \begin{pmatrix} R_e & R_{g_1 g_2}^{-1} \cdots R_{g_1 g_n}^{-1} \\ R_{g_2 g_1}^{-1} & R_e & \cdots & R_{g_2 g_n}^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ R_{g_n g_1}^{-1} & R_{g_n g_2}^{-1} \cdots & R_e \end{pmatrix}. \]

Define \( \varphi : R \to T \) as follows: if \( a = \sum_{g \in G} a_g, a_g \in R_g \) for all \( g \in G \), then
It is clear that \( \varphi \) is additive and injective. It is straightforward to check that if \( b \in R \), then we have \( \varphi(ab) = \varphi(a)\varphi(b) \). Consequently, \( \varphi \) is a ring morphism. Let \( S = \varphi(R) \). Hence \( S \) is a subring of \( T \). Consider the elements:

\[
\varphi(a) = \begin{pmatrix}
    a_0 & a_0 a_1^{-1} & \cdots & a_0 a_n^{-1} \\
    a_0 a_1^{-1} & a_0 & \cdots & a_0 a_n^{-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_0 a_1^{-1} & a_0 a_2^{-1} & \cdots & a_0 a_n^{-1}
\end{pmatrix}
\]

i.e. \( p_{s_k} \) is a matrix with 1 at the intersection of the \( k \)-th row with the \( k \)-th column, all its other entries being zero. Then the system of elements \( \{p_{s_k}\}_{1 \leq k \leq n} \) is a system of orthogonal idempotents whose sum is 1. It is clear that

\[
T = Sp_{s_1} + Sp_{s_2} + \ldots + Sp_{s_n} = p_{s_1}S + p_{s_2}S + \ldots + p_{s_n}S
\]

Let us prove that \( \{p_{s_k}\}, 1 \leq k \leq n \), is a linear independent system over the ring \( S \). Let \( \sum_{k=1}^{n} s_k p_{s_k} = 0 \), where \( s_k \in S \). Hence

\[
s_k = \begin{pmatrix}
    a_0^k & a_0^k a_1^{-1} & \cdots & a_0^k a_n^{-1} \\
    a_0^k a_1^{-1} & a_0^k & \cdots & a_0^k a_n^{-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_0^k a_1^{-1} & a_0^k a_2^{-1} & \cdots & a_0^k a_n^{-1}
\end{pmatrix}
\]
Then

\[ \sum_{k=1}^{n} s_{k} p_{x_{k}} = \begin{pmatrix} a_{1}^{1} & 0 & \cdots & 0 \\ a_{2g_{1}^{-1}}^{1} & 0 & \cdots & 0 \\ 0 & a_{2g_{2}^{-1}}^{2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1g_{1}^{-1}}^{2} & 0 & \cdots & 0 \\ a_{2g_{n}^{-1}}^{2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1g_{n}^{-1}}^{2} & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ a_{2g_{2}^{-1}}^{2} \\ \vdots \\ a_{2g_{n}^{-1}}^{2} \\ 0 & a_{2g_{2}^{-1}}^{2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{2g_{n}^{-1}}^{2} & 0 & \cdots & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & a_{2g_{n}^{-1}}^{n} \\ \vdots & \vdots & 0 & \cdots & 0 \\ 0 & a_{2g_{n}^{-1}}^{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{2g_{n}^{-1}}^{n} \end{pmatrix} \]

and hence \( a_{k}^{k} = a_{g_{1}g_{k}^{-1}}^{k} = \ldots = a_{g_{n}g_{k}^{-1}}^{k} = 0 \), for all \( k, 1 \leq k \leq n \). Thus \( s_{1} = s_{2} = \ldots = s_{n} = 0 \). The ring \( S \) is a graded ring of type \( G \) with the grading \( \{ S_{g} : g \in G \} \), where

\[ S_{g} = \begin{pmatrix} 0 & \cdots & R_{g} & 0 & \cdots & 0 \\ 0 & R_{g} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \cdots & R_{g} \end{pmatrix} \]

and \( R_{g} \) is on the first row on the \( k_{1} \)-th position, \( k_{1} \) being such that \( g = g_{1}g_{k_{1}^{-1}} \), on the second row on the \( k_{2} \)-th position such that \( g = g_{2}g_{k_{2}^{-1}} \), \ldots, on the \( n \)-th row on the \( k_{n} \)-th position such that \( g = g_{n}g_{k_{n}^{-1}} \). Thus it is clear that \( \varphi : R \to S \) becomes an isomorphism of graded rings. To finish the proof we need to show that:

\[ (s_{p_{g}})(t_{p_{h}}) = st_{p_{g}^{-1}p_{h}}, \text{ for all } s, t \in S. \]

To see this let \( g = g_{m}, h = g_{e} \) and

\[ t = \begin{pmatrix} b_{e} & b_{g_{1}g_{2}^{-1}} & \cdots & b_{g_{1}g_{n}^{-1}} \\ b_{g_{2}g_{2}^{-1}} & b_{e} & \cdots & b_{g_{2}g_{n}^{-1}} \\ \vdots & \vdots & \ddots & \vdots \\ b_{g_{n}g_{n}^{-1}} & b_{g_{e}} & \cdots & b_{e} \end{pmatrix}. \]
We have that
\[ p_{m} \cdot t = \begin{pmatrix}
0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & \cdots & 0 \\
b_{m,g_{m}^{-1}} & b_{m,g_{m}^{-1}} & \cdots & b_{m,g_{n}^{-1}} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & \cdots & 0
\end{pmatrix} \equiv \text{m-th row}
\]
and thus
\[ p_{m}(tp_{s}) = \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix} \equiv \text{m-th row}.
\]

On the other hand, since \( t_{m,g_{i}^{-1}} \) is a matrix which has on the first row a single non zero element, \( b_{m,g_{i}^{-1}} \), on the \( k_{1} \)-th position, for which \( g_{m}g_{i}^{-1} = g_{1}g_{k_{1}} \), on the second row a single non zero element, \( b_{m,g_{i}^{-1}} \), on the \( k_{2} \)-th position, where \( g_{m}g_{i}^{-1} = g_{2}g_{k_{2}} \), \( \ldots \), on the \( n \)-th row a single non zero element, \( b_{m,g_{i}^{-1}} \), on the \( k_{n} \)-th position such that \( g_{m}g_{i}^{-1} = g_{n}g_{k_{n}} \), it is clear that \( k_{m} = 1 \) and \( k_{s} \neq 1 \) for \( s \neq m, 1 \leq s \leq n \). We deduce that
\[ t_{m,g_{i}^{-1}} \cdot p_{s} = \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix} \equiv \text{m-th row}
\]
and thus \( p_{s}(tp_{h}) = t_{d_{h^{-1}}p_{h}} \), so clearly \((sp_{s})(tp_{h}) = st_{d_{h^{-1}}p_{h}}\), for all \( s, t \in S \) and \( g, h \in G \).

2. Cohen-Montgomery duality theorems.

The notation in this section will be the same as the one in section 1.
THEOREM 2.1. Let \( R = \bigoplus_{\sigma \in G} R_{\sigma} \) be a graded ring of type \( G \), where \( G \) is a finite group. If we denote \( U = \bigoplus_{\sigma \in G} R(\sigma) \), then the functor \( M \mapsto \text{Hom}_{R-gr}(U, M) \) is an equivalence from the category \( R-gr \) to the category \( \text{End}_{R-gr}(U) \)-mod.

PROOF. Since \( \{R(\sigma)\}_{\sigma \in G} \) is a set of generators for \( R-gr \) [2], then \( U \) is a generator for \( R-gr \). On the other hand, \( U \) is a finitely generated projective \( R \)-module. Hence \( U \) is a small projective generator for \( R-gr \) and therefore, after a classical result of B. Mitchel (see [3]), the functor \( M \mapsto \text{Hom}_{R-gr}(U, M) \) is an equivalence between the categories \( R-gr \) and \( \text{End}_{R-gr}(U) \)-mod. Q.E.D.

REMARKS:

1) Bearing in mind Theorem 1.2, Theorem 2.1 is nothing else that Theorem 2.2 of [1].

2) If \( M = \bigoplus_{\sigma \in G} M_{\sigma} \), then \( \text{Hom}_{R-gr}(U, M) \simeq \bigoplus_{\sigma \in G} \text{Hom}_{R-gr}(R(\sigma), M) \).

Let \( R = \bigoplus_{\sigma \in G} R_{\sigma} \) be a graded ring. \( R \) is called a crossed product if for every \( \sigma \in G \) there exists a homogeneous invertible element \( u_{\sigma} \in R_{\sigma} \). The structure of crossed products is given in Theorem I.3.23 of [2].

THEOREM 2.2. Let \( R = \bigoplus_{\sigma \in G} R_{\sigma} \) be a graded k-algebra which is a crossed product, where \( G \) is a finite group with \( n = |G| \). Then

\[
R \# k[G]^* \simeq M_n(R_{e}).
\]

PROOF. Since \( R \) is a crossed product, then \( R \simeq R(\sigma) \) in the category \( R-gr \) for any \( \sigma \in G \). Therefore \( U = \bigoplus_{\sigma \in G} R(\sigma) \simeq R^{(e)} \).

LEMMA. Let \( C \) be an abelian category and \( M \) an object of \( C \). Then, for \( n > 0 \)

\[
\text{End}_C(M^{(e)}) \simeq M_n(\text{End}_C(M)).
\]

PROOF. Straightforward (see [3]).

We may use the Lemma to obtain that \( \text{End}_{R-gr}(U) \simeq M_n(\text{End}_{R-gr}(R)) \).

But \( \text{End}_{R-gr}(U) \simeq R_{e} \) and therefore \( \text{End}_{R-gr}(U) \simeq M_n(R_{e}) \). We apply
now Theorem 1.3 and obtain that

\[ R \neq k[G]^* \simeq M_n(R_e). \quad \text{Q.E.D.} \]

**REMARK.** This result is a slight extension of Theorem 3.2 (Duality Theorem for Action) of Cohen and Montgomery [1], which is given in the case \( R = S \star G \) is a skew group ring.

**THEOREM 2.3.** (Duality for Coactions) (see [1]). Let \( \text{End}_k(R_e) \) be a graded \( k \)-algebra, where \( G \) is a finite group with \( n = |G| \). Then

\[ (R \neq k[G]^*) \star G \simeq M_n(R). \]

**PROOF.** By Theorem 1.2 and Theorem 1.3 we have that \( \text{End}_R(U) \simeq (R \neq k[G]^*) \star G \). Since \( U = \bigoplus_{\sigma \in G} R(\sigma) \), then in the category \( R \text{-mod} \) it is \( U \simeq R^{(n)} \). Therefore

\[ \text{End}_R(U) \simeq \text{End}_R(R^{(n)}) \simeq M_n(R). \quad \text{Q.E.D.} \]

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