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DĂNUŢ MARCU

Note on the regular digraphs

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Note on the Regular Digraphs.

DĂNUŢ MARCU (*)

ABSTRACT - In this paper, we give two sufficient conditions, in which, the conjecture of [1] is true.

Introduction.

Our graph-theoretic terminology is standard and is based essentially on the book of Harary [3]. For a vertex v of a digraph D, we denote by $id(v)$ and $od(v)$ the indegree and outdegree, respectively, of v. If $id(v) = od(v) = r$, then we speak of the *degree* of v. If every vertex of D has degree r, then D is said to be *regular of degree r* or simply *r-regular*. The *girth* of a digraph D containing directed cycles (circuits) is the length of the smallest cycle in D. For $n \geqslant 2$ and $r \geqslant 1$, $g(r, n)$ is the minimum number of vertices in an r-regular digraph having girth n. From [1], we have

$$g(r, n) \leqslant r(n-1) + 1 \,, \tag{1}$$

and the following

CONJECTURE. *For all* $r \geqslant 1$, $n \geqslant 2$, $g(r, n) = r(n-1) + 1$.

In this paper, we give two sufficient conditions, in which, this conjecture is true.

The main results. Throughout, for any real number x, we use $\lfloor x \rfloor$ to denote the integer less than or equal to x, and all the digraphs

(*) Indirizzo dell'A.: Faculty of Mathematics, University of Bucharest, Academiei 14, 70109 Bucharest, Romania.

considered are finite, without loops or multiple edges. If $D = (V, E)$ is a digraph with V as the vertex set and E as the edge set, then, for every $W \subseteq V$, we denote:

$$V^+(W) = \{v \in V : (w, v) \in E, w \in W\}$$

and

$$V^-(W) = \{v \in V : (v, w) \in E, w \in W\}.$$

THEOREM 1. *Let $D = (V, E)$ be an r-regular digraph of girth n with $r \geqslant 1$ and $n \geqslant 2$. If there exist a vertex $v \in V$ and the sets V_i, $W_i \subseteq V$, $i = 1, 2, \ldots, \lfloor n/2 \rfloor$, such that*

(a) $V_1 = V^-(\{v\})$, $W_1 = V^+(\{v\})$,

(b) $|V_i| = |W_i| = r$, $i = 1, 2, \ldots, \lfloor n/2 \rfloor$,

(c) $V_i \subseteq \left[V^-\left(\bigcup_{j<i} V_j\right) - \bigcup_{j<i} V_j\right]$, $i = 2, 3, \ldots, \lfloor n/2 \rfloor$,

(d) $W_i \subseteq \left[V^+\left(\bigcup_{j<i} W_j\right) - \bigcup_{j<i} W_j\right]$, $i = 2, 3, \ldots, \lfloor n/2 \rfloor$,

then $g(r, n) = r(n-1) + 1$.

PROOF. Obviously, from (c) and (d), we have

$$V_i \cap V_j = \emptyset, \quad i, j = 1, 2, \ldots, \lfloor n/2 \rfloor, \tag{2}$$

$$W_i \cap W_j = \emptyset, \quad i, j = 1, 2, \ldots, \lfloor n/2 \rfloor. \tag{3}$$

Now, we prove, by induction on i, that for every $1 \leqslant i \leqslant \lfloor n/2 \rfloor$ and $x \in V_i$, there exists a directed path from x to v of length $\leqslant i$. Obviously, by (a), that is true for $i = 1$, and consider it to be true for $j \leqslant i - 1$. Since $V_i \subseteq V^-\left(\bigcup_{j<i} V_j\right)$, there exists $z \in \bigcup_{j<i} V_j$, such that $(x, z) \in E$. By induction hypothesis, the path of length $\leqslant i - 1$, from z to v, joined with (x, z), is a path from x to v of length $\leqslant i$. Similarly, for every $1 \leqslant i \leqslant \lfloor n/2 \rfloor$ and $y \in W_i$, there exists a path from v to y of length $\leqslant i$.

Suppose that, for some $1 \leqslant i \leqslant \lfloor n/2 \rfloor$, $v \in V_i$. Thus, there exists a cycle (from v to v) of length $\leqslant i \leqslant \lfloor n/2 \rfloor < n$, contradicting the definition of n. Hence,

$$v \notin V_i, \quad i = 1, 2, \ldots, \lfloor n/2 \rfloor. \tag{4}$$

Similarly, we have

$$v \notin W_i\,, \qquad i = 1, 2, \ldots, \lfloor n/2 \rfloor\,. \tag{5}$$

Suppose that, for some $1 \leqslant i \leqslant \lfloor n/2 \rfloor$ and $1 \leqslant j \leqslant \lfloor (n-1)/2 \rfloor$, $V_i \cap W_j \neq \emptyset$. Let then $t \in V_i \cap W_j$. Thus the union of the path from t to v (of lenght $\leqslant i$) with the path from v to t (of length $\leqslant j$) is a cycle of length $\leqslant i + j \leqslant \lfloor n/2 \rfloor + \lfloor (n-1)/2 \rfloor = n - 1 < n$, contradicting the definition of n. Hence,

$$V_i \cap W_j = \emptyset, \quad i = 1, 2, \ldots, \lfloor n/2 \rfloor, \quad j = 1, 2, \ldots, \lfloor (n-1)/2 \rfloor\,. \tag{6}$$

Thus, from (b) and (2)-(6), we have $g(r, n) \geqslant r\lfloor n/2 \rfloor + r\lfloor (n-1)/2 \rfloor = r[\lfloor n/2 \rfloor + \lfloor (n-1)/2 \rfloor] = r(n-1)$, and, by (1), the theorem is proved (Q.E.D.).

Theorem 2. *Let $D = (V, E)$ be an r-regular digraph of girth n with $r \geqslant 1$ an $n \geqslant 2$. If $r(\lfloor n/2 \rfloor - 1) < n$, then $g(r, n) = r(n-1) + 1$.*

Proof. Let $v \in V$ arbitrary chosen, and $V_1 = V^-(\{v\})$, $W_1 = V^+(\{v\})$. Suppose that, for every $j < i \leqslant \lfloor n/2 \rfloor$, we have constructed the sets V_i, W_i satisfying (b)-(d).

Thus, because V_i's are pairwise disjoint and of cardinality r, we have

$$\Big|\bigcup_{j<i} V_j\Big| = \sum_{j=1}^{i-1} |V_j| = r(i-1)\,, \text{ i.e., } \Big|\bigcup_{j<i} V_j\Big| \leqslant r(\lfloor n/2 \rfloor - 1) < n\,.$$

Hence, the subdigraph of D, induced by $\bigcup_{j<i} V_j$ (the subdigraph of D whose vertex set is $\bigcup_{j<i} V_j$ and whose edge set is the set of those edges of D that have both ends in $\bigcup_{j<i} V_j$), does not contain cycles. But, according to [3], if a digraph has no cycles and which fails to consist only of isolated vertices, then it contains a transmiter (a vertex with positive outdegree and zero indegree) and a receiver (a vertex with positive indegree and zero outdegree). Hence, there exists $x \in \bigcup_{j<i} V_j$, such that $V^-(\{x\}) \cap \Big(\bigcup_{j<i} V_j\Big) = \emptyset$. Obviously, $V_i := V^-(\{x\})$ satisfies (b) and (c). Similarly, we can construct W_i. In this way, we come in the hypothesis of the theorem 1, and, therefore, we have $g(r, n) = r(n-1) + 1$ (Q.E.D.).

Corollary 1. [1]. *For the following values of r and n, the conjecture is true:*

$$r = 1 \text{ and } n \text{ arbitrary } (n \geqslant 2),$$

$$r \text{ arbitrary } (r \geqslant 1) \text{ and } n = 2,\ 3.$$

Proof. It follows from the theorem 2 (Q.E.D.).

Corollary 2. [2]. *The conjecture is true for $r = 2$ and arbitrary $n (n \geqslant 2)$.*

Proof. It follows from the theorem 2 (Q.E.D.).

Corollary 3. *The conjecture is true for the following values of r and n:*

$$r = 3, \quad n = 4, 5, 7 .$$

$$r = 4, \quad n = 5 .$$

Proof. It follows from the theorem 2 (Q.E.D.).

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