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Rings $\mathcal{S}$-Radical Over PI-Subrings.

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1. A ring $R$ is said to be radical over a subring $A$ if, for every $x \in R$, there exists an integer $n(x) \geq 1$ such that $x^{n(x)} \in A$. One of the results concerning the structure of radical extensions is a result due to Herstein and Rowen. In [5] they proved: if $R$ is a ring with no nil right ideals, radical over a subring $A$ and $A$ satisfies a polynomial identity, then $R$ satisfies the same multilinear identities. In [6] Zelmanov showed that the conclusion still holds if we merely assume that $R$ is without nil ideals.

In this paper we shall be concerned with the same problem of lifting polynomial identities in the setting of rings with involution. If $R$ is a ring with involution and $S$ the set of symmetric elements of $R$, we say that $R$ is $S$-radical over a subring $A$ if, given $s \in S$, then $s^{n(s)} \in A$ for some integer $n(s) \geq 1$.

$S$-radical extensions were studied in [1] where it was shown that if $R$ is a division ring $S$-radical over a proper subring $A$ then, for all $x \in R$, $xx^*$ is central in $R$ and so, $R$ is at most 4-dimensional over its center.

Here we shall prove the following: let $R$ be a prime ring with no nil right ideals and char $R \neq 2, 3$. If $R$ is $S$-radical over a subring $A$ and $A$ satisfies a polynomial identity of degree $d$, then $R$ satisfies a polynomial identity (PI) and PI-deg ($R$) $< d$.

We remark that if every element in $S$ is nilpotent then $R$ contains

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a nonzero nil right ideal; however it is not known if $R$ contains a nonzero nil ideal (this is tied in with a conjecture due to McCrimmon [4]).

Throughout this paper $R$ will denote a ring with involution $\ast$, $Z$ its center and $S = \{x \in R : x = x^\ast\}$ and $K = \{x \in R : x = -x^\ast\}$, the set of symmetric and skew elements respectively. Finally $N = \{x^{**} : x \in R\}$ will denote the set of norms of $R$.

If $R$ is a prime ring satisfying a polynomial identity, then its ring of central quotients, $Q$, is a central simple algebra of dimension $n^2$ over its center and we define $\text{PI-deg}(R) = n$.

2. We first prove a result of independent interest which will be very useful in proving the main theorem, namely:

**Theorem 1.** Let $R$ be a ring with no nonzero nil right ideals. If $R$ is $S$-radical over a division ring $A$, $A \neq R$ then either

1) $R$ is a direct sum of a division ring and its opposite with the exchange involution or

2) $R$ is simple, $N \subseteq Z$ and $\dim_Z R \leq 4$.

**Proof.** Since $R$ is also $S$-radical over $A \cap A^\ast$, we may assume $A = A^\ast$. Let $U = U^\ast$ be a proper $\ast$-ideal of $R$. Since $U$ is proper and $A$ is a division ring, $U \cap A = 0$. Thus $U \cap S$ consists of nilpotent elements. Let $s \in U \cap S$ be such that $s^2 = 0$. If $r \in R$, $sr + r^\ast s \in U \cap S$, so, for a suitable $n$, $0 = (sr + r^\ast s)^n = (sr)^n + (r^\ast s)^n + \text{sys}$ for some $y \in R$. Hence $(sr)^n s = 0$. This shows that $sR$ is nil and so, $sR = 0$ consequently $s = 0$. Therefore we get $U \cap S = 0$. Let $x \in U$, then $x + x^\ast = 0$ implies $x = -x^\ast \in K$ and so $x^2 \in U \cap S = 0$. Thus every element in $U$ is nilpotent of index 2. It follows that $U = 0$.

We have proved that $R$ is $\ast$-simple. Since $J(R)$, the Jacobson radical of $R$, is a $\ast$-ideal and $J(A) = 0$, we immediately get $J(R) = 0$ that is $R$ is semisimple. Now each $s \in S$ is either nilpotent or invertible so by ([4], Theorem 2. 3. 4) $R$ is one of the following types:

(i) a division ring,

(ii) a direct sum of a division ring and its opposite with the exchange involution,

(iii) the $2 \times 2$ matrices over a field $F$, or

(iv) a commutative ring with trivial involution.
If the first case occurs, by the result of Chacron and Herstein [1] we are done. In case (ii) or (iii) we are obviously done. In case (iv) \( R \) is radical over a division ring and so by ([2], Theorem 1.1) \( R \) is a field. This completes the proof of the theorem.

We now state our main theorem.

**Theorem 2.** Let \( R \) be a prime ring with involution of characteristic \( \neq 2, 3 \) which is \( S \)-radical over a subring \( A \). If \( R \) has no nonzero nil right ideals and \( A \) satisfies a polynomial identity of degree \( d \), then \( R \) satisfies a polynomial identity and PI-deg \( (R) < d \).

The proof of theorem 2 requires several lemmas; we first make a few preliminary remarks and then state and prove the required lemmas.

In what follows \( A \subset R \) will be rings satisfying the hypotheses of the theorem and \( f(X_1, \ldots, X_d) \) will be a multilinear polynomial identity of degree \( d \) satisfied by \( A \). Moreover we assume, as we may, that \( A = A^* \).

We remark that, by a theorem of Giambruno [3], either \( S \subset Z(R) \) or \( Z(A) \subset Z(R) \). In the former case \( R \) satisfies the standard identity of degree 4 and there is nothing to show. Hence, we shall always assume that \( Z(A) \subset Z(R) \). In particular since \( R \) is prime, every nonzero element in \( Z(A) \) is regular in \( R \).

We begin with

**Lemma 1.** If \( A \) is a domain then \( R \) is PI.

**Proof.** By ([4], Theorem 1.4.2) we have that \( Z(A) \neq 0 \). If we localize \( A \) and \( R \) at \( Z(A) \) we get rings with induced involution \( A_1, R_1 \) respectively. Then \( R_1 \) has no non-zero nil right ideals and is \( S \)-radical over \( A_1 \). Moreover, since \( A \) is a domain, by ([4], Theorem 1.3.4), \( A_1 \) is a division algebra. From theorem 1 we get that either \( A_1 = R_1 \) or \( S = S(R_1) \subset Z(R_1) \). In any case \( R_1 \), and so \( R \), is PI.

**Lemma 2.** If \( R \) is PI then PI-deg \( (R) < d \).

**Proof.** By ([4], Theorem 1.4.2), \( Z(R) \neq 0 \). Hence, since \( Z(R) \) is \( S \)-radical over \( Z(A) \), \( Z(A) \neq 0 \). If we localize \( R \) at \( Z(R) \) and \( A \) at \( Z(A) \subset Z(R) \), we get rings \( R_1, A_1 \) respectively. Then, by ([4], Theorem 1.4.3), \( R_1 \) is a finite dimensional central simple algebra with induced involution which is \( S \)-radical over \( A_1 \). Moreover, \( A_1 \) satisfies the polynomial identity \( f(X_1, \ldots, X_d) \). Thus, in order to complete the proof of the lemma, we may assume that \( R \) is a finite dimensional central simple algebra. Therefore, \( R = D_n \), the ring of \( n \times n \) matrices over a division ring \( D \), and the involution \( * \) is either symplectic or of transpose type.
Suppose first that \( * \) is symplectic. Then \( D \) is a field, moreover, since \( S \notin Z(R) \), \( n > 2 \). Let \( e_{ij} \) be the usual matrix units in \( R \). For \( \alpha \in D \) and \( i > 1 \) odd, the elements

\[
e_{11} + e_{22},
\]

\[
e_{11} + e_{22} + \alpha(e_{1i} + e_{i+1,2}),
\]

and

\[
e_{11} + e_{22} + \alpha(e_{1i} + e_{2,i+1})
\]

lie in \( A \) since they are symmetric idempotents. Hence \( \alpha(e_{1i} + e_{i+1,2}), \alpha(e_{1i} + e_{2,i+1}) \in A \) and multiplying these elements first from the left and then from the right by \( e_{11} + e_{22} \) we conclude that

\[
(1) \quad De_{1i} + De_{i+1,2} + De_{1i} + De_{2,i+1} \subseteq A \quad (i > 1 \text{ odd}).
\]

Similarly, since for \( i > 2 \) even the elements

\[
e_{11} + e_{22} + \alpha(e_{1i} - e_{i-1,2})
\]

\[
e_{11} + e_{22} + \alpha(e_{1i} - e_{2,i-1})
\]

are symmetric idempotents, we obtain

\[
(2) \quad De_{1i} + De_{i-1,2} + De_{11} + De_{2,i-1} \subseteq A, \quad (1 > 2 \text{ even}).
\]

From (1) and (2) since \( e_{11} + e_{22} \in A \), it follows that \( De_{ij} \subseteq A \) for all \( i, j \). Thus \( A = R \) and we are done.

Suppose now that \( * \) is of transpose type, that is, there exists an invertible diagonal matrix \( C = \text{diag} \{ c_1, \ldots, c_n \} \in D_n \) with \( c_i = c_i^* \in D \) such that \( (x_{ij})^* = C(x_{ij}^*)C^{-1} \) for all \( (x_{ij}) \in D_n \). In this case \( e_{ii} (i = 1, \ldots, n) \) is a symmetric idempotent and so lies in \( A \).

We claim that for every \( e_{ii} \) there exists \( 0 \neq \alpha = \alpha_{ii} \in Z \), the center of \( D \), such that \( \alpha \cdot e_{ii} \in A \). Since \( A \) is a subring and \( e_{ii} \in A \) \( (i = 1, \ldots, n) \), it is enough to show that this holds for \( e_{i,i+1} \) and \( e_{i+1,i} \) \( (i = 1, \ldots, n-1) \).

Moreover, since \( * \) restricted to the diagonal \( 2 \times 2 \) block \( De_{ii} + De_{i,i+1} + De_{i+1,i} + De_{i,i+1} \) is still an involution of transpose type, in order to prove the claim, we may assume that \( R = D_n \).

Now, since \( D \) is \( S \)-radical over \( A \cap D \), it follows by [1] that either \( S(D) \subseteq Z \) or \( D \subseteq A \). Moreover, by [3], since \( e_{11} \notin Z \), there exists \( s \in S \)
such that, for some $k$, $e_{11} s^k \neq s^k e_{11}$ and $s^k \in A$. In particular $s^k = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is not a diagonal matrix, say $b \neq 0$.

If $S(D) \subseteq Z$, then $C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \in Z_2$ and $*$ induces an involution on $Z_2$.

Thus, in this case we may assume that $s \in Z_2$. Hence $e_{11} s^k e_{22} = b e_{12} \in A$ and $(b e_{12})^* = b' e_{21} \in A$ with $b, b' \in Z$.

On the other hand, if $D \subseteq A$, $b e_{12} = e_{11} s^k e_{22} \in A$ and $e_{12} \in A$. Hence $c_2 c_1^{-1} e_{21} = e_{12}^* \in A$ and $e_{21}$ lies also in $A$. Thus the claim is established; in other words, there exist $0 \neq \alpha_{ij} \in Z$ such that $\alpha_{ij} e_{ij} \in A(i, j = 1, \ldots, n)$.

Now, if $D \subseteq A$, then clearly $D_n = A$ and there is nothing to prove. Therefore we may assume that $S(D) \subseteq Z$ and so $\text{PI-deg} (D_n) \leq 2n$.

Let $f$ be the multilinear identity for $A$ of degree $d$. If $d < 2n$, then

$$f(\alpha_{11} e_{11}, \alpha_{12} e_{12}, \alpha_{22} e_{22}, \ldots) \neq 0,$$

a contradiction. Hence $d > 2n > \text{PI-deg} (D_n)$ and the lemma is proved.

**Lemma 3.** If $R$ satisfies a generalized polynomial identity (GPI), then $R$ is PI and $\text{PI-deg}(R) < d$.

**Proof.** Suppose that $R$ is not a PI ring. Then, by a theorem of Montgomery ([4], Corollary to Theorem 2.5.1), for every positive integer $n$, $R$ contains a *-subring $R(n)$ which is a prime PI ring with $\text{PI-deg}(R(n)) > n$. But $R(n)$ is $S$-radical over $R(n) \cap A$ and $R(n) \cap A$ satisfies the polynomial identity $f(X_1, \ldots, X_d)$ of degree $d$. By Lemma 2, $d > \text{PI-deg}(R(n)) > n$, for every positive integer $n$, a contradiction. Thus $R$ is PI and by Lemma 2, $\text{PI-deg}(R) < d$.

We are finally able to prove our main theorem.

**Proof of Theorem 2.** Since, by assumption, $S \notin Z(R)$, by ([4], Theorem 2.2.1), either $S$ contains non-zero nilpotent elements or the involution is positive definite, that is $xx^* = 0$ in $R$ forces $x = 0$.

Suppose first that there exists $s \neq 0$ in $S$ with $s^2 = 0$. If $x \in R$, let $n(x, s) > 1$ be such that $(sx + x^* s)^{n(x, s)} \in A$ and let $A_1$ be the subring of $R$ generated by all $(sx)^{n(x, s)}$, $x \in R$. Then $R_1 = s R$ is radical over $A_1$. Now, if $b \in A_1$, say

$$b = \sum (sx_i)^{m_i} (sx_i')^{m_1} \ldots (sx_i)^{m_k}$$

then, since $s^2 = 0$,

$$bs = \sum (sx_i + x_i^* s)^{m_i} (sx_i + x_i^* s)^{m_1} \ldots (sx_i + x_i^* s)^{m_k} s = as$$
where \( a \in A \). From this it easily follows that if \( b_1, \ldots, b_d \in A_1 \) then 
\[
(b_1 \ldots b_d) s = (a_1 \ldots a_d) s \quad \text{where} \quad a_1, \ldots, a_d \in A.
\]
Hence,
\[
f(b_1, \ldots, b_d)s = \sum a_{\sigma(b_1)} \ldots a_{\sigma(b_d)} s \\
= \sum a_{\sigma(a_1)} \ldots a_{\sigma(a_d)} s \\
= f(a_1, \ldots, a_d)s = 0.
\]

In other words \( A_1 \) satisfies the polynomial identity \( f(X_1, \ldots, X_d)X_{d+1} \).

Let \( R_2 = R_1/N(R_1) \) where \( N(R_1) \) is the nil radical of \( R_1 \). Since \( R \) has no non-zero nil right ideals, neither does \( R_2 \). Moreover, \( R_2 \) is radical over \( A_2 \), the image of \( A_1 \) in \( R_2 \). Since \( A_1 \), and so \( A_2 \), satisfies \( f(X_1, \ldots, X_d)X_{d+1} \) by [5], \( R_2 \) also satisfies \( f(X_1, \ldots, X_d)X_{d+1} \). Therefore \( R \) satisfies a GPI and by Lemma 3 the result follows.

Suppose now that \( * \) is positive definite. We proceed by induction on the degree of the multilinear polynomial identity \( f(X_1, \ldots, X_d) \) satisfied by \( A \).

Since \( * \) is positive definite, \( A \) is semiprime. Moreover, since the center of a prime ring is a domain, \( Z(A) \subseteq Z(R) \) is also a domain. But in a semiprime PI-ring, every ideal hits the center non-trivially ([4], Corollary to Theorem 1.4.2), therefore \( A \) is prime.

If \( A \) has no non-zero nilpotent elements, then \( A \) is a domain and we are done by Lemma 1. Hence we may assume that there exists \( a \neq 0 \) in \( A \) with \( a^2 = 0 \).

Let \( R' = aRa^*; \) then \( R' \) is a *-subring of \( R \), \( S \)-radical over \( A' = aRa^* \cap A \), and, since \( * \) is positive definite, \( R' \) is a prime ring.

Let
\[
f(X_1, \ldots, X_d) = X_dh(X_1, \ldots, X_{d-1}) + g(X_1, \ldots, X_d)
\]
where \( X_d \) never appears as first variable in any monomial of \( g \). Since \( a^2 = 0 \), if \( x_1, \ldots, x_{d-1} \in A' \) and \( x_d \in A \), we have
\[
0 = af(x_1, \ldots, x_{d-1}, x_d) = ax_d h(x_1, \ldots, x_{d-1})
\]
Hence \( aAh(x_1, \ldots, x_{d-1}) = 0 \) and, since \( a \neq 0 \), the primeness of \( A \) forces \( h(x_1, \ldots, x_{d-1}) = 0 \). In other words \( A' \) satisfies \( h(x_1, \ldots, x_{d-1}) \).

By our induction hypothesis, \( R' \) is PI. From this we get that \( R \) satisfies a GPI. By Lemma 3, the result follows.
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