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A Stefan Problem for the Heat Equation
Subject to an Integral Condition (*).

ELENA COMPARINI - DOMINGO A. TARZIA (**)  

SUMMARY - We prove existence, uniqueness and continuous dependence and we study the behaviour of the free boundary of the solution of a Stefan problem for the heat equation when the integral condition \( E(t) = \int_0^{s(t)} u(x, t) \, dx \) is assigned.

1. Introduction.

In [6] the heat conduction in a slab of variable thickness \( 0 < x < s(t) \) is studied in the case in which no boundary conditions are assigned on the face \( x = 0 \), but the integral of the temperature \( E(t) = \int_0^{s(t)} u(x, t) \, dx \) is prescribed as a function of time.

Obviously \( E(t) \) represents the thermal energy at time \( t \) if we assume that the heat capacity of the material is constant and equal to 1.

In [3] the same problem is considered assuming that the slab is made of a material undergoing a change of phase at a fixed temperature (say \( u = 0 \)). In this case \( x = s(t) \) represents the interphase and it is assumed that \( u = 0 \) for \( x > s(t) \).

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In this problem the total thermal energy at time $t$ consists of two terms: one is the « latent energy », $Ls(t)$ ($L$ is the latent heat), while the « diffusing energy » is still given by the integral $E(t)$.

In [3] the well posedness of the problem is proved when $E > 0$, $\dot{E} > 0$, the initial temperature $\varphi(x)$ satisfies $0 < \varphi(x) < N(b - x)$ for all $0 < x < b$ ($N$ constant $> 0$), and $\varphi'(x) < 0$.

Here we consider the case of general data (without sign specification) and we prove that a $T > 0$ exists such that the problem is well posed in the time interval $(0, T)$.

The possible non existence of a global solution (i.e. for arbitrary $T$) is outlined in sec. 5, where we show that if $E < 0$, $\dot{E} < 0$, $u(x, 0) < 0$, then a finite time $T_0$ exists such that $\lim_{t\to T_0} \dot{s}(t) = -\infty$.

2. Formulation and results.

Let us consider the following problem: find a triple $(T, s, u)$ such that

i) $T > 0$;

ii) $s(t)$ is a positive function, continuously differentiable in $[0, T]$;

iii) $u(x, t) \in C_1(D_T)$, $u_{xx}$, $u_t$ are continuous in $D_T$, where $D_T = \{(x, t) : 0 < x < s(t), 0 < t < T\}$ and $\overline{D_T}$ is its closure;

iv) the following conditions are satisfied:

\begin{align}
(2.1) & & u_{xx} - u_t = 0 & & \text{in } D_T \\
(2.2) & & s(0) = b, & & b > 0 \\
(2.3) & & u(x, 0) = \varphi(x), & & 0 < x < b \\
(2.4) & & u(s(t), t) = 0 & & 0 < t < T \\
(2.5) & & \int_0^{s(t)} u(x, t) \, dx = E(t), & & 0 < t < T \\
(2.6) & & u_x(s(t), t) = -\dot{s}(t), & & 0 < t < T.
\end{align}
Here we assume

(A) \( \varphi(x) \in C_1[0,b] \quad \text{and} \quad \varphi(b) = 0 \)

(B) \( E(t) \in C_1[0,T] \quad \text{and} \quad E(0) = \int_0^b \varphi(x) \, dx \).

First, we state our existence and uniqueness theorem.

**Theorem 1.** Under assumptions (A), (B) there exists a solution \((T, s, u)\) of problem (i)-(iv), which is unique in \((0, T)\).

Moreover we have

**Theorem 2.** Let \((T, s, u)\) be the solution of problem (i)-(iv), then \(s \in C_1[0,T) \cap C_\infty(0,T)\).

We need some more regularity on the data to state the continuous dependence theorem.

Let us consider two solutions \((T_1, s_1, u_1)\), \((T_2, s_2, u_2)\) of problem (i)-(iv) corresponding to data \(\varphi_1, E_1\) and \(\varphi_2, E_2\) respectively.

If we replace assumptions (A), (B) with

(A') \( \varphi(x) \in C_2[0,b] \), \( \varphi(b) = 0 \)

(B') \( E(t) \in C_2[0,T] \), \( E(0) = \int_0^b \varphi(x) \, dx \)

then we prove

**Theorem 3.** If assumption (A)', (B)' are satisfied then two constants \(k, \bar{T}\) can be found a priori such that:

\[
\|s_1 - s_2\|_{C_1(0, \bar{T})} \leq k \{ \|\varphi_1 - \varphi_2\|_{C_1(0,b)} + \|E_1 - E_2\|_{C_1(0,\bar{T})} \}.
\]

The notation of spaces and norms used here and in the following are the same as in [12]. For sake of simplicity we often use the symbol \(\| \cdot \|_N, N \geq 0\) integer instead of \(\| \cdot \|_{C_N}, \| \cdot \|_\alpha, \alpha \in (0,1)\) instead of \(\| \cdot \|_{H}\), to denote the Hölder norm of order \(\alpha\); and \(\| \cdot \|_{N+\alpha}\), instead of \(\| \cdot \|_{H^{N+\alpha}}\), to denote the Hölder norm of the \(N\)-th derivative.
3. Proof of Theorem 1.

I. An equivalent formulation.

We begin with stating the following

**Lemma 3.1.** Let \((T, s, u)\) be a solution of (2.1)-(2.6) then

\[
(3.1) \quad u_x(0, t) = -\hat{E}(t) - \hat{s}(t), \quad 0 < t < T.
\]

**Proof.** From Green's identity

\[
\int_{D_t} (vLu - uL^*v) \, dx \, d\tau = \int_{\partial D_t} \left[ (u_xv - u v_x) \, d\tau + u v \, dx \right]
\]

where \(L\) is the heat operator and \(L^*\) its adjoint, with \(u = u(x, t)\) and \(v = 1\), we obtain

\[
(3.2) \quad s(t) = b - \int_0^t u_x(0, \tau) \, d\tau - E(t) + E(0)
\]

from which (3.1) follows.

**Lemma 3.2.** Let \(v(t) = u_x(s(t), t)\), where \(u, s\) solves (2.1)-(2.6), then

\[
(3.3) \quad v(t) = 2\int_0^t N_x(s(t), t; s(\tau), \tau) \, v(\tau) \, d\tau - 2\int_0^t N_x(s(t), t; 0, \tau) \, v(\tau) \, d\tau - 2E(0)N_x(s(t), t; 0, 0) - 2\int_0^t N_x(s(t), t; 0, \tau) E(\tau) \, d\tau + 2\int_0^b g(s(t), t; \xi, 0) q'(\xi) \, d\xi
\]

with

\[
(3.4) \quad s(t) = b - \int_0^t v(\tau) \, d\tau.
\]
PROOF. (3.3)-(3.4) are proved following the methods of [4]. Here 
$G(x, t; \xi, \tau)$ and $N(x, t; \xi, \tau)$ are Green's and Neuman's functions for 
the heat operator.

Thus we have reduced (2.1)-(2.6) to a system of integral equa-
tions such that if $u(x, t)$, $s(t)$ satisfy (2.1)-(2.6) then $v(t)$, $s(t)$ satisfy 
(3.3)-(3.4).

Conversely, if $v(t)$ is a continuous solution of (3.3) and if $s(t)$, given 
by (3.4), is positive, we can define $u(x, t)$ replacing $u_x(s(\tau), \tau)$ with $v(\tau)$ 
and $u_x(0, \tau)$ with $-E(\tau) + v(\tau)$ in the formal rappresentation for 
the solution of the problem. Now it is easy to show (see [5]) that $u(x, t)$ 
so defined satisfies (2.1)-(2.6).

Moreover, it is known that the initial boundary problem (2.1)-(2.5), 
for given Lipschitz continuous and positive $s(t)$, admits a unique 
solution [6].

It is so proved that problem (2.1)-(2.6) is equivalent to the problem 
of finding a continuous solution of the integral equations (3.3), (3.4).

II. Existence and uniqueness.

Now we prove that the system (3.3), (3.4) has a unique solution 
for $0 < t < T$ where $T$ is sufficiently small.

We consider

$$X_{T,M} = \{v(t) \in C[0, T] : \|v\|_0 = \max_{0 \leq t \leq T} |v(t)| < M\}.$$ 

On the set $X_{T,M}$ we define a transformation

$$\tilde{v} = \mathcal{G}(v)$$ 

as follows

\begin{align*}
\tilde{v}(t) &= 2\int_{0}^{t} N_x(s(t), t; s(\tau), \tau) v(\tau) \, d\tau - 2\int_{0}^{t} N_x(s(t), t; 0, \tau) v(\tau) \, d\tau - \\
&\quad - 2E(0) N_x(s(t), t; 0, 0) - 2\int_{0}^{t} N_x(s(t), t; 0, \tau) E(\tau) \, d\tau + \\
&\quad + 2\int_{0}^{b} G(s(t), t; \xi, 0) p'(\xi) \, d\xi
\end{align*}

(3.5)
where

\begin{equation}
(3.6) \quad s(t) = b - \int_0^t v(\tau) \, d\tau .
\end{equation}

We shall prove that there exists a fixed point of \( \mathcal{G} \). Chosen a \( T \) such that

\begin{equation}
(3.7) \quad \frac{b}{2} < s(t) < \frac{3}{2} b , \quad 0 < t < T
\end{equation}

we have that \( \|v\|_0 < M \) implies immediately

\begin{equation}
(3.8) \quad |s(t) - s(\tau)| < M(t - \tau) , \quad 0 < \tau < t
\end{equation}

and, from (3.5),

\begin{equation}
(3.9) \quad \|\dot{s}\|_0 < k + c M^2 T^4
\end{equation}

having posed \( T < 1, \quad M > 1 \). \( k \) is a constant depending on \( \|E\|, \|\varphi\|, b \), and \( c \) is a constant depending on \( b \) only.

Thus chosen a set \( X_{T,M} \) with e.g. \( M = 2k \) and \( T \) such that

\[ T^4 < \frac{1}{c^4 k} \]

then \( \mathcal{G} \) maps \( X_{T,M} \) into itself.

Now we prove that \( \mathcal{G} \) is a contraction.

For any \( v_1, v_2 \in X_{T,M} \) let us consider the difference \( v_1 - v_2 \).

Denote

\begin{equation}
(3.10) \quad \|v_1 - v_2\|_0 = \varepsilon , \quad \varepsilon < 2M .
\end{equation}

From (3.6) we have,

\begin{equation}
(3.11) \quad |s_1(t) - s_2(t)| < \varepsilon t , \quad \|\dot{s}_1 - \dot{s}_2\|_0 < \varepsilon , \quad 0 < t < T .
\end{equation}
From (3.5):

\[
\begin{align*}
&\tilde{I}_1 - \tilde{I}_2 = 2 \int_0^t \left[ N_x(s_1(t), t; s_1(\tau), \tau) v_1(\tau) - N_x(s_2(t), t; s_2(\tau), \tau) v_2(\tau) \right] d\tau \\
&\quad - 2 \int_0^t \left[ N_x(s_1(t), t; 0, \tau) v_1(\tau) - N_x(s_2(t), t; 0, \tau) v_2(\tau) \right] d\tau - \\
&\quad - 2 E(0) \left[ N_x(s_1(t), t; 0, 0) - N_x(s_2(t), t; 0, 0) \right] - \\
&\quad - 2 \int_0^t \left[ N_{x\tau}(s_1(t); t; 0, \tau) - N_{x\tau}(s_2(t), t; 0, \tau) \right] E(\tau) d\tau + \\
&\quad + 2 \int_0^b \left[ G(s_1(t), t; \xi, 0) - G(s_2(t), t; \xi, 0) \right] \varphi'(\xi) d\xi.
\end{align*}
\]

To estimate the first integral on the right-hand side of (3.12), say $I_1$, we consider that:

\[
(3.13) \quad I_1 = - \int \frac{s_1(t) - s_1(\tau)}{t - \tau} \Gamma(s_1(t), t; s_1(\tau), \tau) v_1(\tau) d\tau + \\
\quad + \int \frac{s_2(t) - s_2(\tau)}{t - \tau} \Gamma(s_2(t), t; s_2(\tau), \tau) v_2(\tau) d\tau + I'_1
\]

where $I'_1$ denotes the sum of the first two terms on the right-hand side of (3.13) but with $s_1(\tau)$ and $s_2(\tau)$ replaced by $-s_1(\tau), -s_2(\tau)$ respectively.

We can write

\[
(3.14) \quad I_1 = - \int \frac{s_1(t) - s_1(\tau)}{t - \tau} - \frac{s_2(t) - s_2(\tau)}{t - \tau} \left[ \Gamma(s_2(t), t; s_2(\tau), \tau) v_2(\tau) d\tau + \\
\quad + \int \frac{s_1(t) - s_1(\tau)}{t - \tau} \left[ \Gamma(s_2(t), t; s_2(\tau), \tau) - \Gamma(s_1(t), t; s_1(\tau), \tau) \right] v_2(\tau) d\tau - \\
\quad - \int \frac{s_1(t) - s_1(\tau)}{t - \tau} \Gamma(s_1(t), t; s_1(\tau), \tau) \left[ v_1(\tau) - v_2(\tau) \right] d\tau + I'_1.
\]
From (3.14)

\[
|I_1| \leq c \left\{ \int_0^t \int_0^t \frac{|v_1(\eta) - v_2(\eta)| d\eta}{(t-\tau)^{\frac{3}{2}}} M d\tau + \right. \\
+ \int_0^t \frac{M^2}{(t-\tau)^{\frac{3}{2}}} \left[ 1 - \exp \left( \frac{(s_1(t) - s_1(\tau))^2}{4(t-\tau)} - \frac{(s_2(t) - s_2(\tau))^2}{4(t-\tau)} \right) \right] d\tau + \\
\left. \int_0^t \frac{M}{(t-\tau)^{\frac{1}{2}}} \epsilon d\tau \right\} + |I'_1| \leq c \{ M \epsilon t^\frac{1}{2} + M^2 \epsilon t^\frac{3}{2} + M \epsilon t^1 \} + |I'_1|.
\]

Here and in the following \( c \) will denote a constant depending on \( b \) and possibly on \( M \). The estimate for \( |I'_1| \) is obtained by means of the mean value theorem

\[
|I'_1| \leq c M \epsilon t^1.
\]

Finally

\[
|I_1| \leq c M \epsilon T^1.
\]

The second integral in (3.12), call it \( I_2 \), is easily estimated as follows

\[
|I_2| \leq 2 \int_0^t \left| N_x(s_1(t), t; 0, \tau) - N_x(s_2(t), t; 0, \tau) \right| v_1(\tau) d\tau + \\
+ \int_0^t \left| N_x(s_2(t), t; 0, \tau) \right| v_1(\tau) - v_2(\tau) | d\tau \leq \\
\leq 2M\|s_1 - s_2\|_0 \int_0^t \left| N_{xx}(\bar{s}(t), t; 0, \tau) \right| d\tau + 2\epsilon \int_0^t \left| N_x(s_2(t), t; 0, \tau) \right| d\tau
\]

with

\[
\min (s_1, s_2) \leq \bar{s} \leq \max (s_1, s_2)
\]

that is

\[
|I_2| \leq c \epsilon T.
\]
Estimates like these hold for the third and the fourth terms on the right-hand side of (3.12), from which we obtain

\begin{equation}
|I_3 + I_4| < c\|E\|_0 e^T.
\end{equation}

For the last integral, say \(I_5\), we have (see [4])

\begin{equation}
|I_5| < c\|\varphi\|_1 T^1.
\end{equation}

From (3.17), (3.19), (3.20) and (3.21) we obtain

\begin{equation}
\|\tilde{\varphi}_1 - \tilde{\varphi}_2\|_0 < cT^1\|v_1 - v_2\|_0.
\end{equation}

Thus, there exists a time \(\bar{T} < T\) such that \(c\bar{T}^1 < 1\), then (3.22) implies that \(G\) is a contractive mapping in the norm of \(C[0, \bar{T}]\).

Therefore we proved that there exists a unique fixed point \(v(t)\) of \(G\) in \(X_{\bar{T}, M}\), and then \(v(t)\) is the unique solution of the integral equation (3.3), with \(s(t)\) defined by (3.4).

**Remark 3.1.** Note that in the case in which \(s(t)\) is monotone non-decreasing (this happens for example for \(\varphi(x) > 0, E(t) > 0, 0 < \bar{E}(t) < A\) (see [3])) we can apply the proof of § 3 step by step, to obtain a solution \(u(x, t), s(t)\) (or \(v(t)\)) for all times [4].

Now we prove

**Theorem 2.** If \((T, s, u)\) solves (i)-(iv) then \(s \in C_1[0, T) \cap C(0, T)\).

**Proof.** Recalling [7] we can assert that \(s \in C^\alpha(\varepsilon, T)\) for any \(\varepsilon > 0\). Moreover, performing the limit for \(t \to 0\) in (3.3), we can easily prove that

\begin{equation}
v(0) = \varphi'(b)
\end{equation}

that is \(s(t)\) is continuous at \(t = 0\).

**Remark 3.2.** Let us define

\begin{equation}
w(x, t) = \int_{s(0)}^{x} u(\xi, t) d\xi.
\end{equation}
By straightforward computation one verifies that if $(T, s, u)$ solves (2.1)-(2.6) then $(T, s, \tilde{u}, u)$ solves

\begin{align}
(3.25) & \quad w_{xx} - w_t = -\dot{s} \quad \text{in } D_T \\
(3.26) & \quad w(x, 0) = \int_{b}^{a} \phi(\xi) \, d\xi, \quad 0 < x < b \\
(3.27) & \quad w(0, t) = -E(t), \quad 0 < t < T \\
(3.28) & \quad w(s(t), t) = 0, \quad 0 < t < T \\
(3.29) & \quad w_x(s(t), t) = 0, \quad 0 < t < T.
\end{align}

Thus we proved existence and uniqueness of the solution of problem (3.25)-(3.29), which is a problem with Cauchy data assigned on $x = s(t)$, which differs from those studied in [8] where the right-hand member of the parabolic equation was not allowed to depend on $\dot{s}$.

4. Proof of Theorem 3.

**Lemma 4.1.** Assume $(A)', (B)'$ and let $(T, s, u)$ be the solution of problem (i)-(iv), then

\begin{equation}
|t^k \dot{s}(t)| < e, \quad 0 < t < T.
\end{equation}

**Proof.** Replacing assumptions $(A), (B)$ with $(A)', (B)'$, we can repeat the arguments of sec. 3 to prove a contraction on the set

\[ \mathcal{X}_{\tilde{r}, a} = \{ v(t) \in H_1[0, T]: \|v\|_1 < H, \tilde{T} < T \} \]

where $v(t)$ is defined by (3.3), (3.4).

This implies $s \in H_{1+\beta}[0, T]$.

The estimates of $\|v\|_1$ and $\|v - v_s\|_1$ are obtained following the methods of [9].

**Lemma 4.2.** Under the hypothesis of Lemma 4.1 we have

\begin{equation}
\max_{0 \leq x \leq s(t)} |u_{xx}(x, t)| \leq c t^{-\frac{1}{2}} \quad 0 < t < \tilde{T}.
\end{equation}
Proof. In [10] it is proved that the solution \( z(x, t) \) of the first boundary problem in DT for the heat equation has a bounded second order derivative \( z_{xx} \), when \( z(s(t), t) = 0 \), \( z(0, t) \) is Lipschitz continuous, \( z(x, 0) \) satisfies assumptions like \((\Lambda')\).

It can be shown that in our case \( u(0, t) \in H_1 \) and that it is possible to modify the estimates of [10] to prove inequality (4.2). Details are omitted for sake of brevity.

Let \((T_1, s_1, u_1), (T_2, s_2, u_2)\) be two solutions of problem (i)-(iv), with assumptions \((\Lambda')', (B)'\), corresponding to the data \( \varphi_1, E_1, b \) and \( \varphi_2, E_2, b \) respectively. We perform the transformation (for \( i = 1, 2 \))

\[
y = x/s_i, \quad w_i(y, t) = u_i(s_i y, t), \quad \bar{\varphi}_i(y) = \varphi_i(b y)
\]

leading to

\[
w_{it} = -s_i^{-2}w_{iyy} + y \delta_i \delta_i^{-1}w_{iy} \quad \text{on} \quad D_T, \quad T = \min (T_1, T_2) < \tilde{T}
\]

\[
w_i(y, 0) = \bar{\varphi}_i(y), \quad 0 < y < 1
\]

\[
w_{iy}(0, t) = -s_i(t)[\delta_i(t) + \tilde{E}_i(t)], \quad 0 < t < \tilde{T}
\]

\[
w_i(1, t) = 0, \quad 0 < t < \tilde{T}
\]

\[
s_i(t) = -s_i^{-1}(t)w_{iy}(1, t), \quad 0 < t < \tilde{T}.
\]

Obviously problem (4.4)-(4.8) is equivalent to (2.1)-(2.6).

Let us introduce the following notation

\[
\delta(t) = s_2(t) - s_1(t), \quad \delta(t) = \delta_2(t) - \delta_1(t)
\]

\[
W(y, t) = w_2(y, t) - w_1(y, t)
\]

\[
\Delta \varphi(y) = \bar{\varphi}_2(y) - \bar{\varphi}_1(y)
\]

\[
\Delta E(t) = E_2(t) - E_1(t), \quad \Delta \tilde{E}(t) = \tilde{E}_2(t) - \tilde{E}_1(t),
\]

\[
\|\delta\| = \max_{0 \leq \tau \leq t} |\delta(\tau)|, \quad \|\delta\|, \max_{0 \leq \tau \leq t} |\delta(\tau)|
\]

\( W(y, t) \) defined by (4.9) solves:

\[
W_t = A(t) W_{yy} + B(y, t) W_y + F(y, t) \quad \text{in} \quad D_\tilde{T}
\]

\[
W(y, 0) = \Delta \varphi(y), \quad 0 < y < 1
\]
\[ W_s(0, t) = -s_4(\Delta \varepsilon + \delta) - \delta(E_1 + \delta_1) = G(t), \quad 0 < t < \hat{T} \]
\[ W(1, t) = 0, \quad 0 < t < \hat{T} \]

where

\[ A(t) = s_1^{-2} \]
\[ B(y, t) = y\delta_1s_1^{-1} \]
\[ F(y, t) = -\delta((s_1 + s_2)s_1^{-2}s_2^{-2}w_{2yy} + ys_1^{-1}s_2^{-1}\delta_2w_{2y}) + \delta ys_1^{-1}w_{2y}. \]

We are going to study the difference

\[ \delta = \delta_2 - \delta_1 = -s_2^{-1}W_s(1, t) - \delta s_1^{-1}s_2^{-1}w_{1y}(1, t) \]

for which we need an estimate of \( W_s(1, t) \).

We split \( W \) into the sum

\[ W = W_1 + W_2 \]

where \( W_1 \) solves problem:

\[ W_{1t} = A(t)W_{1yy} \quad \text{in } D_{\hat{T}} \]

with conditions (4.12), and \( W_2 \) solves:

\[ W_{2t} = A(t)W_{2yy} + B(y, t)W_{2y} + F_0(y, t), \quad \text{in } D_{\hat{T}} \]

with zero initial and boundary conditions.

In (4.16)

\[ F_0(y, t) = F(y, t) + B(y, t)W_{1y}. \]

As to \( W_1 \), we split it again into the sum \( W_1 = z_1 + z_2 \), where \( z_1 \) is the solution in the half plane \( x > 0 \) of

\[ z_{1t} = A(t)z_{1yy} \]

with

\[ z_1(y, 0) = 0, \quad z_{1y}(0, t) = G(t), \]
while $z_2$ solves the same equation (4.18) with

$$z_2(y, 0) = \Delta \varphi, \quad z_{2y}(0, t) = 0, \quad z_2(1, t) = -z_1(1, t). \tag{4.19}$$

Introducing the fundamental solution for the operator $\partial / \partial y - \Delta(t) (\partial^2 / \partial y^2)$, say $\Gamma(y, t; \xi, \tau)$, by means of the parametrix method of E. E. Levi, we have for $z_1$:

$$z_1(y, t) = -2 \int_0^t \Gamma(y, t; 0, \tau) G(\tau) \, d\tau. \tag{4.20}$$

From (4.20) we have immediately the estimate

$$|z_{1y}(1, t)| < c t (\|\delta\|_t + \|\Delta \varphi\|_t). \tag{4.21}$$

An estimate like (4.21) holds for $z_{2y}(1, t)$.

Now we consider $z_2$ as the restriction to $[0, 1] \times (0, \hat{T})$ of the solution of

$$z_2(t) = A(t) z_{2y}, \quad \text{in } (-1, 1) \times (0, \hat{T}) \tag{4.22}$$

$$z_2(y, 0) = \overline{\Delta \varphi}, \quad -1 < y < 1 \tag{4.23}$$

$$z_2(-1, t) = z_2(1, t) = -z_1(1, t), \quad 0 < t < \hat{T} \tag{4.24}$$

with

$$\overline{\Delta \varphi} = \begin{cases} \Delta \varphi(y), & y > 0 \\ \Delta \varphi(-y), & y < 0. \end{cases}$$

We can estimate $z_{2y}(1, t)$ knowing $z_2(y, t)$ on $\partial D_x$, by means of Lemma 3.1 p. 535 of [11].

Making use of the estimate on $z_1$, we obtain

$$|z_{2y}(1, t)| < c \{ \|\Delta \varphi\|_1 + t (\|\delta\|_1 + \|\Delta \varphi\|_1) \}. \tag{4.25}$$

From (4.21) and (4.25) we get the estimate

$$|W_{1y}(1, t)| < c \{ \|\Delta \varphi\|_1 + t (\|\delta\|_1 + \|\Delta \varphi\|_1) \}. \tag{4.26}$$
Moreover, applying the maximum principle in $D_T$, we get also the estimate

$$
|W_{1v}(y, t)| < c[\|\Delta \varphi\|_1 + \|\delta\|_t + \|\Delta E\|_1].
$$

Finally, let us consider $W_2$ as the restriction in $[0, 1) \times (0, \hat{T})$ of the solution of

$$
W_{zt} = A(t) W_{2v} + \overline{F}(y, t) \quad \text{in } (-1, 1) \times (0, \hat{T})
$$

$$
W_2(y, 0) = W_2(-1, t) = W_2(1, t) = 0
$$

with

$$
\overline{F}(y, t) = \begin{cases}
B(y, t) W_{2v}(y, t) + F_0(y, t), & y > 0 \\
B(-y, t) W_{2v}(y, t) + F_0(-y, t), & y < 0.
\end{cases}
$$

Using the methods of [12], sec. 4 we obtain the estimate

$$
\max_{\nu \in [-1, 1]} |W_{2v}(y, t)| < c \left\{ \int_0^t (t - \tau)^{-\frac{1}{2}} \max_{\nu} |W_{2v}(y, t)| d\tau +
\right.
$$

$$
\left. + t^4 (\|\delta\|_t + \|\Delta E\|_t + \|\Delta \varphi\|_1) \right\}
$$

which gives, with (4.26),

$$
|W_2(1, t)| < c \left\{ t^4 \|\delta\|_t + \|\Delta \varphi\|_1 + t^4 \|\Delta E\|_1 \right\}.
$$

From (4.14)

$$
\|\delta\|_t < c \left\{ \|\Delta \varphi\|_1 + t^4 \|\Delta E\|_1 + t^4 \|\delta\|_t \right\}
$$

which proves (2.7).

5. Behaviour of the free boundary.

It has been proved in [3] that if one assumes positive data $\varphi(x)$, $E(t)$ with $\varphi'(x) < 0$ and $0 < \hat{E}(t) < A$, then $s(t)$ is monotone non decreasing in $t$ and $\hat{s}(t)$ is bounded so that a solution can exist with arbitrarily large $T$. 
In this section we want to study the problem of the continuation of the solution and to analyze the behavior of the free boundary when the sign restriction imposed in [3] are no longer valid.

We will assume besides of (A)', (B)' that

\begin{align}
q(x) &< 0 \\
E(t) &< 0, \quad \dot{E}(t) < 0. 
\end{align}

We first prove

**Lemma 5.1.** Let \((T, s, w)\) be the solution of (i)-(iv) with assumption (A)', (B)'. Let the data satisfy (5.1), (5.2), then

\begin{align}
u(x, t) &< 0, \quad \dot{s}(t) < 0. 
\end{align}

**Proof.** From (5.1) it is \(q'(b) > 0\), that is since \(\dot{s}(t)\) is continuous,

\begin{align}
\dot{s}(0) = -u_{\alpha}(b, 0) < 0.
\end{align}

Consider the solution corresponding to

\begin{align}
q^{(n)}(x) = q(x) - \frac{1}{n}(b - x)
\end{align}

for which \(\dot{s}_n(0) < 0\).

Assume that there exists a first time \(\tilde{t}_n\) such that \(\dot{s}_n(\tilde{t}_n) = 0\), and then

\begin{align}
u_{\alpha}(s_n(\tilde{t}_n), \tilde{t}_n) = 0.
\end{align}

From the maximum principle in \(D_{\tilde{t}_n}\), it is \(u_n(x, t) < 0\), that is, as \(u_n(s_n(\tilde{t}_n), \tilde{t}_n) = 0\), \((s_n(\tilde{t}_n), \tilde{t}_n)\) is an isolated maximum for \(u_n\), and the parabolic Hopf's Lemma [13] ensures \(u_{\alpha}(s_n(\tilde{t}_n), \tilde{t}_n) > 0\), contradicting (5.6).

Then

\begin{align}
u_n(x, t) < 0, \quad \dot{s}_n(t) < 0
\end{align}

and performing the limit for \(n \to \infty\) we obtain (5.3).
REMARK 5.1. If one excludes the trivial case in which $\dot{s} = 0$, corresponding to data $E = \varphi = 0$, then immediately

\begin{equation}
\dot{s}(t) < 0, \quad 0 < t < T.
\end{equation}

THEOREM 4. Let us consider two sets of data $(E_1, \varphi_1, b_1), (E_2, \varphi_2, b_2)$ for problem (i)-(iv), and assume that both of them satisfy assumptions (A)', (B)' and (5.1), (5.2).

Let $(T_1, s_1, u_1), (T_2, s_2, u_2)$ be the correspondent solutions and assume:

\begin{align}
E_2 &> E_1, \quad \varphi_2 > \varphi_1, \quad b_2 > b_1 \\
\int_0^{b_s} [\varphi_2(y) + 1] \, dy &> 0, \quad 0 < x < b_2 \\
\int_0^{b_s} [\varphi_2(y) + 1] \, dy &> 0.
\end{align}

Then

\begin{equation}
s_1(t) < s_2(t), \quad t < T_0
\end{equation}

where $T_0 = \min \{T_1, T_2, \sup \tilde{t}: s_2(\tilde{t}) > -E_2(\tilde{t})\}$.

PROOF. Lemma 3.1 ensures

\begin{equation}
u_{2i}(0, t) = -E_2(t) - \dot{s}(t), \quad i = 1, 2.
\end{equation}

Thus we can consider $(T_1, s_1, u_1), (T_2, s_2, u_2)$ as solutions of two boundary problems with assigned flux, and then (see [14], Lemma 2.10) (5.11) holds.

We conclude this section with the following

THEOREM 5. Let the hypothesis of Lemma 5.1 hold. Then there exists a finite time $T_0$ such that:

\begin{equation}
\lim_{t \to T_0^-} \dot{s}(t) = -\infty.
\end{equation}
PROOF. The existence of a solution of problem (i)-(iv) with arbitrarily large $T$ implies (see [14], Cor. 2.12) that

\begin{equation}
\int_{D} |u(x, t)| \, dx \, dt = \int_{0}^{t} |E(\tau)| \, d\tau < + \infty
\end{equation}

for all $t > 0$, which is contradictory with (5.2).

Now if we suppose that a time $\bar{t}$ exists such that $\lim_{t \to \bar{t}} s(t) = 0$ then

\begin{equation}
\lim_{t \to \bar{t}} s(t) = - \infty .
\end{equation}

Indeed if (5.15) is not true then

\[ \lim_{t \to \bar{t}} u_x(s(t), t) \quad \text{and} \quad \lim_{t \to \bar{t}} u_x(0, t) \]

exist and are bounded, and then $u$ is continuous in $(s(\bar{t}), \bar{t})$ (and equal to 0).

That implies

\begin{equation}
E(\bar{t}) = \lim_{t \to \bar{t}} \int_{0}^{s(t)} u(x, t) \, dx = 0
\end{equation}

which cannot hold because of (5.2).

Of course we excluded the trivial case $E = 0$.

The theorem is proved recalling [15], Theorem 8.

REFERENCES


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