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## Positive Functionals and the Axiom of Choice.

NORBERT BRUNNER (\*)

In this note we prove, that a proposition which is useful in integration theory, is equivalent to a weak axiom of choice.

**THEOREM.** The axiom  $MC^\omega$  of countable multiple choice is equivalent to the assertion, that every positive linear functional on a  $B$ -lattice is continuous.

Our set theory is  $ZF^0$ , the Zermelo-Fraenkel system minus the axioms of choice and foundation.  $MC^\omega$  says, that a countable sequence of nonempty sets  $(S_n)_{n \in \omega}$  contains a sequence of nonempty finite subsets  $F_n \subseteq S_n$ . In  $ZF^0$   $MC^\omega$  is not provable, and  $MC^\omega$  does not imply the countable axiom of choice  $AC^\omega$ . It is unknown, whether in  $ZF$  ( $= ZF^0 + \text{foundation}$ )  $MC^\omega \Leftrightarrow AC^\omega$ .

A  $B$ -lattice ( $AB$ -lattice in [4]) is a vector lattice together with a Frechet-complete (Cauchy sequences are convergent, c.f. [3] for a discussion of diverse completeness properties) Riesz-norm  $\|\cdot\|$  (c.f. [7], p. 61 and p. 101), i.e. (i)  $-y \leq x \leq y$  implies  $\|x\| \leq \|y\|$  and (ii) if  $\|x\| < 1$  there is a  $y \geq 0$  such that  $-y \leq x \leq y$  and  $\|y\| < 1$ . A positive linear functional for  $X$  is a linear mapping  $f: X \rightarrow \mathbf{R}$  such that  $fx \geq 0$  whenever  $x \geq 0$ .

**PROOF OF THE THEOREM.** « $\Rightarrow$ »: Assume  $MC^\omega$ . Let  $f: X \rightarrow \mathbf{R}$  be positive and linear. If  $f$  is not continuous, the standard argument proves, that in  $ZF^0$   $f$  is not bounded. We observe, that  $S_n = \{x \in X:$

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$\|x\| \leq 1, fx \geq n, x \geq 0\}$  is nonempty. Since  $f$  is not bounded, there is a  $x \in X, \|x\| \leq 1$ , such that  $|fx| \geq 2n$ . According to the Riesz decomposition theorem,  $x = x^+ - x^-, |x| = x^+ + x^-$ , where  $x^+ = \sup \{x, 0\}$ ,  $x^- = (-x)^+, |x| = \sup \{x, -x\}$ . Since  $\|\cdot\|$  is a Riesz norm, lemma 9.16 of [7] applies, which shows:  $\|x^+\| \leq 1, \|x^-\| \leq 1$ . Since  $f$  is positive  $fx^+ \geq 0$ . If  $fx^+ < n$  and  $fx^- < n$ , then  $|fx| = |fx^+ - fx^-| < 2n$ , a contradiction. So  $x^+ \in S_n$  or  $x^- \in S_n$ .  $MC^\omega$  provides us with a sequence  $F_n \subseteq S_n$  of finite sets  $F_n \neq \emptyset; x_n$  is the arithmetic mean of  $F_n: x_n \geq 0, \|x_n\| \leq 1$  and  $fx_n \geq n$ . We set  $z = \sum_{n=1}^{\infty} (1/n^2)x_n$ . As this series converges absolutely and  $X$  is complete, the limit  $z$  exists;  $z = \lim y_n, y_n = \sum_{i=1}^n (1/i^2)x_i$ . Since  $y_n \leq z, \sum_{i=1}^n 1/i \leq fy_n \leq fz$ . This is a contradiction: The harmonic series is unbounded.

The above proof is an adaption of [6], p. 273, where this result was proved for a class of bornological ordered vector spaces. Examples of  $B$ -lattices are the spaces  $L_p, p \geq 1$ , and  $C(X), X$  compact  $T_2$ , with the usual order. If the cone of a  $B$ -lattice has a base, there is—in  $ZF^0 + MC^\omega$ —a positive-, continuous-linear functional  $f$ , such that  $B = C \cap f^{-1}(1)$  ( $B$  the base,  $C = \{x \in X: x \geq 0\}$  the cone). In [6], p. 272, it was shown without the axiom of choice, that a positive linear functional on a topological ordered vector space is continuous, if the cone has interior points. Since this result applies to  $C(X)$ , it was of interest, whether its extension to  $B$ -lattices depends on the axiom of choice. This is shown next. A modification of the argument in [5], p. 24, combined with « $\Leftarrow$ » proves, that the following assertion is equivalent to  $MC^\omega$ . In a  $B$ -lattice a convex, balanced set which absorbs all order bounded sets absorbs all bounded sets, too. This property also implies the continuity of positive linear functionals (in  $ZF_0$ ).

« $\Leftarrow$ »: We first observe, that  $MC^\omega$  is equivalent to the following weaker principle  $PMC^\omega$ : If  $(S_n)_{n \in \omega}$  is a sequence of nonempty sets, there exists a subsequence  $(S_{n(k)})_{k \in \omega}$  and a sequence  $(E_k)_{k \in \omega}$  of nonempty finite  $E_k \subseteq S_{n(k)}$ . For if  $PMC^\omega$  is applied to the sequence  $(T_n)_{n \in \omega}$ ,  $T_n$  the set of all choice functions on  $(S_i)_{i \in n}$ , we define a finite  $\emptyset \neq F_i \subseteq S_i$  through  $F_i = \{f(i): f \in E_{k(i)}\}$ , where  $E_k \subseteq T_{n(k)}$  is finite and  $k(i) = \min \{j: i < n(j)\}$ .

We next assume, that  $MC^\omega$ —and hence  $PMC^\omega$ —is false and let  $(S_n)_{n \in \omega}$  be a counterexample of  $PMC^\omega$ , consisting of infinite and pairwise disjoint sets. We define a  $B$ -lattice  $X$  with a positive, un-

bounded functional  $\varphi$ .  $S = \bigcup_{n \in \omega} S_n$  and  $X = \{f \in S^\omega : \|f\| < \infty\}$ , where  $\|f\|^2 = \sum_{i=1}^{\infty} \left( \sum_{s \in S_i} |f(s)| \right)^2$ .  $X$  is the  $l_2$ -sum of the spaces  $l_1(S_i)$ . As a normed space,  $X$  is complete, as can be proved in  $ZF^0$  (c.f. [4]). The ordering  $f \leq g$ , iff  $f(s) \leq g(s)$  for all  $s \in S$ , defines a vector lattice on  $X$  and  $\|\cdot\|$  is a Riesz-norm with respect to this structure. We want define  $\varphi$  as  $\varphi(f) = \sum_{s \in S} f(s)$ .

This definition is correct, i.e.:  $\sum f(s)$  is convergent. For set  $F_n = \{s \in S : |f(s)| \geq 1/n\}$ . Since  $\|f\| < \infty$ ,  $F_n$  is finite. If  $F = \bigcup_{n \in \omega} F_n$ ,  $M = \{n \in \omega : S_n \cap F \neq \emptyset\} = \{n \in \omega : \exists x \in S_n : f(x) \neq 0\}$  is finite, for otherwise we can define a partial multiple choice function  $(E_n)_{n \in M} : E_n = S_n \cap F_{n(k)}$ ,  $n(k) = \min \{m : S_n \cap F_m \neq \emptyset\}$ . Hence  $\varphi(f) = \sum_{n \in M} \sum_{s \in S_n} f(s)$  is

absolutely convergent by the definition of  $X$  and hence the series which defines  $\varphi(f)$  is (unconditionally) convergent, by completeness.

That  $\varphi$  is linear and positive is obvious. But  $\varphi$  is not bounded.

For otherwise we choose  $N$  so that  $\sum_{n=1}^N 1/n > \pi/\sqrt{6} \cdot \|\varphi\|$ . Choose  $s_i \in S_i$  for  $i \in N$  and set

$$f(s_i) = \frac{1}{i+1}, \quad i \in N, \quad f(s) = 0 \text{ otherwise.}$$

Since

$$\|f\| = \left( \sum_{n=1}^N \frac{1}{n^2} \right)^{\frac{1}{2}} \leq \frac{\pi}{\sqrt{6}}$$

we obtain

$$\frac{\pi}{\sqrt{6}} \cdot \|\varphi\| < \sum_{n=1}^N \frac{1}{n} = |\varphi(f)| \leq \|\varphi\| \cdot \|f\| \leq \|\varphi\| \cdot \frac{\pi}{\sqrt{6}},$$

a contradiction. This proves the theorem.

Axioms similar to  $PMC^\omega$  were studied in [2]. Another application of  $l_p$ -sums of  $l_q$ -spaces to the axiom of choice appeared in [1].

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