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Linearly Compact Rings and Selfcogenerators.

CLAUDIA MENINI (*)

0. Introduction.

0.1. Throughout this paper all rings are associative with identity $1 \neq 0$ and all modules are unitary.

In our previous work [5] we outlined the connection between linearly compact rings and quasi-injective modules. (Definitions and main results we got in [5] are listed below).

In this paper we give some applications of these results.

First of all, in section 1 we get a further characterization of linearly compact rings. More precisely we prove that a ring R admits a left linearly compact ring topology iff $R = \text{End}(K_A)$ where A is a ring and K_A is a right A -module which is strongly quasi-injective, with essential socle and whose cyclic right A -submodules are linearly compact in the discrete topology. Moreover, in this case, R is linearly compact in the topology τ_* having as a basis of neighbourhoods of 0 the left ideals $\text{Ann}_R(L)$ where L ranges among all linearly compact discrete submodules of K_A . This topology τ_* is the finest topology in the equivalence class of the K -topology of R . This generalizes a previous result by F.L. Sandomierski ([9]), who proved it in the discrete case and gives a method to build linearly compact rings.

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The existence of a finest equivalent topology for a linearly compact ring has been recently proved by P.N. Ánh (see [1], Proposition 3.1.) in a rather different way and without the representation method.

In section 2 we deal with strictly linearly compact rings and, more in general, with topologically left artinian rings (a linearly topologized ring is called topologically left artinian—see [2] and [7]—if it has a basis of neighbourhoods of 0 consisting of left ideals with artinian residue). These rings have been extensively studied by Ballet in [2] and by A. Orsatti and the author in [7].

Here we point out the relation between such rings and Σ -quasi-injective modules (a module is Σ -quasi-injective iff any direct sum of copies of itself is quasi-injective). We prove that a left selfcogenerator ${}_R K$ over a ring R has a strictly linearly compact biendomorphism ring (which coincides with the Hausdorff completion $\hat{R}$ of R in its K -topology) iff K , regarded as a right module over its endomorphism ring, is Σ -quasi-injective.

Moreover we give an analogous version of characterization theorem above for strictly linearly compact rings. We prove that a ring R admits a left strictly linearly compact ring topology iff $R = \text{End}(K_A)$ where A is a ring and K_A is a right A -module which is Σ -strongly quasi-injective, with essential socle and whose cyclic right A -submodules are linearly compact in the discrete topology. Theorem 2.9 describes those Σ -strongly quasi-injective left modules over a ring R which are also Σ -strongly quasi-injective as right modules over their endomorphism ring. Theorem 2.12 gives a characterization of topologically left artinian rings in terms of Σ -quasi injective modules. Finally in Proposition 2.14 we prove that a commutative linearly topologized ring (R, τ) is a topologically artinian ring iff τ coincides with the Leptin topology τ^* of τ and the minimal cogenerator of the hereditary pretorsion class associated with τ is Σ -strongly quasi-injective.

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0.2. We conclude this introduction giving some notations and recalling some definitions and results of [5].

Let R be a ring. $R\text{-Mod}$ will denote the category of left R -modules and $\text{Mod-}R$ that of right R -modules. The notation ${}_R M$ will be used to emphasize that M is a left R -module. Morphisms between modules will be written on the opposite side to that of the scalars and the composition of morphisms will follow this convention. For every $M \in R\text{-Mod}$,

$E_R(M)$, or simply $E(M)$, will denote the injective envelope of M in $R\text{-Mod}$ and $\text{Soc}({}_R M)$, or simply $\text{Soc}(M)$, the socle of M .

$\mathbb{N}$ will denote the set of positive integers.

Let R be a ring and let $M \in R\text{-Mod}$. ${}_R M$ is *quasi-injective* (for short q.i.) if for every submodule $L \leqslant {}_R M$ and for every morphism $f: L \rightarrow {}_R M$, f extends to an endomorphism $\bar{f}$ of ${}_R M$. ${}_R M$ is a *selfcogenerator* if, for every $n \in \mathbb{N}$, given a submodule L of ${}_R M^n$ and an element $x \in M^n \setminus L$, there exists a morphism $f: {}_R M^n \rightarrow {}_R M$ such that $(L)f = 0$, and $(x)f \neq 0$. ${}_R M$ is called *strongly quasi-injective* (for short s.q.i.) if given any submodule B of ${}_R M$, a morphism $f: B \rightarrow {}_R M$ and an element $x \in M \setminus B$, f extends to an endomorphism $\bar{f}$ of ${}_R M$ such that $(x)\bar{f} \neq 0$. Clearly if ${}_R M$ is both quasi-injective and selfcogenerator, then ${}_R M$ is strongly quasi-injective. The converse is true as well (see [6] Corollary 4.5 and [3] Lemma 2.5).

Let ${}_R K_A$ be a bimodule. ${}_R K_A$ is *faithfully balanced* if $A \cong \text{End}({}_R K)$ and $R \cong \text{End}(K_A)$ canonically.

Let R be a ring and let $M \in R\text{-Mod}$. The *M -topology of R* is the left linear ring topology defined by taking as a basis of neighbourhoods of 0 in R the annihilators in R of finite subsets of M .

Recall that a linearly topologized left module M over a discrete ring R is said to be *linearly compact* (for short l.c.) if M is Hausdorff and if any finitely solvable system of congruences $x \equiv x_i$, where the x_i are closed submodules of ${}_R M$, is solvable. We will say that a left R -module is *linearly compact discrete* (for short l.c.d.) iff it is linearly compact in the discrete topology.

All ring and module topologies are assumed to be linear. By a *topological ring* (R, τ) we mean a ring R endowed with a left linear topology τ . $\mathcal{F}_\tau$ denotes the filter of open left ideals of R and $\mathcal{C}_\tau$ the class of τ -torsion modules:

$$\mathcal{C}_\tau = \{M \in R\text{-Mod} : \forall x \in M, \text{Ann}_R(x) \in \mathcal{F}_\tau\}$$

$\mathcal{C}_\tau$ is an hereditary pretorsion class of $R\text{-Mod}$.

For every left ideal I of R it is:

$$I \in \mathcal{F}_\tau \Leftrightarrow R/I \in \mathcal{C}_\tau$$

For every $M \in R\text{-Mod}$, $t_\tau(M)$ denotes the τ -torsion submodule of M . Analogous notations hold for a right linear topology.

The following two theorems are essentially the main results in [5]. We state them here in a slightly different way, which is more appropriate for our purposes.

0.3. THEOREM. *Let (R, τ) be a topological ring and let $(\hat{R}, \hat{\tau})$ the Hausdorff completion of (R, τ) . The following statements are equivalent:*

- (a) $(\hat{R}, \hat{\tau})$ is linearly compact.
- (b) If ${}_R K$ is a cogenerator of $\mathfrak{C}_\tau$ and $A = \text{End}({}_R K)$, then K_A is quasi-injective and ${}_{\hat{R}} K_A$ is faithfully balanced.
- (c) There exists a cogenerator ${}_R K$ of $\mathfrak{C}_\tau$ such that, setting $A = \text{End}({}_R K)$, K_A is quasi-injective and ${}_{\hat{R}} K_A$ is faithfully balanced.
- (d) Let ${}_R U$ be the minimal cogenerator of $\mathfrak{C}_\tau$, $T = \text{End}({}_R U)$. Then ${}_{\hat{R}} U_T$ is faithfully balanced.

PROOF. It is easy too see that every left R -module in $\mathfrak{C}_\tau$ has a natural structure of $\hat{R}$ -module, that every morphism between two modules of $\mathfrak{C}_\tau$ is an $\hat{R}$ -morphism and that $\mathfrak{C}_\tau = \mathfrak{C}_{\hat{\tau}}$ (see [6], Proposition 6.5). It follows that a module ${}_R K \in \mathfrak{C}_\tau$ is a cogenerator of $\mathfrak{C}_\tau$ iff it is a cogenerator of $\mathfrak{C}_{\hat{\tau}}$. Moreover the minimal cogenerator of $\mathfrak{C}_\tau$ and that one of $\mathfrak{C}_{\hat{\tau}}$ coincide. From these remarks and from the Main Theorem of [5] the proof follows.

0.4. THEOREM. *Let R be a ring, ${}_R K$ a strongly quasi-injective left R -module, $A = \text{End}({}_R K)$, $\hat{R}$ the Hausdorff completion of R in this K -topology. Then $\text{Soc}(K_A)$ is essential in K_A , the bimodule ${}_{\hat{R}} K_A$ is faithfully balanced and the following conditions are equivalent:*

- (a) K_A is strongly quasi-injective.
- (b) $\hat{R}$ is linearly compact in its K -topology and $\hat{R}$ separates points and submodules of K_A :
- (c) $\hat{R}$ is linearly compact in its K -topology and $\text{Soc}({}_R K)$ is essential in ${}_R K$.

Moreover, if these conditions hold, then A is linearly compact in its K -topology.

PROOF. If ${}_R K$ is s.q.i. and $A = \text{End}({}_R K)$, then by Proposition 6.10 of [6] $\text{Soc}(K_A)$ is essential in K_A and by Corollary 7.4 of [6], $\text{End}(K_A) \cong \hat{R}$. Moreover it is easy to see (see [6], Theorem 6.7) that ${}_R K$ is s.q.i. iff ${}_R \hat{K}$ is s.q.i. Apply now Theorem 10 of [5].

1. A further characterization of linearly compact rings.

1.1. LEMMA. Let R be a ring, ${}_R K$ a quasi-injective R -module, $A = \text{End}({}_R K)$. Let L be a submodule of ${}_R K$, $\{a_1, \dots, a_n\}$ a finite subset of A , and set

$$I = a_1 A + \dots + a_n A + \text{Ann}_A(I).$$

Then $I = \text{Ann}_A \text{Ann}_K(I)$.

PROOF. Straightforward.

1.2. LEMMA. Let R be a ring, ${}_R K$ a selfcogenerator, $A = \text{End}({}_R K)$. Let L be a finitely generated A -submodule of an A -module M_A which is cogenerated by K_A .

Then every morphism of L in K_A extends to a morphism of M in K_A .

PROOF. See Corollary 2.3 of [6].

Let R be a ring. A left ideal I of R is *completely irreducible* if R/I is an essential submodule of the injective envelope $E(S)$ of a left simple R -module S . A left R -module M is *finitely embedded* if its socle is finitely generated and it is essential in M .

1.3. LEMMA. Let R be a ring, ${}_R K$ a quasi-injective left R -module with essential socle, $A = \text{End}({}_R K)$. Let I be a right ideal of A such that $I = \text{Ann}_A \text{Ann}_K(I)$. If $\text{Ann}_K(I)$ is l.c.d., then for every right ideal H of A containing I it is $H = \text{Ann}_A \text{Ann}_K(H)$.

PROOF. Let H be a left ideal of A containing I and let $a \in \text{Ann}_A \text{Ann}_K(H)$. Then $\text{Ann}_K(a) \supseteq \text{Ann}_K(H)$ and $L = \text{Ann}_K(I) \supseteq \text{Ann}_K(H)$. Thus $\text{Ann}_L(a) \supseteq \text{Ann}_L(H) = \bigcap_{h \in H} \text{Ann}_L(h)$.

Since L is l.c.d., $L/\text{Ann}_L(a) \cong La$ is l.c.d. and since $\text{Soc}({}_R K)$ is essential in ${}_R K$, La is finitely embedded.

Thus, by Lemma 8 of [5], there exists $h_1, \dots, h_n \in H$ such that

$$(1) \quad \text{Ann}_L(a) \supseteq \bigcap_{i=1}^n \text{Ann}_L(h_i).$$

Since, by Lemma 1.1, the right ideal of A

$$J = (h_1A + \dots + h_nA) + \text{Ann}_A(L)$$

is closed in the K -topology of A and $J = \text{Ann}_A \text{Ann}_K(J)$, if $a \notin J$ there exists an $x \in \text{Ann}_K(J)$ such that $xa \neq 0$. Since $\text{Ann}_K(J) = \bigcap_{i=1}^n \text{Ann}_K(h_i) \cap L$, by (1) this is impossible.

Let R be a ring, ${}_R K$ a left R -module. We will say that ${}_R K$ is *finitely linearly compact discrete* (for short f.l.c.d.) if every cyclic (and hence every finitely generated) left R -submodule of ${}_R K$ is l.d.c.

1.4. LEMMA. *${}_R K$ is f.l.c.d. iff the Hausdorff completion $\hat{R}$ of R in its K -topology is linearly compact.*

PROOF. Straightforward.

The following proposition unifies some known results ([9], Corollary 2 page 342, [6] Theorem 9.4, [8] Proposition 3.4 a)) which were proved by the use of the same technique.

1.5. PROPOSITION. *Let R be a ring, ${}_R K \in R\text{-Mod}$ a selfcogenerator, $A = \text{End}({}_R K)$ and $L \leqslant {}_R K$. Then L is linearly compact discrete iff for any right ideal I of A and for any morphism $f: I \rightarrow K_A$ such that $\text{Ker}(f) \supseteq \text{Ann}_A(L)$, f extends to a morphism $\bar{f}: A \rightarrow K_A$. In particular ${}_R K$ is f.l.c.d. iff K_A is quasi-injective, while ${}_R K$ is l.c.d. iff K_A is injective.*

PROOF. Assume that L is l.c.d. and let I be a right ideal of A and $f: I \rightarrow K_A$ a morphism such that $\text{Ker}(f) \supseteq \text{Ann}_A(L)$. Then f induces, in a natural way, a morphism $\bar{f}: I/\text{Ann}_A(L) \rightarrow K_A$. Let $(H_j)_{j \in J}$ be the family of finitely generated A -submodules of $I/\text{Ann}_A(L)$. Since $A/\text{Ann}_A(L)$ is embeddable in $\prod_{x \in L} xA$ which is embeddable in K_A^L , by Lemma 1.2, for every $j \in J$, $\bar{f}|_{H_j}$ extends to a morphism $A/\text{Ann}_A(L) \rightarrow K_A$ so that there exists an $x_j \in \text{Ann}_K \text{Ann}_A(L) = L$ such that $\bar{f}|_{H_j}$ coincides with the left multiplication by x_j . Now, write $H_j = I_j + \text{Ann}_A(L) \text{Ann}_A(L)$ where I_j is a finitely generated ideal of A . Then

it is easy to prove that the system

$$(1) \quad X \equiv x_j \pmod{\text{Ann}_L(I_j)} \quad j \in J$$

is finitely solvable in L and hence—since L is l.c.d.—it is solvable in L . Let $x \in L$ be a solution of (1). Then $x - x_j \in \text{Ann}_L(I_j)$, for every $j \in J$, so that the left multiplication by x gives a morphism $A \rightarrow K_A$ which extends f .

Conversely assume that $L \leq_R K$ is such that for every right ideal I of A and any morphism $f: I \rightarrow K_A$ such that $\text{Ker}(f) \supseteq \text{Ann}_A(L)$, f extends to a morphism $A \rightarrow K_A$. Let

$$(2) \quad X \equiv x_j \pmod{(L_j)}$$

be a finitely solvable system of congruences in L .

Then the morphism

$$g: \sum_{j \in J} \text{Ann}_A(L_j) \rightarrow K_A$$

defined by setting $g\left(\sum_{j \in F} a_j\right) = \sum_{j \in F} x_j$, where F is a finite subset of J and $a_j \in \text{Ann}_A(L_j)$ for every $j \in F$, is well defined and $\text{Ker}(g) \supseteq \text{Ann}_A(L)$.

By hypothesis g extends to a morphism $A \rightarrow K_A$ so that there exists an $x \in K$ such that $x - x_j \in \text{Ann}_K \text{Ann}_A(L_j) = L_j$. Note that, since $x_j \in L$ and $L_j \leq L$, $x \in L$. Thus (2) is solvable in L .

The two last statements follows by the Baer's criterion for quasi-injectivity (see e.g. Proposition 6.6 of [6]) and that one for injectivity.

Let R be a ring. Recall that two left linear ring topologies on R are called *equivalent* if they have the same closed ideals.

1.6. THEOREM. *Let A be a ring. A admits a right linearly compact ring topology iff $A = \text{End}({}_R K)$ where R is a ring and ${}_R K$ is a finitely linearly compact discrete and strongly quasi-injective left R -module with essential socle. In this case*

- 1) A is linearly compact in the topology τ_* having the right annihilators of submodules of ${}_R K$ which are linearly compact discrete as a basis of neighbourhoods of 0.
- 2) τ_* is the finest topology in the equivalence class of the K -topology of A .
- 3) K_A is strongly quasi-injective with essential socle.

PROOF. Assume that (A, τ) is right linearly compact and let U_A be an injective cogenerator of $\mathfrak{C}_\tau$ with essential socle, $R = \text{End}(U_A)$. By Lemma 6 of [5] τ is equivalent to the U -topology τ_U of A so that (A, τ_U) is right linearly compact. Thus, by Theorem 0.4 ${}_R U$ is f.l.c.d. and s.q.i. with essential socle.

Conversely, assume R is a ring and ${}_R K$ is an f.l.c.d. and s.q.i. left R -module with essential socle. Thus by Lemma 1.4 and by Theorem 0.4 K_A is s.q.i. with essential socle and A is linearly compact in its K -topology τ . Now to prove statements 1) and 2) it is enough to show that τ_* is the finest topology in the equivalence class of τ . By Lemma 1.3 every open—and hence every closed—right ideal of τ_* is closed in τ . Let τ' be a topology equivalent to τ . To complete our proof let us show that every open right ideal I of τ' is open in τ_* . Since I is open in τ' , which is equivalent to τ , A/I is l.c.d. and moreover every right ideal of A containing I is closed in τ . In particular I is closed in τ so that, as K_A is s.q.i., it is $I = \text{Ann}_A(L)$ where $L = \text{Ann}_K(I)$. Let us prove that L is l.c.d. To do this we use Proposition 1.5. Let H be a right ideal of A and let $f: H \rightarrow K_A$ be a morphism such that $\text{Ker}(f) \supseteq I$. Then, since A/I is l.c.d. and K_A has essential socle, $\text{Im}(f)$ is finitely embedded. Thus, there exists a finite number $S_1, \dots, S_n$ of simple A -submodules of K_A such that $\text{Im}(f) \leq \bigoplus_{i=1}^n E(S_i)$ and hence f extends to a morphism $\bar{f}: A \rightarrow \bigoplus_{i=1}^n E(S_i)$. Let $x = \bar{f}(1)$. Then $x = x_1 + \dots + x_n$ where $x_i \in E(S_i)$ for every i , and $\bigcap_{i=1}^n \text{Ann}_A(x_i) = \text{Ann}_A(x) = \text{Ker}(\bar{f}) \supseteq \text{Ker}(f) \supseteq I$. Thus each $\text{Ann}(x_i)$ is closed in τ and, since it is completely irreducible, it is also open in τ . Thus $\text{Ker}(f) = H \cap \text{Ker}(\bar{f})$ is open in the relative topology on H of τ and, as K_A is q.i., f extends to a morphism $A \rightarrow K_A$. Hence, by Proposition 1.5., L is l.c.d.

1.7. COROLLARY (Theorem 3.10 of [9]). *A ring A is right linearly compact discrete if and only if $A = \text{End}({}_R K)$ where ${}_R K$ is l.c.d. and s.q.i. with essential socle.*

1.8. COROLLARY (Proposition 4.3 of [8]). *Let R be a ring and let ${}_R K$ be an s.q.i. R -module, $A = \text{End}({}_R K)$. The following conditions are equivalent:*

- (a) ${}_R K$ is l.c.d. and $\text{Soc}({}_R K)$ is essential in ${}_R K$.
- (b) K_A is an injective selfcogenerator.

(c) K_A is an injective cogenerator of $\text{Mod-}A$.

(d) A is linearly compact in the discrete topology which is equivalent to the K -topology of A and $\text{Soc}({}_R K)$ is essential in ${}_R K$.

PROOF. (a) $\Leftrightarrow$ (b) follows by Lemma 1.4, Theorem 0.4 and Proposition 1.5.

(c) $\Rightarrow$ (b) is trivial.

(a) $\Rightarrow$ (c) and (a) $\Rightarrow$ (d). Since we already proved (a) $\Leftrightarrow$ (b), K_A is injective. By Theorem 1.6 A is linearly compact in the discrete topology which is equivalent to the K -topology of A . Thus K_A is a cogenerator of $\text{Mod-}A$. (d) $\Rightarrow$ (a). By Theorem 1.6.

2. A characterization of strictly linearly compact rings.

2.1. LEMMA. *Let R be a ring and let $(M_i)_{i \in I}$, $M_i \in R\text{-Mod}$, be a family of selfcogenerators. If $\bigoplus_{i \in I} M_i$ is quasi-injective, then it is strongly quasi-injective.*

PROOF. One easily sees that it is enough to give a proof for I finite. In this case a proof similar to that one of Lemma 2.5 in [3] works.

Let R be a ring and let $M \in R\text{-Mod}$. We will say that ${}_R M$ is Σ -quasi-injective (for short Σ -q.i.) if every direct sum of copies of ${}_R M$ is q.i. Moreover we will say that ${}_R M$ is ω -quasi-injective (for short ω -q.i.) if ${}_R M^{(\mathbb{N})}$ is quasi-injective. The definitions of Σ -strongly quasi-injective (for short Σ -s.q.i.) and ω -strongly quasi-injective (for short ω -s.q.i.) module are given in an analogous way.

The following useful lemma is a trivial consequence of Lemma 2.1.

2.2. LEMMA *Let R be a ring and let $M \in R\text{-Mod}$ be a selfcogenerator. Then ${}_R M$ is Σ -s.q.i. (ω -s.q.i.) iff ${}_R M$ is Σ -q.i. (ω -q.i.).*

2.3. PROPOSITION. *Let R be a ring, ${}_R K \in R\text{-Mod}$ a selfcogenerator, $A = \text{End}({}_R K)$ and $L \leqslant {}_R K$. The following statements are equivalent:*

(a) ${}_R L$ is artinian.

(b) *For every right ideal I of A , for every set X , and for every morphism $f: I \rightarrow K_A^{(X)}$ such that $\text{Ker}(f) \geqslant \text{Ann}_A(L)$, f extends to a morphism $A \rightarrow K_A^{(X)}$.*

- (c) For every right ideal I of A and for every morphism $f: I \rightarrow K_A^{(\mathbb{N})}$ such that $\text{Ker}(f) \supseteq \text{Ann}_A(L)$, f extends to a morphism $A \rightarrow K_A^{(\mathbb{N})}$.

PROOF. (a) $\Rightarrow$ (b). Assume ${}_R L$ is artinian and let $f: I \rightarrow K_A^{(x)}$ as in (b). Proceeding in a similar way as in Proposition 1.4 define $\tilde{f}: I/\text{Ann}_A(L) \rightarrow K_A^{(x)}$, $(H_j)_{j \in J}$ and I_j and note that for $j \in J$, $\tilde{f}(H_j)$ is contained in an A -submodule M of $K_A^{(x)}$ which is a finite direct sum of copies of K_A . Thus since ${}_R K$ is a selfcogenerator, using Lemma 1.2 it is easy to prove that $\tilde{f}|_{H_j}$ extends to a morphism $A/\text{Ann}_A(L) \rightarrow M$. Hence, for every $j \in J$, there exists an $x_j \in K_A^{(x)}$ such that $\tilde{f}|_{H_j}$ coincides with the left multiplication by x_j . It is $x_j(\text{Ann}_A(L)) = 0$ and hence $x_j \in \text{Ann}_{K^{(x)}} \text{Ann}_A(L) = (\text{Ann}_K \text{Ann}_A(L))^{(x)} = L^{(x)}$. Hence the system

$$(1) \quad X = x_j \quad \text{mod } \text{Ann}_{L^{(x)}}(I_j) \quad j \in J$$

is finitely solvable in $L^{(x)}$. Let $j_0 \in J$ be such that $\text{Ann}_L(I_{j_0})$ is a minimal element of the non empty family $\{\text{Ann}_L(I_j)\}_{j \in J}$ of submodules of the the artinian left R -module L . Then x_{j_0} is a solution of the system (1) so that left multiplication by x_{j_0} gives a morphism $A \rightarrow K^{(x)}$ which extends f .

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (a) Assume $L_0 = L \supseteq L_1 \supseteq \dots \supseteq L_n \supseteq \dots$ is a strictly decreasing sequence of submodules of L and, for any $n \in \mathbb{N}$, let $y_n \in L_n \setminus L_{n+1}$. Since ${}_R K$ is a selfcogenerator and $A = \text{End}({}_R K)$, left multiplication by y_n defines a morphism $\mu_n: A \rightarrow K_A$ such that

$$\mu_n(\text{Ann}_A(L_n)) = 0 \quad \text{and} \quad \mu_n(\text{Ann}_A(L_{n+1})) \neq 0.$$

Let $I = \bigcup_{n \in \mathbb{N}} \text{Ann}_A(L_n)$ and let $\mu: A \rightarrow K^{\mathbb{N}}$ be the diagonal morphism of the μ_n 's. It is easy to check that $\mu(I) \subseteq K^{(\mathbb{N})}$ so that μ induces a morphism $\bar{\mu}: I \rightarrow K^{(\mathbb{N})}$. Note that, since $y_n \in L$ for every $n \in \mathbb{N}$, $\text{Ker}(\bar{\mu}) \supseteq \text{Ann}_A(L)$. By hypothesis $\bar{\mu}$ extends to a morphism $A \rightarrow K^{(\mathbb{N})}$. Thus there exists an $x \in K^{(\mathbb{N})}$ such that $\text{Im}(\bar{\mu}) \subseteq xA$. For every $n \in \mathbb{N}$, let $\pi_n: K^{(\mathbb{N})} \rightarrow K$ be the canonical projection. Then there is a $k \in \mathbb{N}$ such that $\pi_n \circ \bar{\mu} = 0$ for every $n \geq k$. Since $\pi_n \circ \bar{\mu} = \mu_n \neq 0$ for every n , we get a contradiction.

2.4. REMARK. Let R , ${}_R K$, L and A be as in Proposition 2.3 and let X be an infinite set. Assume that for every right ideal I of A and

for every morphism $f: I \rightarrow K_A^{(x)}$ such that $\text{Ker}(f) \supseteq \text{Ann}_A(L)$, f extends to a morphism $A \rightarrow K_A^{(x)}$. Then L satisfies (c) of Proposition 2.3 and hence it is artinian.

We will say that a left R -module M is *artinian finitely generated* (for short artinian f.g.) if every cyclic, and hence every finitely generated, left R -submodule of M is artinian.

The definition of *noetherian finitely generated* (for short noetherian f.g.) module is given in an analogous way.

Recall (see [2] and [7]) that a topological ring (R, τ) is a *topologically left artinian ring* (for short (R, τ) is a TA -ring) if, for every $I \in \mathcal{F}_\tau$, R/I is left artinian.

The definition of *topologically left noetherian ring* (for short TN -ring) is given in an analogous way (see [1]).

(R, τ) is a *strongly topologically left artinian ring* (for short (R, τ) is an STA -ring) iff it is both a TA -ring and a TN -ring (see [7]).

Finally recall that (R, τ) is *strictly linearly compact* (for short s.l.c.) iff it is a complete and Hausdorff TA -ring.

For technical convenience we state the following lemma whose proof is straightforward.

2.5. LEMMA. *Let R be a ring, let ${}_R K \in R\text{-Mod}$ and let τ be the K -topology of R . Denote by $(\hat{R}, \hat{\tau})$ the Hausdorff completion of (R, τ) . Then:*

- (a) (R, τ) is a TA -ring $\Leftrightarrow (\hat{R}, \hat{\tau})$ is a TA -ring $\Leftrightarrow {}_R K$ is artinian f.g.
- (b) (R, τ) is a TN -ring $\Leftrightarrow (\hat{R}, \hat{\tau})$ is a TN -ring $\Leftrightarrow {}_R K$ is noetherian f.g.
- (c) (R, τ) is an STA -ring $\Leftrightarrow (\hat{R}, \hat{\tau})$ is an STA -ring $\Leftrightarrow {}_R K$ is both artinian f.g. and noetherian f.g.
- (d) (R, τ) is strictly linearly compact $\Leftrightarrow (R, \tau) = (\hat{R}, \hat{\tau})$ and ${}_R K$ is artinian f.g. $\Leftrightarrow (R, \tau)$ is linearly compact and ${}_R K$ is artinian f.g.

Moreover, if ${}_R K$ is a selfcogenerator then:

- (e) K_A artinian f.g. $\Rightarrow {}_R K$ noetherian f.g.
- (f) K_A noetherian f.g. $\Rightarrow {}_R K$ artinian f.g.

REMARK. Since ${}_R K$ has a natural structure of $\hat{R}$ -module and since, with respect to this structure, every subgroup of K is an R -submodule iff it is an $\hat{R}$ -submodule, it is clear that all the statements of lemma above hold if one writes ${}_R K$ instead of ${}_R K$.

2.6. PROPOSITION. *Let (R, τ) be a linearly compact ring, ${}_R U$ an injective cogenerator of $\mathcal{G}_\tau$ with essential socle, $A = \text{End}({}_R U)$. Let τ_* be the finest topology in the equivalence class of τ . Then (R, τ_*) is strictly linearly compact iff every linearly compact discrete submodule of U_A is finitely generated. In this case every topology equivalent to τ coincides with τ .*

PROOF. Recall (see Theorem 1.6) that ${}_R U_A$ is faithfully balanced and that U_A is f.l.c.d. and s.q.i. with essential socle. Moreover τ_* has the left annihilators of submodules of U_A which are linearly compact discrete as a basis of neighbourhoods of 0.

Now if (R, τ_*) is strictly linearly compact then it is clear that there is not any topology in the equivalence class of τ_* coarser than τ_* . Thus $\tau_* = \tau$ and every topology equivalent to τ coincides with τ . In particular τ_* coincides with the U -topology of R . Hence if L is a linearly compact discrete submodule of U_A , there exists a finitely generated submodule H of U_A such that $\text{Ann}_R(L) \supseteq \text{Ann}_R(H)$ so that $L \subseteq H$. As (R, τ) is s.l.c., ${}_R U$ is artinian f.g. and hence, by Proposition 2.5 e) U_A is noetherian f.g. It follows that L is a finitely generated A -module.

Conversely, assume that every linearly compact discrete submodule of U_A is finitely generated. Then τ_* coincides with the U -topology of R and U_A is noetherian f.g. By Lemma 2.5 e) and a) (R, τ_*) is strictly linearly compact.

The following result was suggested to me by D. Dikranjan.

2.7. COROLLARY. *Let (R, τ) be a linearly compact ring and assume that the Jacobson radical of R , $J(R)$, is zero. Then (R, τ) is strictly linearly compact.*

Consequently (R, τ) is topologically isomorphic to a topological product $\prod_{\lambda \in A} \text{End}_{D_\lambda}(V_\lambda)$ where, for every $\lambda \in A$, V_λ is a vector space over the division ring D_λ and $\text{End}_{D_\lambda}(V_\lambda)$ is endowed with the finite topology.

PROOF. It is easy to see (cf. [5] Theorem 14) that for the minimal

injective cogenerator ${}_R U$ of $\mathfrak{C}_\tau$ we have, in this case,

$${}_R U = \bigoplus_{\lambda \in A} S_\lambda$$

where $(S_\lambda)_{\lambda \in A}$ is a system of representatives of the isomorphism classes of the left simple R -modules of $\mathfrak{C}_\tau$. Thus each S_λ is fully invariant in ${}_R U$ and hence, setting $A = \text{End}({}_R U)$, it is straightforward to prove that U_A is semisimple. Then every l.c.d. submodule of U_A has finite length.

By Proposition 2.6, (R, τ) is strictly linearly compact and, in particular, τ coincides with its Leptin topology. Thus the last assertion of the Corollary follows by the classical Leptin's result (see [5], Theorem 14).

In the following theorem we sum up all the main relations between the properties of a selfcogenerator ${}_R K \in R\text{-Mod}$ and those of the Hausdorff completion of R in its K -topology.

2.8. THEOREM. *Let R be a ring, ${}_R K \in R\text{-Mod}$ a selfcogenerator, $A = \text{End}({}_R K)$, $(\hat{R}, \hat{\tau})$ the Hausdorff completion of R in its K -topology. Then:*

- a) $(\hat{R}, \hat{\tau})$ is l.c. $\Leftrightarrow {}_R K$ is f.l.c.d. $\Leftrightarrow K_A$ is q.i.
- b) $(\hat{R}, \hat{\tau})$ is s.l.c. $\Leftrightarrow {}_R K$ is artinian f.g. $\Leftrightarrow K_A$ is Σ -q.i. $\Leftrightarrow K_A$ is ω -q.i.
- c) ${}_R K$ is l.c.d. $\Leftrightarrow K_A$ is injective.
- d) ${}_R K$ is artinian $\Leftrightarrow K_A$ is Σ -injective $\Leftrightarrow K_A$ is ω -injective.

If moreover ${}_R K$ is s.q.i. then:

- α) $(\hat{R}, \hat{\tau})$ is l.c. and $\text{Soc}({}_R K)$ is essential in ${}_R K \Leftrightarrow K_A$ is s.q.i.
- β) $(\hat{R}, \hat{\tau})$ is s.l.c. $\Leftrightarrow K_A$ is Σ -s.q.i. $\Leftrightarrow K_A$ is ω -q.i.
- γ) ${}_R K$ is l.c.d. with essential socle $\Leftrightarrow K_A$ is an injective cogenerator of $\text{Mod-}A$.
- δ) ${}_R K$ is artinian $\Leftrightarrow K_A$ is a Σ -injective cogenerator of $\text{Mod-}A \Leftrightarrow A_A$ is noetherian.

PROOF. Assume ${}_R K$ is a selfcogenerator. Then

- a) follows by Proposition 1.5,

- b) follows by Proposition 2.3 and by Proposition 6.6 of [6] after observing that, for a non-empty set X , the K -topology and the $K^{(X)}$ -topology of A coincide,
- c) follows by Proposition 1.5,
- d) follows by Proposition 2.3.

Assume now ${}_R K$ is s.q.i. Then

- α) follows by Theorem 0.4,
- β) follows by Theorem 0.4, Lemmata 2.5 and 2.2 and by b),
- γ) follows by Corollary 1.8.
- δ) The first equivalence of δ) follows by d) and γ). Now if $\text{Mod-}A$ has a Σ -injective cogenerator then it is well known that A is right noetherian. Conversely if A is right noetherian then, as ${}_R K$ is a selfcogenerator and $A = \text{End}({}_R K)$ it is straightforward to prove that ${}_R K$ is artinian.

REMARK. Statement c) is Theorem 9.4 of [6].

Statement d) could be deduced from Theorem 9.4 of [6] and Proposition 3 of [4]. Statement δ) was already proved, in a different way, in [7] (see [7], Lemma 4.12).

It is natural to ask when, in the hypothesis of Theorem 2.8, ${}_R K$ and K_A are both Σ -s.q.i. Following theorem gives an answer to this question.

2.9. THEOREM. *Let R be a ring and let ${}_R K \in R\text{-Mod}$ be an s.q.i. module. Then, in the notations of Theorem 2.6, the following statements are equivalent:*

- (a) *Every finitely generated submodule of ${}_R K$ has finite length.*
- (b) *Every finitely generated submodule of K_A has finite length.*
- (c) *$\hat{R}$ and A are both strictly linearly compact in their K -topologies.*
- (d) *${}_R K$ and K_A are both Σ -strongly quasi-injective.*

PROOF. (a) $\Rightarrow$ (b) Since ${}_R K$ is artinian f.g. by Theorem 2.8 K_A is Σ -s.q.i. Since ${}_R K$ is s.q.i., $\hat{R} = \text{End}(K_A)$ by Theorem 0.4. Thus (b) follows by Lemma 2.5 e) and f).

- (b) $\Rightarrow$ (a) Since ${}_R K$ is s.q.i., (a) follows by Lemma 2.5 e) and f).
 (a) $\Rightarrow$ (c) Since (a) $\Leftrightarrow$ (b), (c) follows by Theorem 0.4.
 (c) $\Rightarrow$ (d) As $\hat{R}$ is s.l.c. in its K -topology, by Theorem 0.4 ${}_R K_A$ is faithfully balanced and K_A is s.q.i. (d) follows now from Theorem 2.8.
 (d) $\Rightarrow$ (a) By Theorem 2.8 ${}_R K$ and K_A are both artinian f.g. Since ${}_R K$ is s.q.i. (a) follows by Lemma 2.5 e).

2.10. PROPOSITION. *Let R be a ring and let ${}_R K \in R\text{-Mod}$ be an f.l.c.d. and s.q.i. module with essential socle, $A = \text{End}({}_R K)$. Then A is s.l.c. in its K -topology $\Leftrightarrow K_A$ is artinian f.g. $\Leftrightarrow {}_R K$ is noetherian f.g. $\Leftrightarrow {}_R K$ is Σ -q.i. $\Leftrightarrow {}_R K$ is Σ -s.q.i.*

PROOF. By Lemma 1.4 and Theorem 0.4, ${}_R K_A$ is faithfully balanced, K_A is an s.q.i. module and A is linearly compact in its K -topology. Thus by Lemma 2.5 d) A is s.l.c. in this topology $\Leftrightarrow K_A$ is artinian f.g. Now, by Lemma 2.5 e) and f) K_A is artinian f.g. $\Leftrightarrow {}_R K$ is noetherian f.g. The other equivalences follow by Theorem 2.8.

As a corollary we get the following result which is analogous to Theorem 1.6.

2.11. COROLLARY. *Let A be a ring. A admits a right strictly linearly compact ring topology iff $A = \text{End}({}_R K)$ where R is a ring and ${}_R K$ an f.l.c.d. Σ -strongly quasi-injective left R -module with essential socle. In this case A is s.l.c. in its K -topology.*

PROOF. Follows by Proposition 2.10 and Theorem 1.6.

The following theorem, which is analogous to Theorem 0.3, characterizes topologically left artinian rings.

2.12. THEOREM. *Let (R, τ) be a topological ring, $(\hat{R}, \hat{\tau})$ the Hausdorff completion of (R, τ) and τ^* the Leptin topology of τ . Then, with the notations introduced in 0.2, the following statements are equivalent:*

- (a) (R, τ) is a TA -ring (i.e. $(\hat{R}, \hat{\tau})$ is strictly linearly compact).
 (b) If ${}_R K$ is a cogenerator of $\mathfrak{C}_\tau$ and $A = \text{End}({}_R K)$, then K_A is Σ -q.i. (ω -q.i.).
 (c) $\tau = \tau^*$ and there exists a cogenerator K of $\mathfrak{C}_\tau$ such that, setting $A = \text{End}({}_R K)$, K_A is Σ -q.i. (ω -q.i.).

(d) Let ${}_R U$ be the minimal cogenerator of $\mathfrak{C}_\tau$, $T = \text{End}({}_R U)$.

Then U_τ is Σ -s.q.i. (ω -s.q.i.) and $\tau = \tau^*$.

PROOF. (a) $\Rightarrow$ (b) Since (R, τ) is a TA -ring every module of $\mathfrak{C}_\tau$ is artinian f.g. Now (b) follows from Theorem 2.8.

(b) $\Rightarrow$ (a) Let ${}_R K = \bigoplus_{I \in \mathcal{F}_\tau} t_\tau(E(R/I))$. Then ${}_R K$ is a cogenerator of $\mathfrak{C}_\tau$. By Theorem 2.8 ${}_R K$ is artinian f.g. so that, for every $I \in \mathcal{F}_\tau$, R/I is left artinian.

(a) $\Rightarrow$ (c) Since (a) $\Leftrightarrow$ (b), this is trivial.

(c) $\Rightarrow$ (a) By Theorem 2.8 ${}_R K$ is artinian f.g. Since ${}_R K$ contains the minimal cogenerator of $\mathfrak{C}_\tau$, by Lemma 5 of [5], (a) follows.

(a) $\Rightarrow$ (d) Since (a) $\Leftrightarrow$ (b), (d) follows by Theorem 0.3 and Lemma 2.2.

(d) $\Rightarrow$ (c) is trivial.

The following proposition characterizes STA -rings.

2.13. PROPOSITION. *In the hypothesis of Theorem 2.1, let ${}_R U$ be the minimal cogenerator of $\mathfrak{C}_\tau$, $T = \text{End}({}_R U)$. The following statements are equivalent:*

- (a) (R, τ) is an STA -ring.
- (b) $\tau = \tau^*$ and ${}_R U$ and U_T are both Σ -s.q.i. (ω -s.q.i.).
- (c) ${}_R U$ is s.q.i. and T , endowed with its U -topology is a strongly topologically right artinian ring.
- (d) (R, τ) is a TA -ring and ${}_R U$ is Σ -s.q.i. (ω -s.q.i.).

PROOF. (a) $\Rightarrow$ (b) By Theorem 2.1 of [7] ${}_R U$ is s.q.i. Thus Theorem 2.7 applies.

(b) $\Rightarrow$ (c) By Theorem 2.9.

(c) $\Rightarrow$ (d) By Theorem 2.9.

(d) $\Rightarrow$ (b) By Theorem 2.8.

(b) $\Rightarrow$ (a) By Theorem 2.9.

Commutative TA -rings were extensively studied in [7] (see [7], section 6). Following proposition gives a further characterization of these rings.

2.14. PROPOSITION. *Let (R, τ) be a commutative topological ring, ${}_R U$ the minimal cogenerator of $\mathfrak{C}_\tau$. The following statements are equivalent:*

- (a) (R, τ) is a (strongly) topologically artinian ring.
- (b) $\tau = \tau^*$ and ${}_R U$ is Σ -s.q.i.
- (c) $\tau = \tau^*$ and (R, τ) is a topologically noetherian ring.

PROOF. First of all note that (R, τ) is a topologically artinian ring iff it is a strongly topologically artinian ring (see Proposition 3.9 of [7]). Thus

- (a) $\Rightarrow$ (b) follows from Proposition 2.13.
- (b) $\Rightarrow$ (c) Since ${}_R U$ is Σ -s.q.i. it is straightforward to prove that (R, τ) is a topologically noetherian ring (see Theorem 15 of [1]).
- (c) $\Rightarrow$ (a) Let $x \in {}_R U$, $x \neq 0$, $I = \text{Ann}_R(x)$. Then Rx is noetherian and thus the ring R/I is a commutative noetherian ring. Since Rx is a finitely embedded R/I -module, from the general theory of commutative noetherian rings, it follows that Rx is artinian.

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