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A remark on the spectra of rings with Gabriel dimension

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All rings considered in this Note will be commutative and unitary. For such a ring $\mathfrak{J}(A)$ will denote the lattice of ideals of A and Spec (A) the topological space (endowed with the Zariski topology) of prime ideals of A.

Năstăsescu [N] has proved that if A and B are two commutative Noetherian rings having the lattices $\mathfrak{J}(A)$ and $\mathfrak{J}(B)$ isomorphic, then Spec (A) and Spec (B) are homeomorphic topological spaces. He gives also a counter-example to show that the result fails for arbitrary commutative rings.

The aim of this Note is to present a very short proof of the above result in a more general setting: A or B has a Gabriel dimension, and $\mathfrak{J}(A)$ is isomorphic to $\mathfrak{J}(B)$ not necessarily as lattices but only as partially ordered sets.

Denote by Spec $(A)_0$ the set of maximal ideals of A, and if $\alpha > 0$ is an ordinal, denote by Spec $(A)_\alpha$ the set of prime ideals P of A such that each prime ideal Q of A properly containing P belongs to Spec $(A)_\beta$ for some $\beta < \alpha$. If A has a Gabriel dimension then it is well-known that Spec$(A) =$ Spec$(A)_\alpha$ for an ordinal α; in this case the least such α is called the classical Krull dimension of A, denoted by cl K dim A. If the ring A has a Gabriel dimension as defined in [G-R], this dimension will be denoted by G dim A.

LEMMA [G-R]. If the ring A has a Gabriel dimension, then A is Gabriel simple if and only if A is a domain. ■

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THEOREM. Let A and B two commutative rings with unit element such that there exists an isomorphism

$$\varphi\colon \mathfrak{I}(A) \to \mathfrak{I}(B)$$

of partially ordered sets. Suppose that A has a Gabriel dimension. If P is an ideal of A and $\alpha \geqslant 0$ is an ordinal, then:

(1) B has a Gabriel dimension and $G \dim B = G \dim A$.

(2) B is Gabriel simple if and only if A is Gabriel simple.

(3) $P \in \operatorname{Spec}(A)$ if and only if $\varphi(P) \in \operatorname{Spec}(B)$.

(4) $P \in \operatorname{Spec}(A)_\alpha$ if and only if $\varphi(P) \in \operatorname{Spec}(B)_\alpha$.

(5) $\operatorname{cl} K \dim B = \operatorname{cl} K \dim A$.

PROOF. (1) and (2) follow at once from the inductive noncategorical definition of the Gabriel dimension of a module, sketched in [G] and explicited in [L]. A proof that this definition is indeed the same with the original definition of the Gabriel definition by means of quotient categories may be found in [A].

(3) Suppose that $P \in \operatorname{Spec}(A)$; then A/P is a domain with Gabriel dimension, and by the Lemma, A/P is Gabriel simple. But the isomorphism

$$\varphi\colon \mathfrak{I}(A) \to \mathfrak{I}(B)$$

of posets induces an isomorphism

$$\varphi_P\colon \mathfrak{I}(A/P) \to \mathfrak{I}(B/\varphi(P))$$

of posets. According to (2), $B/\varphi(P)$ is Gabriel simple, and so, by Lemma, $B/\varphi(P)$ is a domain. Consequently $\varphi(P) \in \operatorname{Spec}(B)$.

(4) We proceed by transfinite induction. If $\alpha = 0$ and $P \in \operatorname{Spec}(A)_0$ then P is a maximal ideal of A, and then, since φ is an isomorphism of posets, $\varphi(P)$ has the same property; so $\varphi(P) \in \operatorname{Spec}(B)_0$. Let now $\alpha > 0$ and suppose that $P \in \operatorname{Spec}(A)_\alpha$. Denote $P' = \varphi(P)$ and let $Q' \in \operatorname{Spec}(B)$ with $Q' \supsetneqq P'$. Then $Q' = \varphi(Q)$ for same $Q \in \mathfrak{I}(A)$ with $Q \supsetneqq P$. By (3), $Q \in \operatorname{Spec}(A)$, hence $Q \in \operatorname{Spec}(A)_\beta$ for some $\beta < \alpha$.

By the induction hypothesis $\varphi(Q) = Q' \in \operatorname{Spec}(B)_\beta$, and consequently $P' = \varphi(P) \in \operatorname{Spec}(B)_\alpha$.

(5) follows immediately from (4). ■

COROLLARY 1. If the rings A and B are as in the theorem above then Spec (A) and Spec (B) are homeomorphic.

PROOF. The isomorphism φ: $\mathfrak{J}(A) \to \mathfrak{J}(B)$ of posets induces by the theorem a bijective map

$$\bar{\varphi}: \operatorname{Spec}(A) \to \operatorname{Spec}(B)$$

which is clearly bicontinuous. ■

COROLLARY 2 [N]. If A and B are commutative Noetherian rings with unit element for which there exists an isomorphism φ: $\mathfrak{J}(A) \to \mathfrak{J}(B)$ of lattices, then Spec (A) and Spec (B) are homeomorphic. ■

REMARKS. (1) The proof of Corollary 2 given in [N] uses essentially both the Noetherian condition on A and the fact that φ: $\mathfrak{J}(A) \to \mathfrak{J}(B)$ is a lattice isomorphism.

(2) The condition « A has a Gabriel dimension » is essential for Corollary 1 be true. To see this it is sufficient to consider the example given in [N]:

Let A be a rank 1 nondiscrete valuation ring with value group the additive group $\mathbb{R}$ of real numbers; if v: $A \to \mathbb{R} \cup \{\infty\}$ is the valuation on A and $I = \{x \in A | v(x) \geqslant \pi/2\}$, then I is a nonzero ideal of A, $\mathfrak{J}(A)$ and $\mathfrak{J}(A/I)$ are isomorphic lattices, but Spec $(A) = \{0, M\}$ and Spec $(A/I) = \{M/I\}$, where M is the unique maximal ideal of A. Note that A has not Gabriel dimension. ■

REFERENCES

[A] T. ALBU, *Gabriel dimension of partially ordered sets,* Bull. Math. Soc. Sci. Math. R. S. Roumanie (N.S.), **28** (76) (1984).

[G] R. GORDON, *Gabriel and Krull dimension,* in B. R. McDonald, A. R. Magid and K. C. Smith, *Ring theory: Proceedings of the Oklahoma Con-*

ference, Lecture Notes in Pure and Applied Mathematics 7, Marcel Dekker, Inc., New York and Basel, 1974.

[G-R] R. Gordon - J. C. Robson, *The Gabriel dimension of a module*, J. Algebra, **29** (1974), pp. 459-473.

[L] C. Lanski, *Gabriel dimension and rings with involution*, Houston J. Math., **4** (1978), pp. 397-415.

[N] C. Năstăsescu, *Collegamento tra il reticolo degli ideali di un anello e il suo spettro*, Ann. Univ. Ferrara, Sez. VII (N.S.), **19** (1974), pp. 87-92.

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