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On the B^λ Almost Periodic behaviour of Certain Arithmetical Convolutions.

PAOLO CODECÀ (*)

0. Introduction.

First of all let us remember the definition of B^λ almost periodic function. Let $g: [1, +\infty) \to \mathbf{C}$ be integrable on every bounded interval and let $\lambda > 0$. We say that g is a B^λ almost periodic function if, for every $\varepsilon > 0$, there is a trigonometric polynomial $P_\varepsilon(x) = \sum a_j \exp(2\pi i \alpha_j x)$ with $\alpha_j \in \mathbf{R}$ such that

$$\limsup_{x \to +\infty} \frac{1}{x} \int_1^x |g(t) - P_\varepsilon(t)|^\lambda dt \leqslant \varepsilon .$$

Consider now convolutions of the kind $\sum_{n \leqslant y(x)} (a(n)/n) f(x/n)$, where $f(x)$ is a bounded variation periodic function, $a(n)$ is a bounded sequence of real numbers and $y(x)$ is a suitable divergent function. The aim of the present paper is to prove that such convolutions are always B^λ almost periodic for every $\lambda > 0$ (1).

We note that convolutions of the type considered arise naturally in problems of number theory and Lambert summability of series.

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(1) A general result of B^2 almost periodicity for convolutions of the type considered with $y(x) = x$ and suitable $a(n)$ has already been obtained, by a different method, by the author (cf. [3], [4]).

Precisely, in connection with questions of summability, Hardy and Littlewood consider the two functions

$$P(x) = \sum_{n \leqslant x} \frac{1}{n} \cos \frac{x}{n} \tag{0.1}$$

$$Q(x) = \sum_{n \leqslant x} \frac{1}{n} \sin \frac{x}{n} \tag{0.2}$$

and prove that these functions are unbounded (cf. [8] p. 266).

Other examples of such convolutions are given by the remainder terms of summatory functions of important arithmetical functions.

To be precise put $\sigma_0(n) = \sigma(n)/n = \sum_{d|n} 1/d$ and consider $S_0(x) = \sum_{n \leqslant x} \sigma_0(n)$: it is not difficult to see that (cf. [12] p. 100)

$$S_0(x) = \frac{\pi^2}{6} x - \frac{1}{2} \lg x - \sum_{n \leqslant x} \frac{1}{n} \left(\left\{ \frac{x}{n} \right\} - \frac{1}{2} \right) + O(1) . \tag{0.3}$$

Now let $\varphi(n)$ be Euler's function: if we remember that $\varphi(n)/n = \sum_{d|n} \mu(d)/d$ we have immediately

$$S_1(x) = \sum_{n \leqslant x} \frac{\varphi(n)}{n} = \frac{6}{\pi^2} x - \sum_{n \leqslant x} \frac{\mu(n)}{n} \left(\left\{ \frac{x}{n} \right\} - \frac{1}{2} \right) + O(1) . \tag{0.4}$$

In both cases it is known that the remainder terms

$$R_1(x) = - \sum_{n \leqslant x} \frac{1}{n} \left(\left\{ \frac{x}{n} \right\} - \frac{1}{2} \right) \tag{0.5}$$

and

$$R_2(x) = - \sum_{n \leqslant x} \frac{\mu(n)}{n} \left(\left\{ \frac{x}{n} \right\} - \frac{1}{2} \right) \tag{0.6}$$

are unbounded ([2]).

The present paper is divided into two parts; in the first one we prove the following two theorems:

([2]) For further and more precise information about the functions $P(x)$, $Q(x)$, $R_1(x)$ and $R_2(x)$ see examples 1), 2) and 3) at the end of this paper.

THEOREM 1. Let $f(x)$ be a real function, periodic with period 1, of bounded variation on $[0, 1]$ and such that $\int_0^1 f(x)\,dx = 0$. Let $a(n)$ be a bounded sequence of real numbers and $y(x)$ a strictly increasing function defined on $[1, +\infty)$ satisfying the following conditions:

i) $\lim_{x\to+\infty} y(x) = +\infty$

ii) $\lim_{x\to+\infty} y(x)x^{-\varepsilon} = 0$ for every $\varepsilon > 0$.

If $g(x) = \sum_{n\leqslant y(x)} \big(a(n)/n\big) f(x/n)$ then the limit

$$M(g^k) = \lim_{x\to+\infty} \frac{1}{x}\int_1^x \big(g(t)\big)^k dt \tag{0.7}$$

exists and is finite for every positive integer k.

THEOREM 2. Let $g(x)$ be as in theorem 1: then $g(x)$ is a B^λ almost periodic function for every real positive λ.

In the second part of the paper we will apply theorems 1 and 2 to the functions $P(x)$, $Q(x)$, $R_1(x)$ and $R_2(x)$ defined in (0.1), (0.2), (0.5) and (0.6). To be precise we will prove the following

COROLLARY 1. Let $E(x)$ be any of the functions $P(x)$, $Q(x)$, $R_1(x)$ and $R_2(x)$: then in each case we can write

$$E(x) = E_1(x) + O(1) \tag{0.8}$$

where $E_1(x)$ is a B^λ almost periodic function for every $\lambda > 0$. We have also

$$\frac{1}{x}\int_1^x |E(t)|^\lambda dt = O(1) \text{ [3]} \tag{0.9}$$

for every $\lambda > 0$, though the function $E(x)$ is unbounded in each of the four cases (see examples 1, 2 and 3).

[3] Obviously the constant implied by O in (0.9) depends on λ.

The proof of corollary 1 is based on theorems 1 and 2 and on well known estimates for $P(x)$, $Q(x)$, $R_1(x)$ and $R_2(x)$. The estimates for $P(x)$ and $Q(x)$ were obtained by T. M. Flett (see (6.7) of example 1) by means of Van der Corput's method and those for $R_1(x)$ and $R_2(x)$ by A. Walfisz (see example 2 (7.5) and lemma 6 (8 11)) by means of Weyl's and Vinogradov's methods.

In what follows if $a, b, \ldots, l$ are positive integers the symbols $(a, b, \ldots, l)$ and $[a, b, \ldots, l]$ will indicate, respectively, the greatest common divisor and the least common multiple of $a, b, \ldots, l$.

If x is a real number $[x]$ will denote the integral part of x and $\{x\}$ the fractional part i.e. $\{x\} = x - [x]$. With $\mu(n)$ we will indicate the Möbius function and with $\mathbf{Z}'$ the set of all relative integers without zero.

Part I. We begin by proving some lemmata.

LEMMA 1. Let $a_1, \ldots, a_k$ be k positive integers. Put

$$\sum (a_1, \ldots, a_k) = \sum{}' \frac{1}{|m_1| \ldots |m_k|}$$

where the dash indicates that the sum is extended to all k-tuples $m_1, \ldots, m_k$ such that $m_j \in \mathbf{Z}'$ for $j = 1, \ldots, k$ and $\sum_{j=1}^{k} m_j a_j = 0$. Then we have

$$\sum (a_1, \ldots, a_k) \leqslant M$$

where the constant M is independent of the choice of $a_1, \ldots, a_k$ and depends only on the dimension k.

PROOF. Consider the function $f(x) = -\lg |2 \sin x/2|$ and its Fourier series (cf. [11] p. 93)

$$f(x) = -\lg \left| 2 \sin \frac{x}{2} \right| = \sum_{k=1}^{\infty} \frac{1}{k} \cos kx = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n \exp(inx) \tag{1.1}$$

where $c_n = (2|n|)^{-1}$.

It is easy to see, integrating by parts, that $f \in L^p[0, 2\pi]$ for every natural p. If we remember classical results on the product of Fourier series (see [1] vol. 1, p. 76) from (1.1) we obtain the expansion

$$(1.2)\qquad f(a_1x)f(a_2x)\dots f(a_kx)=\sum_{n=-\infty}^{\infty} b_n\exp(inx)$$

where

$$b_n=2^{-k}\sum{}'\frac{1}{|m_1||m_2|\dots|m_k|}$$

and the dash indicates that the sum is extended to all k-tuples m_1, $m_2\dots m_k$ such that $m_j\in\mathbf{Z}'$ and $\sum_{j=1}^{k} m_ja_j=n$. An empty sum means obviously that the corresponding b_n is zero.

From (1.2) we have

$$(1.3)\qquad \int_0^{2\pi} f(a_1x)f(a_2x)\dots f(a_kx)\,dx=\int_0^{2\pi}\left(\sum_{n=-\infty}^{\infty} b_n\exp(inx)\right)dx=$$

$$=2\pi b_0=2\pi 2^{-k}\sum(a_1,\dots,a_k)$$

where termwise integration is justified by classical theorems (cf. [1], vol. 1, p. 116).

Lemma 1 follows immediately from (1.3) if we apply repeatedly Schwarz inequality and observe that $\int_0^{2\pi}|f(a_jx)|^p\,dx=\int_0^{2\pi}|f(x)|^p\,dx$

Lemma 2. Let $n_1,\dots,n_k$ be k positive integers and put $N_j=\prod_{\substack{i=1\\ i\neq j}}^{k} n_i$ for $j=1,\dots,k$. Put

$$\sum(n_1,\dots,n_k)=\sum{}'\frac{1}{|m_1||m_2|\dots|m_k|}$$

where the dash indicates that the sum is extended to all k-tuples m_1, $m_2,\dots,m_k$ such that $m_j\in\mathbf{Z}'$ for $j=1,\dots,k$ and $\sum_{j=1}^{k} m_jN_j=0$. Then we have the estimate

$$\sum (n_1, \ldots, n_k) \leqslant M \frac{(n_1, N_1)(n_2, N_2) \ldots (n_k, N_k)}{n_1 n_2 \ldots n_k} . \quad [4]$$

where the constant M is independent of the choice of $n_1, n_2, \ldots, n_k$ and depends only on the dimension k.

PROOF. Let us consider the equation in integers

$$m_1 N_1 + m_2 N_2 + \ldots + m_k N_k = 0 \tag{2.1}$$

Put $\nu_j = (n_j, N_j)$: we have obviously $\nu_j | N_i$ for every $i, j = 1, \ldots, k$. This implies that every solution $m_1, m_2, \ldots, m_k$ of (2.1) must also be a solution of

$$m_1 \frac{N_1}{\nu_j} + m_2 \frac{N_2}{\nu_j} + \ldots + m_k \frac{N_k}{\nu_j} = 0 \tag{2.2}$$

for $j = 1, 2, \ldots, k$.

We note that $(n_j/\nu_j, N_j/\nu_j) = 1$ and, if $i \neq j$, $(n_j/\nu_j) | (N_i/\nu_j)$. From this we conclude that if (2.2) is satisfied then we must have $(n_j/\nu_j) | m_j$ for $j = 1, 2, \ldots, k$. But this implies that condition (2.1) can be written equivalently

$$\frac{n_1}{\nu_1} m_1' N_1 + \frac{n_2}{\nu_2} m_2' N_2 + \ldots + \frac{n_k}{\nu_k} m_k' N_k = 0 \tag{2.3}$$

and Lemma 2 follows from (2.3) and Lemma 1.

LEMMA 3. Let $n_1, n_2, \ldots, n_k$ be positive integers and let, as in the preceding lemma, $N_j = \prod_{\substack{i=1 \\ i \neq j}}^{k} n_i$ for $j = 1, 2, \ldots, k$.

Then the series

$$\sum \frac{(n_1, N_1)(n_2, N_2) \ldots (n_k, N_k)}{n_1^2 n_2^2 \ldots n_k^2}$$

where the sum is extended to all k-tuples $n_1, n_2, \ldots, n_k$ of positive integers, is convergent.

[4] We remember that (n_J, N_J) denotes the greatest common divisor of n_J and N_J.

PROOF. Put

$$L = \frac{n_1^2 n_2^2 \dots n_k^2}{(n_1, N_1)(n_2, N_2) \dots (n_k, N_k)}$$

then we have

$$L = n_i^2 \left[\prod_{\substack{j=1 \\ j \neq i}}^{k} \frac{n_j}{(n_j, N_j)}\right] \frac{N_i}{(n_i, N_i)} \tag{3.1}$$

and this obviously implies $n_i^2 | L$ for $i = 1, \dots, k$. From this follows that $[n_1^2, n_2^2, \dots, n_k^2] | L$ where $[n_1^2, n_2^2, \dots, n_k^2]$ is the least common multiple of $n_1, n_2, \dots, n_k$.

We conclude that

$$\sum \frac{(n_1, N_1)(n_2, N_2) \dots (n_k, N_k)}{n_1^2 n_2^2 \dots n_k^2} \leqslant \sum \frac{1}{[n_1^2, n_2^2, \dots, n_k^2]} \leqslant \sum_{m=1}^{\infty} \frac{(d(m))^k}{m^2} \tag{3.2}$$

where $d(m)$ is the number of divisors of m.

From (3.2) obviously follows lemma 3.

We now pass to the proof of theorem 1.

PROOF OF THEOREM 1. Let y^{-1} be the inverse function of y. We have (5)

$$\int_1^x g^k(t)\,dt = \sum_{m=1}^{[y(x)]-1} \int_{y^{-1}(m)}^{y^{-1}(m+1)} \left(\sum_{n \leqslant m} \frac{a(n)}{n} f\left(\frac{t}{n}\right)\right)^k dt + \int_{y^{-1}([y(x)])}^{x} \left(\sum_{n \leqslant y(x)} \frac{a(n)}{n} f\left(\frac{t}{n}\right)\right)^k dt = \tag{4.1}$$

$$= \sum_{1 \leqslant n_j \leqslant y(x)} \frac{a(n_1) a(n_2) \dots a(n_k)}{n_1 n_2 \dots n_k} \int_{\lambda}^{x} f\left(\frac{t}{n_1}\right) f\left(\frac{t}{n_2}\right) \dots f\left(\frac{t}{n_k}\right) dt$$

where $\lambda = \lambda(n_1, n_2, \dots, n_k) = \max\left(y^{-1}(n_1), \dots, y^{-1}(n_k)\right)$.

Let $f(x) \sim \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} A(n) \exp(inx)$ be the Fourier series of $f(x)$; put $N = n_1, n_2 \dots n_k$, $N_j = N/n_j$ for $j = 1, \dots, k$ and $t = Nu$ and consider

(5) Obviously we can suppose $y(1) = 1$ without loss of generality.

the integral

$$(4.2)\qquad \int_{\lambda}^{x} f\left(\frac{t}{n_1}\right) f\left(\frac{t}{n_2}\right) \dots f\left(\frac{t}{n_k}\right) dt = N \int_{\lambda/N}^{x/N} f(N_1 u) f(N_2 u) \dots f(N_k u)\, du\,.$$

We observe that

$$(4.3)\qquad f(N_1 u) f(N_2 u) \dots f(N_k u) \sim \sum_{m=-\infty}^{\infty} B(m) \exp(2\pi i m x)$$

where $B(m)$ is given by

$$(4.4)\qquad B(m) = \sum_{\substack{m_1 N_1 + m_2 N_2 + \dots + m_k N_k = m \\ m_j \in \mathbf{Z}'}} A(m_1) A(m_2) \dots A(m_k)$$

and an empty sum means $B(m) = 0$.

From (4.2) and (4.3) follows

$$(4.5)\qquad \int_{\lambda}^{x} f\left(\frac{t}{n_1}\right) f\left(\frac{t}{n_2}\right) \dots f\left(\frac{t}{n_k}\right) dt = B(0)(x - \lambda) + O(N)\,.$$

Obviously $B(0)$ depends on $n_1, \dots, n_k$ so that we write $B(0) = B(0, n_1, \dots, n_k)$. We will now prove that

$$(4.6)\qquad \lim_{x \to +\infty} \frac{1}{x} \int_{1}^{x} g^k(t)\, dt = \sum \frac{a(n_1) a(n_2) \dots a(n_k)}{n_1 n_2 \dots n_k} B(0, n_1, n_2, \dots, n_k)$$

where the sum is extended to all k-tuples $n_1, \dots, n_k$ of positive integers.

First of all we note that the series on the right of (4.6) is convergent. In fact if $f(x) \sim \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} A(n) \exp(2\pi i n x)$ is a bounded variation function we have $A(n) = O(1/n)$ (cf. [1] vol. 1, p. 71) so that from (4.4) follows

$$(4.7)\qquad B(0, n_1, \dots, n_k) = O\left(\sum_{\substack{m_1 N_1 + m_2 N_2 + \dots + m_k N_k = 0 \\ m_j \in \mathbf{Z}'}} \frac{1}{|m_1| |m_2| \dots |m_k|} \right).$$

If we remember lemma 2 we obtain from (4.7)

$$(4.8) \qquad B(0, n_1, \dots, n_k) = O\left(\frac{(n_1, N_1)(n_2, N_2) \dots (n_k, N_k)}{n_1 n_2 \dots n_k}\right).$$

The convergence of the series on the right of (4.6) follows immediately from (4.8) and from lemma 3 if we remember that the sequence $a(n)$ is bounded.

In the second place we prove that

$$(4.9) \qquad \lim_{x\to+\infty} \frac{1}{x} \sum_{1\leqslant n_j \leqslant y(x)} \frac{a(n_1)\,a(n_2)\dots a(n_k)}{n_1 n_2 \dots n_k} \cdot$$
$$\cdot B(0, n_1, n_2, \dots, n_k)\,\lambda(n_1, n_2, \dots, n_k) = 0$$

We have $\lambda(n_1, \dots, n_k) = \max\left(y^{-1}(n_1), \dots, y^{-1}(n_k)\right)$ so that

$$(4.10) \qquad \left|\frac{1}{x}\left(\sum_{1\leqslant n_j\leqslant y(x)} \frac{a(n_1)\,a(n_2)\dots a(n_k)}{n_1 n_2 \dots n_k} B(0, n_1, \dots, n_k)\,\lambda(n_1, \dots, n_k)\right)\right| =$$
$$= O\left[\left(\sum_{1\leqslant n_j\leqslant y(x/\lg x)} \frac{(n_1, N_1)(n_2, N_2)\dots(n_k, N_k)}{n_1^2 n_2^2 \dots n_k^2}\right)\frac{1}{\lg x} + \right.$$
$$\left. + \sum{}' \frac{(n_1, N_1)\dots(n_k, N_k)}{n_1^2 n_2^2 \dots n_k^2}\right]$$

where the dash in the second sum indicates that the sum is extended to all k-tuples $n_1, \dots, n_k$ of positive integers such that

$$y\left(\frac{x}{\lg x}\right) < \max_{1\leqslant j\leqslant k}(n_j) \leqslant y(x)\,.$$

From the convergence of this last series (see lemma 3) follows (4.9). In order to complete the proof of (4.6) we observe that

$$(4.11) \qquad \sum_{1\leqslant n_j\leqslant y(x)} \frac{a(n_1)\,a(n_2)\dots a(n_k)}{n_1 n_2 \dots n_k} N = O\left(y^k(x)\right) = o(x)$$

because $y(x) = o(x^\varepsilon)$ for every $\varepsilon > 0$.

From (4.1), (4.5), (4.9) and (4.11) follows (4.6) and this completes the proof of theorem 1.

We now pass to the proof of theorem 2.

PROOF OF THEOREM 2. Let N be a natural number and put $g_N(t) = \sum_{n\leqslant N}(a(n)/n)f(t/n)$. If m is a positive integer consider the integral

$$(5.1)\quad \int_1^x |g(t)-g_N(t)|^{2m}dt = \sum_{N<n_j\leqslant y(x)} \frac{a(n_1)\dots a(n_{2m})}{n_1\dots n_{2m}} \int_\lambda^x f\left(\frac{t}{n_1}\right)\dots f\left(\frac{t}{n_{2m}}\right)dt + O(1)$$

where the sum is extended to all the $2m$-tuples of positive integers $n_1, n_2, \dots, n_{2m}$ such that $N< n_j\leqslant y(x)$ for $j=1,\dots,2m$ and $\lambda = \lambda(n_1,\dots,n_{2m}) = \max\left(y^{-1}(n_1),\dots,y^{-1}(n_{2m})\right)$.

We observe that the sum on the right of (5.1) is equal to the sum on the right of (4.1) with the only difference that in equality (5.1) we have the limitations $N<n_j\leqslant y(x)$ for $j=1,2,\dots,2m$.

This evidently implies, if we remember the proof of theorem 1, that

$$(5.2)\qquad \lim_{x\to+\infty}\frac{1}{x}\int_1^x |g(t)-g_N(t)|^{2m}dt\leqslant\varepsilon$$

for every $N>N(\varepsilon)$.

Now let $0<\lambda<2m$ and put $p=2m/\lambda$ and $q=2m/(2m-\lambda)$ and $g(t)=g_N(t)=0$ for $0\leqslant t\leqslant 1$. By Hölder's inequality we have

$$(5.3)\qquad \int_0^x |g(t)-g_N(t)|^{\lambda}dt\leqslant x^{1/q}\left(\int_0^x |g(t)-g_N(t)|^{\lambda p}dt\right)^{1/p} = x^{(2m-\lambda)/2m}\left(\int_0^x |g(t)-g_N(t)|^{2m}dt\right)^{\lambda/2m}$$

from which follows

$$(5.4)\qquad \frac{1}{x}\int_0^x |g(t)-g_N(t)|^{\lambda}dt\leqslant\left(\frac{1}{x}\int_0^x |g(t)-g_N(t)|^{2m}dt\right)^{\lambda/2m}.$$

From (5.2) and (5.4) follows

$$\lim_{N\to+\infty}\left(\limsup_{x\to+\infty}\frac{1}{x}\int_0^x |g(t)-g_N(t)|^\lambda dt\right)=0 \tag{5.5}$$

that is $g(t)$ is a B^λ limit of purely periodic functions. If we remember that each $g_N(t)$ is a B^λ limit of the partial sums of its Fourier series (cf. [1] vol. 2, p. 138) we obtain the proof of theorem 2.

Part 2. Applications of the theorems 1 and 2 and proof of corollary 1.

EXAMPLE 1. In a paper of 1936 (cf. [8]) Hardy and Littlewood consider the two functions

$$P(x)=\sum_{n\leqslant x}\frac{1}{n}\cos\frac{x}{n}\quad\text{and}\quad Q(x)=\sum_{n\leqslant x}\frac{1}{n}\sin\frac{x}{n}\,.$$

They prove the following results

$$P(x)=\Omega(\lg\lg x) \tag{6.1}$$

$$Q(x)=\Omega\big((\lg\lg x)^{\frac12}\big) \tag{6.2}$$

In 1949 T. M. Flett (cf. [7]) obtained the following estimates

$$P(x)=O\big((\lg x)^{\frac34}(\lg\lg x)^{\frac12+\varepsilon}\big) \tag{6.3}$$

$$Q(x)=O\big((\lg x)^{\frac34}(\lg\lg x)^{\frac12+\varepsilon}\big)\,. \tag{6.4}$$

More recently these two functions were studied by Segal (cf. [10]) and by H. Delange (cf. [5]) who proved

$$Q(x)=\Omega\pm\big((\lg\lg x)^{\frac12}\big) \tag{6.5}$$

and by means of a result of Saffari and Vaughan

$$Q(x)=O\big((\lg x)^{\frac23}\big) \tag{6.6}$$

In the quoted paper T. M. Flett (cf. [7] p. 6, lemma 2) proves that

$$\sum_{n\leqslant t}\frac{1}{n}\exp\left(i\frac{t}{n}\right)=\sum_{n\leqslant y(t)}\frac{1}{n}\exp\left(i\frac{t}{n}\right)+O(1) \tag{6.7}$$

where $y(t)=\exp\left((\lg t)/(\lg\lg t)\right)$. The proof of (6.7) is based on Van der Corput's method for estimating exponential sums.

At this point we observe that the proof of corollary 1 for the functions $P(x)$ and $Q(x)$ is an immediate consequence of the estimate (6.7) and of theorems 1 and 2.

EXAMPLE 2. Consider, as in (0.3) of the introduction, the error term

$$R_1(x)=S_0(x)-\left(\frac{\pi^2}{6}x-\frac{1}{2}\lg x\right)=-\sum_{n\leqslant x}\frac{1}{n}\left(\left\{\frac{x}{n}\right\}-\frac{1}{2}\right)+O(1) \tag{7.1}$$

$R_1(x)$ cannot be bounded for the obvious reason that $\sigma(n)/n=\sum_{d|n}1/d$ is not bounded. The best known estimates for $R_1(x)$ were obtained by A. Walfisz (cf. [12] p. 88) by means of Weyl's and Vinogradov's methods for estimating exponential sums. Precisely Walfisz proved

$$R_1(x)=O\left(\frac{\lg x}{\lg\lg x}\right) \tag{7.2}$$

and later

$$R_1(x)=O\left((\lg x)^{\frac{2}{3}}\right)\,. \tag{7.3}$$

Walfisz also studied the mean square behaviour of $R_1(x)$ and obtained (cf. [13])

$$\int_0^x R_1^2(u)\,du=\left(\left(\frac{\gamma+\lg 2\pi}{2}\right)^2+\frac{5}{144}\pi^2\right)x+O\left(x^{\frac{1}{2}}\right)\,. \tag{7.4}$$

The estimate (7.2) was obtained by Walfisz by proving (cf. [12] p. 94) that

$$R_1(x)=-\sum_{n\leqslant y(x)}\frac{1}{n}\left(\left\{\frac{x}{n}\right\}-\frac{1}{2}\right)+O(1) \tag{7.5}$$

where $y(x)=x^{2/(X+4)}$ with $X=4[\frac{1}{4}\lg\lg x]$.

As in the preceding example the proof of corollary 1 for the function $R_1(x)$ follows immediately from (7.5) and from theorems 1 and 2.

EXAMPLE 3. Consider, as in (0.4) of the introduction, the error term

$$R_2(x)=\sum_{n\leqslant x}\frac{\varphi(n)}{n}-\frac{6}{\pi^2}x=-\sum_{n\leqslant x}\frac{\mu(n)}{n}\left(\left\{\frac{x}{n}\right\}-\frac{1}{2}\right)+O(1)\,. \tag{8.1}$$

It is known that $R_2(x)$ is not bounded. Precisely Pillai and Chowla (cf. [9]) proved that

$$R_2(x)=\Omega(\lg\lg\lg x) \tag{8.2}$$

and Erdös and Shapiro (cf. [6])

$$R_2(x)=\Omega\pm(\lg\lg\lg\lg x) \tag{8.3}$$

Chowla, following a method of Walfisz, studied the mean square value of $R_2(x)$ and obtained (cf. [2])

$$\int_1^x|R_2(u)|^2du\sim\frac{1}{2\pi^2}x\,. \tag{8.4}$$

In order to prove corollary 1 for the function $R_2(x)$ we need the following

LEMMA 5. The following estimate holds:

$$\sum_{x\exp(-\sqrt{\lg x})<n\leqslant x}\frac{\mu(n)}{n}\left\{\frac{x}{n}\right\}=O(1)\,. \tag{8.5}$$

PROOF. Put $B(n)=\sum_{k\leqslant n}\mu(k)$ and consider the sum

$$\sum_{y<n\leqslant x}\frac{\mu(n)}{n}\left\{\frac{x}{n}\right\}=\sum_{y<n\leqslant x}\frac{B(n)-B(n-1)}{n}\left\{\frac{x}{n}\right\}= \tag{8.6}$$

$$=\sum_{y<n\leqslant x-1}B(n)\left(\frac{1}{n}\left\{\frac{x}{n}\right\}-\frac{1}{n+1}\left\{\frac{x}{n+1}\right\}\right)+\frac{B([x])}{[x]}\left\{\frac{x}{[x]}\right\}-$$

$$-\frac{B([y])}{[y]+1}\left\{\frac{x}{[y]+1}\right\}=\sum_{y<n\leqslant x-1}B(n)\left(\frac{1}{n}\left\{\frac{x}{n}\right\}-\frac{1}{n+1}\left\{\frac{x}{n+1}\right\}\right)+O(1)\,.$$

But

$$\sum_{y<n\leqslant x-1} B(n)\left(\frac{1}{n}\left\{\frac{x}{n}\right\}-\frac{1}{n+1}\left\{\frac{x}{n+1}\right\}\right)= \tag{8.7}$$

$$=\sum_{y<n\leqslant x-1}\frac{B(n)}{n}\left(\left\{\frac{x}{n}\right\}-\left\{\frac{x}{n+1}\right\}\right)+$$

$$+\sum_{y<n\leqslant x-1} B(n)\left(\frac{1}{n}\left\{\frac{x}{n+1}\right\}-\frac{1}{n+1}\left\{\frac{x}{n+1}\right\}\right)=$$

$$=\sum_{y<n\leqslant x-1}\frac{B(n)}{n}\left(\left\{\frac{x}{n}\right\}-\left\{\frac{x}{n+1}\right\}\right)+$$

$$+\sum_{y<n\leqslant x-1}\frac{B(n)}{n(n+1)}\left\{\frac{x}{n+1}\right\}=\sigma_1(x)+\sigma_2(x)\,.$$

Let us remember the prime number theorem in the form $\sum_{n\leqslant x}\mu(n) = O\big(x\exp(-\sqrt{\lg x})\big)$ from this follows

$$\sigma_1(x)=O\left(\sum_{y<n\leqslant x-1}\left(\exp(-\sqrt{\lg y})\right)\left|\left\{\frac{x}{n}\right\}-\left\{\frac{x}{n+1}\right\}\right|\right)= \tag{8.8}$$

$$=O\left(\frac{x}{y}\exp(-\sqrt{\lg y})\right)$$

and

$$\sigma_2(x)=O(1)\,. \tag{8.9}$$

For $y=y(x)=x\exp(-\sqrt{\lg x})$ we have

$$\frac{x}{y}\exp(-\sqrt{\lg y})=O(1)\,. \tag{8.10}$$

From (8.6), (8.7), (8.8), (8.9) and (8.10) follows lemma 5.

Let us now remember the following result obtained by Walfisz (cf. [12] p. 142, lemma 5) by means of Vinogradov's method:

LEMMA 6. Let $X=[\alpha(\lg x)^{\frac{1}{3}}(\lg\lg x)^{-\frac{4}{3}}]$ where α is an absolute constant. We have the estimate

$$\sum_{x^{6/X}\leqslant n<x\exp(-\sqrt{\lg x})}\frac{\mu(n)}{n}\left(\left\{\frac{x}{n}\right\}-\frac{1}{2}\right)=O(1)\,. \tag{8.11}$$

At this point the proof of Corollary 1 for the function $R_2(x)$ follows immediately from lemmata 5 and 6 and from theorems 1 and 2.

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