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Double Resonance and Multiple Solutions for Semilinear Elliptic Equations.

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SUMMARY - In this paper we consider nonlinear elliptic problems under the assumption of Double Resonance in the sense of [2] and prove multiplicity results.

SUNTO - In questo articolo consideriamo problemi ellittici nonlineari sotto l'ipotesi di duplice risonanza nel senso di [2] e proviamo risultati di molteplicità.

1. Introduction.

In this paper we consider second order nonlinear problems like

$$\begin{aligned} \Delta u + g(u) &= 0 \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, Δ is the Laplacian and $g \in C^2(\mathbb{R})$. We suppose that the behaviour of g at infinity is such that we have double resonance in the sense of (2) below (see also Berestycki-De Figueiredo [2]). We prove in this case similar multiplicity results as in [1] and [3] where non-resonating problems are considered.

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2. In this section we state and prove the main theorems. We use X to denote the space $C_0^{0,\alpha}(\bar{\Omega})$, the space of functions which are Hölder continuous in Ω with exponent α and vanish on $\partial\Omega$. We use also $(\lambda_k)_{k\geqslant 1}$ to denote the eigenvalues of $-\Delta$, with zero data on the boundary and we denote by (φ_k) the corresponding eigenfunctions. We also recall the fact $(\varphi_k)_{k\geqslant 2}$, change sign in Ω. Also whenever we write

$$\lambda_{k-1} < \lambda_k < \lambda_{k+1}$$

we implicitly suppose λ_k is a simple eigenvalue.

THEOREM 1. Let $k > 1$ be such that $\lambda_{k-1} < \lambda_k < \lambda_{k+1}$ and let $g\colon \mathbf{R} \to \mathbf{R}$ be a C^2-function such that

$$\lim_{t\to+\infty} \frac{g(t)}{t} = \lambda_{k+1} \qquad (2)$$
$$\lim_{t\to-\infty} \frac{g(t)}{t} = \lambda_k .$$

Suppose further $g(0) = 0$ and $g'(0)$ is such that

$$\lambda_{k-1} < g'(0) < \lambda_k , \qquad (3)$$

then the equation (1) has at least one non-trivial solution.

THEOREM 2. With g as in Theorem 1, we assume further there exists α such that

$$\lambda_{k-1} < \alpha \leqslant \frac{g(s) - g(\sigma)}{s - \sigma} < \lambda_{k+1} , \qquad \forall s, \sigma \in \mathbf{R} \qquad (4)$$

then (1) has at least two non-trivial solutions.

REMARK 1. We will state results of similar type when $k = 1$ (i.e. $\lambda_k = \lambda_1$) in section 3.

PROOF of THEOREM 1. Define $g_n\colon \mathbf{R} \to \mathbf{R} (n \geqslant 1)$ by

$$g_n(t) = \lambda_k t + \left(1 - \frac{1}{n}\right)(g(t) - \lambda_k t) + \frac{\lambda_{k+1} - \lambda_k}{2n} t \qquad (5)$$

clearly,

$$g_n'(+\infty) \equiv \lim_{t\to+\infty} \frac{g_n(t)}{t} = \lambda_{k+1} - \frac{(\lambda_{k+1}-\lambda_k)}{2n}$$

and

$$g_n'(-\infty) \equiv \lim_{t\to-\infty} \frac{g_n(t)}{t} = \lambda_k + \frac{(\lambda_{k+1}-\lambda_k)}{2n}. \tag{6}$$

Hence $g_n'(+\infty)$ strictly increases to λ_{k+1} and $g_n'(-\infty)$ strictly decreases to λ_k as $n \to \infty$.

Consider now the equation

$$\begin{aligned} \Delta u + g_n(u) &= 0 \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{7}$$

clearly $u \equiv 0$ is a solution of (7).

Since

$$g_n'(0) = \lambda_k + \left(1 - \frac{1}{n}\right)(g'(0) - \lambda_k) + \frac{\lambda_{k+1}-\lambda_k}{2n},$$

and since $g_n'(0) \to g'(0)$ as $n \to \infty$, we have from (6) and (3), for large n

$$\lambda_{k-1} < g_n'(0) < \lambda_k < g_n'(\pm\infty) < \lambda_{k+1}. \tag{8}$$

Notice that (8) implies (7) has at least one non-trivial solution. This fact is easy to show using a degree argument. In fact, note that $u \equiv 0$ is a non-singular solution of (7) with Leray-Schauder index $(-1)^{k-1}$.

Defining $L_n u = (I - (-\Delta)^{-1}G_n)u$, where G_n is the Nemytski operator induced by g_n on X one can show that, from $g_n'(\pm\infty)$ belongs to $(\lambda_k, \lambda_{k+1})$, implies there exists an $R_n > 0$, such that

$$\deg(L_n, B_{R_n}, 0) = (-1)^k.$$

(See for ex: [1] page 637). This then implies (7) has at least one non-trivial solution by the additivity of Leray-Schauder degree.

Let (u_n) be a non-zero solution of (7) for each n, i.e. u_n satisfies

$$\begin{aligned} \Delta u_n + g_n(u_n) &= 0 \qquad \text{in } \Omega \\ u_n &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{9}$$

LEMMA 1. There exists a constant C independent of n, such that $\|u_n\|_{C^{0,\alpha}} \leqslant C$.

PROOF. Suppose $\|u_n\|_X \to \infty$, setting $v_n = u_n/\|u_n\|_X$, we have from (8)

$$\int_\Omega \nabla v_n \nabla \omega = \int_\Omega \frac{g_n(u_n)}{\|u_n\|}\, \omega = \int_\Omega \frac{g_n(u_n)}{u_n}\, v_n \omega \qquad \forall \omega \in D(\Omega)\,. \tag{10}$$

Notice that, since $(g_n(u_n)/u_n)$ is bounded, directly from (9) we have that $\|v_n\|_{C^{1,\alpha}}$ is bounded. Hence $v_n \to v$ in C^1-strongly (subsequence) and $v \not\equiv 0$ in Ω. This leads to, from (10),

$$\int_\Omega \nabla v \nabla \omega = \lambda_k \int_\Omega v\omega + \int_\Omega k_v(x)\, v(x)\, \omega(x) \tag{11}$$

where

$$k_v(x) \equiv \begin{cases} \lambda_{k+1} - \lambda_k & \text{if } v(x) > 0 \\ 0 & \text{if } v(x) < 0\,. \end{cases} \tag{12}$$

Hence from (11) we have,

$$\begin{aligned} \Delta v + \lambda_k v + k_v(x)\, v &= 0 \qquad \text{in } \Omega \\ v &= 0 \qquad \text{on } \partial\Omega\,. \end{aligned} \tag{13}$$

Notice that (13) is the same as equation (2.10) of [2]. One can argue as in [2] to show that such a $v \not\equiv 0$ cannot exist. We briefly sketch the arguments.

It can be shown that if v satisfies (13), then $v = v_0 + v_1$, where $v_0 \in \ker\,(\Delta + \lambda_k)$ and $v_1 \in \ker\,(\Delta + \lambda_{k+1})$. Then (12) implies that

$$(\lambda_k - \lambda_{k+1})\, v_1 + k_v(x)\, v = 0$$

i.e.

$$(\lambda_{k+1} - \lambda_k)\, v_1 - k_v(x)\, v_1 = k_v(x)\, v_0\,. \tag{14}$$

Now taking inner product with v_0 and v_1($L^2(\Omega)$ inner product), we obtain

(15) $$\langle[(\lambda_{k+1}-\lambda_k)-k_v(x)]v_1, v_0\rangle_{L^2(\Omega)}=\langle k_v(x)v_0, v_0\rangle_{L^2(\Omega)}$$

and

(16) $$\langle[\lambda_{k+1}-\lambda_k)-k_v(x)]v_1, v_1\rangle_{L^2(\Omega)}=\langle k_v(x)v_0, v_1\rangle_{L^2(\Omega)}\,.$$

Now from (12) and (16), we have

$$\langle k_v(x)v_0, v_1\rangle\geqslant 0\,,$$

whereas from (12) and (15), we have

$$\langle k_v(x)v_1, v_0\rangle\leqslant 0\,.$$

This implies

(17) $$\langle k_v(x)v_0, v_1\rangle=0\,.$$

Now (15) and (17) imply,

(18) $$\langle k_v(x)v_0, v_0\rangle=0$$

and (16) and (17) imply

(19) $$\langle[(\lambda_{k+1}-\lambda_k)-k_v(x)]v_1, v_1\rangle=0\,.$$

Let $A_0=\{x\in\Omega: v_0(x)\neq 0\}$ and $A_1=\{x\in\Omega: v_1(x)\neq 0\}$. From (12) and (18) it follows $k_v(x)\equiv 0$ on A_0. But if $A_0\neq\Phi$ i.e. if v_0 is not identical zero, then from unique continuation property of elements in $\ker(\Delta+\lambda_k)$ this is leads to $k_v(x)\equiv 0$ a.e. on Ω. But this is a contradiction, for $v=v_0+v_1$ being perpendicular to the first eigenfunction must change sing. Hence A_0 must be empty i.e. $v_0\equiv 0$. In this case $k_v(x)=k_{v_1}(x)$, but (19) now leads to a contradiction, for from (19)

$$k_v(x)\equiv(\lambda_{k+1}-\lambda_k)\,.$$

But this again cannot hold for v_1 changes sign in Ω. This proves the Lemma 1.

Proof of Theorem 1 completed. From the Lemma and from the equation (9), we have from the classical estimates that (u_n) is bounded in $C^{1,\alpha}(\bar{\Omega})$. Hence exists a subsequence which we shall not distinguish such that $u_n \to \bar{u}$ in X. We claim $\bar{u} \not\equiv 0$.

Suppose $\bar{u} \equiv 0$, then consider again,

$$\int_\Omega \nabla u_n \nabla \omega = \int_\Omega g_n(u_n)\,\omega \qquad \omega \in D(\Omega)$$

$$= \lambda_k \int_\Omega u_n \omega + (1-1/n)\int_\Omega g(u_n)\,\omega - (1-1/n)\,\lambda_k \int_\Omega u_n \omega + \frac{\lambda_{K+1}-\lambda_k}{2n}\int_\Omega u_n\,\omega\,.$$

Setting $v_n = u_n/\|u_n\|_x$, we have

$$\int_\Omega \nabla v_n \nabla \omega = \lambda_k \int_\Omega v_n \omega + (1-1/n)\int_\Omega \frac{g(u_n)}{u_n} v_n \omega - (1-1/n)\,\lambda_k \int_\Omega v_n \omega + \frac{\lambda_{k+1}-\lambda_k}{2n}\int_\Omega v_n \omega\,.$$

Again from the boundedness of $(g(t)/t)$, we have from the corresponding equation which (v_n) satisfies, that $\|v_n\|_{C^{1,\alpha}}$ is bounded. Hence $v_n \to v$ in C^1-strongly (subsequence) and $v \not\equiv 0$, because $\|v_n\|_X = 1$. Hence taking limit as $n \to \infty$, we have

$$\int_\Omega \nabla v\, \nabla \omega = \int_\Omega g'(0)\, v\omega$$

i.e.

$$(20) \qquad \begin{aligned} -\Delta v &= g'(0)\,v \qquad \text{in } \Omega \\ v &= 0 \qquad \text{on } \partial\Omega\,. \end{aligned}$$

But we know $\lambda_{k-1} < g'(0) < \lambda_k$ by hypothesis (3). Hence a contradiction. Hence $u_n \not\to 0$ in X. From $u_n \to \bar{u} \not\equiv 0$, now proves the existence of at least one nontrivial solution. This completes the proof of Theorem 1.

Before we proceed to prove the Theorem 2, we prove the following Lemma and also make certain observations which we shall use in the course of the proof of Theorem 2.

LEMMA 2. The equation (1), under the hypothesis of Theorem 1, cannot have a solution in the span of $(\varphi_i)_{i \geq k+1}$.

PROOF. First we show (1) cannot have a solution φ, φ satisfying $-\Delta\varphi = \lambda_{k+1}\varphi$.

Suppose this is not true, then this leads to

$$\begin{aligned} \Delta\varphi + g(\varphi) &= 0 \quad \text{in } \Omega \\ \varphi &= 0 \quad \text{on } \partial\Omega . \end{aligned} \tag{21}$$

Since φ is an eigenfunction corresponding to λ_{k+1}, (21) is the same as,

$$\lambda_{k+1}\varphi = g(\varphi) ,$$

But φ changes sign in Ω, hence for x in a neighbourhood of $\{x \in \Omega: \varphi(x) = 0\}$ we have

$$g(t) = \lambda_{k+1} t , \qquad t = \varphi(x) . \tag{22}$$

But this implies $g'(0) = \lambda_{k+1}$ in contradiction with our assumption (3).

Hence, if (1) has a solution in the span of $(\varphi_i)_{i\geqslant k+1}$, then it show be of the form $u = \varphi + \psi$, where φ is an eigenfunction corresponding to λ_{k+1} and ψ in the span of $(\varphi_i)_{i\geqslant k+2}$ and $\psi \not\equiv 0$ because of the above proven fact.

From $u = \varphi + \psi$ is a solution of (1), we have

$$\begin{aligned} -\Delta u &= g(u) \quad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega . \end{aligned}$$

Then taking L^2-inner product with u, we have

$$-\langle \Delta u, u\rangle = \langle g(u), u\rangle .$$

But from $u = \varphi + \psi$, $\psi \not\equiv 0$, we have

$$-\langle \Delta u, u\rangle > \lambda_{k+1}\|u\|^2_{L^2(\Omega)} \tag{23}$$

whereas, by (4), we have

$$\int_\Omega g(u)u = \int_\Omega \frac{g(u)}{u}\, u^2 \leqslant \lambda_{k+1}\|u\|^2_{L^2(\Omega)} \tag{24}$$

hence (23) and (24) lead to a contradiction. This proves the Lemma.

Before we proceed further we specify the notations we employ.

Domain of $(-\Delta) = D(-\Delta) = H^2(\Omega) \cap H_0'(\Omega)$, where $H^2(\Omega)$ and $H_0^1(\Omega)$ are the usual Sobolev Spaces.

$$H_1 = D(-\Delta) \cap \text{Span}\,\{\varphi_1, \ldots, \varphi_{k-1}\}$$

$$H_2 = D(-\Delta) \cap \text{Span}\,\{(\varphi_i)_{i>k+1}\}\,.$$

P will denote the orthogonal projection of $L^2(\Omega)$ on $\varphi_k^\perp$.
P_0 will denote the orthogonal projection of $L^2(\Omega)$ on φ_k.

Moreover, we remark that for g_n defined as in (4), i.e.,

$$g_n(t) = \lambda_k t + (1 - 1/n)(g(t) - \lambda_k t) + \frac{(\lambda_{k+1} - \lambda_k)}{2n} t\,,$$

we have

$$\lambda_{k-1} < \alpha - \frac{\lambda_{k+1} - \lambda_k}{2n} \leqslant \frac{g_n(t) - g_n(s)}{t - s} \leqslant \lambda_{k+1} - \frac{(\lambda_{k+1} - \lambda_k)}{2n} < \lambda_{k+1}\,, \tag{25}$$

for large n. Also from (6) and (8)

$$g_n'(+\infty) = \lambda_{k+1} - \frac{(\lambda_{k+1} - \lambda_k)}{2n}$$

and

$$g_n'(-\infty) = \lambda_k + \frac{(\lambda_{k+1} - \lambda_k)}{2n}$$

and

$$\lambda_{k-1} < g_n'(0) < \lambda_k\,.$$

Using a result proved in [3], the following Lemma can be proved regarding the solutions of the equation

$$\begin{aligned} \Delta u + g_n(u) &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega\,. \end{aligned} \tag{26}$$

Lemma 3. Equations (26) has at least two nontrivial solutions $u_i (i = 1, 2)$ such that if $P_0 u_i = s_i \varphi_k$, then one of the s_i's is strictly positive and the other strictly negative.

SKETCH OF THE PROOF. Notice that solving

$$(27)\qquad \begin{aligned} \Delta u + g_n(u) &= t\varphi_k \quad \text{in } \Omega,\ (t \in \mathbb{R})\\ \mathrm{J} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

is equivalent to solving

$$(28)\qquad \begin{aligned} &(a)\quad \Delta v + Pg_n(v + s\varphi_k) = 0 \qquad v \in \{\varphi_k\}^\perp \cap D(-\Delta)\\ &(b)\quad -\lambda_k s\varphi_k + P_0 g_n(v + s\varphi_k) = t\varphi_k\,. \end{aligned}$$

The idea of the proof is to fix $s \in \mathbb{R}$ and to show that (28. a) admits unique solution $v(s)$ and then analyse the following function F_n defined from $\mathbb{R}$ to $\mathbb{R}$:

$$(29)\qquad F_n(s) = \lambda_k s - \int_\Omega g_n(v(s) + s\varphi_k)\cdot\varphi_k\,.$$

We discuss F_n later and return now to the unique solvability of (28) for we need some of the details later. The unique solvability of (28 a) is established by a saddle point argument which we sketch.

Define $J_n\colon D(A) \to \mathbb{R}$
by

$$J_n(u) = -\tfrac{1}{2}\langle \Delta u, u\rangle - \int_\Omega G_n(u)\, u dx$$

where

$$G_n(t) = \int_0^t g_n(s)\,ds\,.$$

Notice

$$\langle J_n'(u), v\rangle = -\langle \Delta u, v\rangle - \int_\Omega g_n(u) v \qquad \forall u, v \in D(-\Delta)\,.$$

Setting $H_{n,s}(v_1, v_2) = J_n(v_1 + v_2 + s\varphi_k)$, $v_i \in H_i$ we notice that H_n, is strictly concave for fixed v_2 and of course with s fixed, which we have emphazised already. Also $H_n(v_1, \cdot)$ is strictly convex for fixed v_1. These follow because of the nature of g_n. In fact, with obvious notations, we have,

$$(30)\qquad \langle \partial_1 H_{n,s}(u_1, v_2) - \partial_1 H_{n,s}(v_1, v_2), u_1 - v_1\rangle \leqslant \leqslant \left[\lambda_{k-1} - \left(\alpha - \frac{(\lambda_{k+1} - \lambda_k)}{2n}\right)\right] \|u_1 - v_1\|^2_{L^2(\Omega)}$$

and

$$(31)\qquad \langle \partial_2 H_{n,s}(v_1, u_2) - \partial_2 H_{n,s}(v_1, v_2), u_2 - v_2\rangle \geqslant \geqslant \frac{(\lambda_{k+1} - \lambda_k)}{2n} \|u_2 - v_2\|^2_{L^2(\Omega)}.$$

The existence and uniqueness is now obtained through a saddle point argument.

Going back to $F_n(s)$ defined in (29), it can be shown (because of the fact $g_n'(\pm\infty) \in (\lambda_k, \lambda_{k+1})$) that both the following limits are strictly negative

$$(32)\qquad \lim_{s\to\pm\infty} \frac{F_n(s)}{s}.$$

However $F_n'(0)$ because of (8) becomes strictly positive. These facts together with $F_n(0) = 0$ imply the existence of two numbers $s_1^{(n)}$ and $s_2^{(n)}$, one positive and other negative such that $F_n(s_i^{(n)}) = 0$ $(i = 1, 2)$.

Proof of Theorem 2. In the light of our observation above, if we now set $u_n = v_1(s_n) + v_2(s_n) + s_n\varphi_k$ to be a nontrivial solution of (26), then assuming we have chosen u_n with s_n positive, we will show $s_n \not\to 0$. One can argue in an identical way to show $s_n \not\to 0$, if u_n is chosen such that $s_n < 0$. These then imply the existence of two non-trivial solutions from (1). Notice that all the arguments used in the proof of Theorem 1 hold in the setting in which we are working to prove Theorem 2. That is, $\|u_n\|_{L^2(\Omega)}$ is bounded and $u_n \not\to 0$ in $L^2(\Omega)$ do hold. Moreover the following estimate

$$(33)\qquad \|v_1(s_n)\|^2_{L^2(\Omega)} \leqslant C|s_n|.$$

holds, where C is a constant independent of n.

We now assume (33) and finish the proof of Theorem 2. Suppose $s_n \to 0$, then from (33) it follows that $v_1(s_n) \to 0$ in $L^2(\Omega)$. Now, since $\|u_n\|_{L^2(\Omega)}$ is bounded and since u_n cannot converge to zero in $L^2(\Omega)$, we have from

$$\Delta u_n + g_n(u_n) = 0 \quad \text{in } \Omega$$
$$u_n = 0 \quad \text{on } \partial\Omega$$

that $u_n \to u$, $u \neq 0$ in $H_0'(\Omega)$ (Note we use L^2-estimates and the fact $g_n(u_n)$ is bounded in $L^2(\Omega)$). But since $s_n \to 0$ and $v_1(s_n) \to 0$, we have that u belongs that the span of $(\varphi_i)_{i \geq k+1}$. Moreover u satisfies

$$\Delta u + g(u) = 0 \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega .$$

But this contradicts Lemma 2. Hence the theorem follows if we establish (33).

LEMMA 4. The estimate (33) holds.

PROOF. Setting

$$H_{s_n}(v_1, v_2) = J_n(v_1 + v_2 + s_n\varphi_k)$$

we define

$$T_{s_n}(v_1, v_2) = [-\partial_1 H_{s_n}(v_1, v_2),\ \partial_2 H_{s_n}(v_1, v_2)] . \tag{34}$$

Then it is easy to verify

$$\begin{aligned} &-\langle T_{s_n}(0,0), (v_1(s_n), v(s_n))\rangle = \\ &= \langle T_{s_n}(v_1(s_n), v_2(s_n)) - T_{s_n}(0,0), (v_1(s_n), v(s_n)) \rangle \geq \\ &\geq \left[\left(\alpha - \frac{(\lambda_{k+1}-\lambda_k)}{2n}\right) - \lambda_{k-1}\right] \|v_1(s_n)\|^2_{L^2(\Omega)} + \frac{(\lambda_{k+1}-\lambda_k)}{2n} \|v_2(s_n)\|^2_{L^2(\Omega)} . \end{aligned} \tag{35}$$

Note, here we have used $T_s(v_1(s_n), v_2(s_n)) = 0$, because they are the saddle points of $j_n(v_1 + v_2 + s_n\varphi_k)$ in our notation. Now observe that

$$\begin{aligned} &|\langle T_{s_n}(0,0), (v_1(s_n), v_2(s_n))\rangle| = \\ &= \left| \int_\Omega g_n(s_n\varphi_k)\, v_1(s_n) - \int_\Omega g_n(s_n\varphi_k)\, v_2(s_n) \right| \leq \\ &\leq \int_\Omega \left| \frac{g_n(s_n\varphi_k)}{s_n\varphi_k} \right| |s_n\varphi_k|\, |v_1(s_n)| + \int_\Omega \left| \frac{v_n(s_n\varphi_k)}{s_n\varphi_k} \right| |s_n\varphi_k|\, |v_2(s_n)| \end{aligned} \tag{36}$$

using (25) and Hölder's inequality, it follows that

$$(37)\qquad \lambda_{k+1}|s_n|[\|v_1(s_n)\| + \|v_2(s_n)\|]_{L^2(\Omega)} \geq$$
$$\geq\left[\alpha-\left(\frac{\lambda_{k-1}-\lambda_k}{2n}\right)-\lambda_{k-1}\right]\|v_1(s_n)\|^2_{L^2(\Omega)}.$$

Since we know $(\|v_1(s_n)\|+\|v_2(s_n)\|)$ is bounded (this is because $u_n = v_1(s_n) + v_2(s_n) + s_n\varphi_k$ and we have proved $\|u_n\|_{L^2(\Omega)}$ is bounded), we have from (37)

$$(38)\qquad \|v_1(s_n)\|^2_{L^2(\Omega)} \leq \frac{C|s_n|}{[(\alpha-(\lambda_{k+1}-\lambda_k)/2n)-\lambda_{k-1}]}.$$

But since

$$\left[\left(\alpha-\frac{\lambda_{k+1}-\lambda_k}{2n}\right)-\lambda_{k-1}\right]\to\alpha-\lambda_{k-1}>0\ \text{(by (4))}\qquad \text{as } n\to\infty$$

we have

$$\left[\left(\alpha-\frac{(\lambda_{k+1}-\lambda_k)}{2n}\right)-\lambda_{k-1}\right]\geq O_1>0\qquad \text{indipendent of } n.$$

This then implies the existence of a constant which we shall still denote by C such (33) holds. Hence the Lemma. This then completes the proof of Theorem 2.

3. In this section we consider the case of double resonance when $\lambda_k = \lambda_1$ the first eigenvalue. That is, we assume g is such that

$$(39)\qquad \begin{aligned}\lim_{t\to+\infty}\frac{g(t)}{t} &= \lambda_2\\ \lim_{t\to-\infty}\frac{g(t)}{t} &= \lambda_1.\end{aligned}$$

In this case one cannot hope to prove theorems like Theorem 1 and 2 proved above with identical assumptions. This due to the fact that, in proving that (13) has no trivial solution we have used the fact that any nontrivial solution of (13) must change sign in Ω. However if we consider the equation corresponding to (13), in the case when $\lambda_k = \lambda_1$, i.e.

$$(40) \qquad \begin{aligned} \Delta v + \lambda_1 v + k_v(x) v &= 0 \quad \text{in } \Omega \\ v &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

we cannot claim that non-trivial solutions have to change sign. This is because the eigenfunctions corresponding to λ_1 have a constant sign in Ω. To circumvent this difficulty we impose suitable Ladesman-Lazer type conditions as in [2]. In fact we will assume g to be such that:

$$(41) \qquad \begin{aligned} &(a) \quad g(t) \leqslant C_1 + \lambda_1 t \qquad \forall t \leqslant 0, \text{ for some constant } C_1 \\ &(b) \quad 0 > \int_\Omega g_- \varphi_1 \end{aligned}$$

where $\varphi_1 > 0$ is an eigenfunction of $-\Delta$ corresponding to the eigenvalue λ_1

$$(42) \qquad g_- = \lim_{t \to -\infty} \operatorname{Sup} [g(t) - \lambda_1 t] .$$

Once these assumptions on g are made, one can argue as in [2] to prove that (40) has nontrivial solution. Hence, in the light of these observations, it is clear that if suitable hypothesis is made on g then one can prove theorem similar to Theorem 1 and theorem 2, in the case when $\lambda_k = \lambda_1$ we now state the theorems without proof.

THEOREM 3. Suppose $g: \mathbb{R} \to \mathbb{R}$ is a C^2-function satisfying (39) and (41), suppose in addition $g(0) = 0$ and $g'(0)$ is such that

$$(43) \qquad 0 < g'(0) < \lambda_1 ,$$

then the equation (1) has at least one non-trivial solution.

THEOREM 4. Suppose g is as in Theorem 3 and suppose further there exists α so that

$$(44) \qquad 0 < \alpha \leqslant \frac{g(s) - g(\sigma)}{s - \sigma} \leqslant \lambda_2$$

then (1) has at least two non-trivial solutions.

REMARK 2. In [2] only existence theorems are proved in the case when g exhibit double resonance. No multiplicity results are proved. Moreover the existence results are proved using a degree argument which cannot be used in proving multiplicity results.

REMARK 3. Both in [1] and [3] multiplicity results are proved only in the case of non resonance our results show that identical multiplicity results as in [1] and [3] can be proved even if there is double resonance. In this sense our results are completely new and to the best of our knowledge do not seem to have been proved before.

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