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Affinities of Groups.

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There are many examples of non-isomorphic groups G and $\bar{G}$ with isomorphic coset lattices $\mathfrak{S}(G)$ and $\mathfrak{S}(\bar{G})$; the first were given by Baer [1] and Curzio [2]. These isomorphisms between $\mathfrak{S}(G)$ and $\mathfrak{S}(\bar{G})$ in general map cosets of one subgroup of G onto cosets of different subgroups in $\bar{G}$. So it is natural to ask for examples in which this does not happen or to study isomorphisms σ of coset lattices even satisfying

$$(*) \qquad (xH)^\sigma = x^\sigma H^\sigma \quad \text{and} \quad (Hx)^\sigma = H^\sigma x^\sigma$$

for all subgroups H of G and $x \in G$. This was done by Loiko who considered the two conditions in $(*)$ separately and who proved in [5] that in a group G generated by elements of infinite order any isomorphism from $\mathfrak{S}(G)$ to $\mathfrak{S}(\bar{G})$ satisfying one of these conditions is a group-isomorphism.

In a joint paper with W. Gaschütz [3] we studied permutations σ of finite groups satisfying $(*)$ and therefore of course also inducing automorphisms of $\mathfrak{S}(G)$ which we shall call normed affinities. It is the aim of the first three sections of this paper to show that the results of [3] carry over to normed affinities of infinite groups and also to normed affinities between different groups. Moreover, we introduce two characteristic subgroups—Am (G) and Reg (G)—of a group G

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which measure how near to isomorphisms the normed affinities of G are. Am (G) is the group generated by all the $((y^\sigma)^{-1}(x^\sigma)^{-1}(xy)^\sigma)^{\sigma^{-1}}$ where $x, y \in G$ and σ runs through the normed affinities from G to groups $\bar{G}$, and Reg (G) is the set of all $g \in G$ such that $(xg)^\sigma = x^\sigma g^\sigma$ and $(gx)^\sigma = g^\sigma x^\sigma$ for all $x \in G$ and all normed affinities σ. We show that Am (G) is «small»—it is locally cyclic and contained in the centre of G—and that Reg (G) is «big»—it contains all elements of infinite order of G and $G/\mathrm{Reg}\,(G)$ is π-closed for every set π of primes. Hence for many groups every normed affinity is an isomorphism.

In the remaining three sections we give a characterization of Am (G) and Reg (G) inside the group G in terms of p-collecting subgroups and p-collectors, p a prime, defined in § 5. For this in § 4 we present a number of constructions for normed affinities which by the way also yield examples of normed affinities between non-isomorphic groups. Finally, our characterizations of Am (G) and Reg (G) in § 6 show that these characteristic subgroups are invariant under index-preserving projectivities and therefore also under normed affinities.

1. Simple properties of affinities.

We give the exact definitions of the concepts mentioned in the introduction. For this let G and $\bar{G}$ be groups and $\mathfrak{S}(G)$ be the lattice of cosets of G, i.e. the set of all cosets of subgroups of G and the empty set with the set-theoretical inclusion as relation. In [3] we called two cosets X, Y of G *right parallel* (resp. *left parallel*) if there exists an element $g \in G$ such that $X = Yg$ (resp. $X = gY$).

1.1. Definition. (*a*) A map σ from G to $\bar{G}$ is called *normed* if $1^\sigma = 1$.

(*b*) A bijective map σ from G to $\bar{G}$ is an *S-isomorphism* if

$$X \in \mathfrak{S}(G) \Leftrightarrow X^\sigma \in \mathfrak{S}(\bar{G})$$

for every subset X of G, i.e. if σ induces an isomorphism from $\mathfrak{S}(G)$ onto $\mathfrak{S}(\bar{G})$.

(*c*) An S-isomorphism σ from G to $\bar{G}$ is an *affinity* if X right parallel to Y implies X^σ right parallel to Y^σ and X left parallel to Y implies X^σ left parallel to Y^σ for all X, $Y \in \mathfrak{S}(G)$.

It is easily shown (see [3], p. 188) that an affinity σ from G to $\bar{G}$ satisfies X right (left) parallel to Y if and only if X^σ right (left) parallel to Y^σ for all $X, Y \in \mathfrak{S}(G)$. Hence products and inverses of affinities are affinities. Since for every $g \in G$ the right translation $\varrho_g: x \to xg$ $(x \in G)$ and the left translation $\lambda_g: x \to g^{-1}x$ $(x \in G)$ are affinities ([3], p. 189), the existence of an affinity σ between G and $\bar{G}$ immediately leads to the existence of a normed affinity between G and $\bar{G}$. Therefore we shall restrict our attention to normed affinities throughout. We give a first characterization of normed affinities.

1.2. THEOREM. *The bijective map σ from G to $\bar{G}$ is a normed affinity if and only if,*

(*a*) *σ induces a projectivity from G to $\bar{G}$, i.e. satisfies $X \leqslant G$ iff $X^\sigma \leqslant \bar{G}$ for all subsets X of G, and*

(*b*) *$(xH)^\sigma = x^\sigma H^\sigma$ and $(Hx)^\sigma = H^\sigma x^\sigma$ for all $x \in G$, $H \leqslant G$.*

PROOF. If σ is a normed affinity, then (*a*) holds since the cosets containing 1 are exactly the subgroups of G. Furthermore, xH is left parallel to H and therefore $(xH)^\sigma$ is left parallel to H^σ, i.e. $(xH)^\sigma = yH^\sigma$ with $y \in \bar{G}$. Since $x^\sigma \in yH^\sigma$ and $H^\sigma \leqslant \bar{G}$, we get $x^\sigma H^\sigma = yH^\sigma = (xH)^\sigma$. Similarly, $(Hx)^\sigma = H^\sigma x^\sigma$.

If σ satisfies (*a*) and (*b*), then σ is a normed S-isomorphism. If $X = xH$ and $Y = yK$ are left parallel $(x, y \in G;\ H, K \leqslant G)$, then there exists $g \in G$ such that $gyK = gY = X = xH$. This implies $K = H$ and $Y^\sigma = y^\sigma H^\sigma$ is left parallel to $x^\sigma H^\sigma = X^\sigma$. Since every left coset is also a right coset we get in the same way that σ preserves right parallelism and hence is a normed affinity.

Every bijective map σ with $1^\sigma = 1$ from the cyclic group G of order 4 to the elementary abelian group $\bar{G}$ of order 4 satisfies (*b*) of 1.2; hence (*a*) is needed in this theorem. On the other hand, 1.2 shows that the normed affinities are exactly the S-isomorphisms satisfying condition (*) of the introduction. An easy consequence of 1.2 is the following result.

1.3. THEOREM. *If $\sigma: G \to \bar{G}$ is a normed affinity and $h \in G$ is an involution, then $(xh)^\sigma = x^\sigma h^\sigma$ and $(hx)^\sigma = h^\sigma x^\sigma$ for all $x \in G$. Hence if G is generated by involutions, every normed affinity of G is an isomorphism.*

PROOF. Let $H = \{1, h\}$. Then $Hx = \{x, hx\}$ and by 1.2 we have

$$\{x^\sigma, (hx)^\sigma\} = (Hx)^\sigma = H^\sigma x^\sigma = \{x^\sigma, h^\sigma x^\sigma\} .$$

Hence $(hx)^\sigma = h^\sigma x^\sigma$, and in the same way one gets the other equation. By a trivial induction then, if $h_1, \ldots, h_n \in G$ are involutions, $(h_1 \ldots h_n)^\sigma = h_1^\sigma \ldots h_n^\sigma$. So if G is generated by involutions and $x, y \in G$, there are involutions $h_i, k_j \in G$ such that $x = h_1 \ldots h_r$, $y = k_1 \ldots k_s$ and then

$$(xy)^\sigma = h_1^\sigma \ldots h_r^\sigma k_1^\sigma \ldots k_s^\sigma = x^\sigma y^\sigma .$$

Hence σ is an isomorphism.

In [3] we introduced the *amorphy* $\mathfrak{a}\colon G \times G \to \bar{G}$ of a map $\sigma\colon G \to \bar{G}$ defining

$$(1.4) \qquad \mathfrak{a}(x, y) = (y^\sigma)^{-1}(x^\sigma)^{-1}(xy)^\sigma \quad \textit{for all } x, y \in G .$$

Using $((xy)z)^\sigma = (x(yz))^\sigma$ one gets the associativity identities

$$(1.5) \qquad \mathfrak{a}(x, y)^{z^\sigma} \mathfrak{a}(xy, z) = \mathfrak{a}(y, z)\,\mathfrak{a}(x, yz) \quad \textit{for all } x, y, z \in G .$$

Normed affinities are characterized by the behaviour of their amorphies. In order to prove this we need the following result due to Loiko; for a proof see [5], p. 151 or [6], p. 293.

1.6. Lemma. *If $\sigma\colon G \to \bar{G}$ is a normed S-isomorphism and $u \in G$ has infinite order, then $(u^\sigma)^k = (u^k)^\sigma$ for all $k \in \mathbb{Z}$.*

1.7. Theorem. *The bijective map σ from G to $\bar{G}$ with amorphy $\mathfrak{a}$ is a normed affinity if and only if*

(a) σ induces a projectivity from G to $\bar{G}$ and

(b) $\mathfrak{a}(x, y) \in \langle x \rangle^\sigma \cap \langle y \rangle^\sigma$ for all $x, y \in G$.

If σ is a normed affinity, then $(xy)^\sigma = x^\sigma y^\sigma$ for all $x, y \in G$ with $o(x)$ or $o(y)$ infinite.

Proof. By 1.2, if σ is a normed affinity, (a) holds and for $x, y \in G$ we have $(xy)^\sigma \in (x\langle y \rangle)^\sigma = x^\sigma \langle y \rangle^\sigma$ and hence $\mathfrak{a}(x, y) \in \langle y \rangle^\sigma$ since $\langle y \rangle^\sigma$ is a subgroup of $\bar{G}$. Similarly, $(xy)^\sigma \in \langle x \rangle^\sigma y^\sigma$ and so $\mathfrak{a}(x, y)^{(y^\sigma)^{-1}} \in \langle x \rangle^\sigma$. Since $\langle y \rangle^\sigma$ is cyclic, $\mathfrak{a}(x, y)^{(y^\sigma)^{-1}} = \mathfrak{a}(x, y)$ and hence (b) holds.

Now assume that σ satisfies (a) and (b) and let $x \in G$, $H \leqslant G$. For $h \in H$ we have $(xh)^\sigma = x^\sigma h^\sigma \mathfrak{a}(x, h) \in x^\sigma H^\sigma$ since $\mathfrak{a}(x, h) \in \langle h \rangle^\sigma \leqslant H^\sigma \leqslant \bar{G}$. Hence

$$(1) \qquad (xH)^\sigma \subseteq x^\sigma H^\sigma \quad \textit{for all } x \in G,\ H \leqslant G .$$

If $g \in G$ is of infinite order and $U = \langle g \rangle$, then U^σ is infinite cyclic. Using induction on $|U:H|$ we show that $(xH)^\sigma = x^\sigma H^\sigma$ for all $x \in U$, $H \leqslant U$. This is true for $|U:H| = 1$; so assume it to be true for subgroups of smaller index than $|U:H| = n$ and let $U = x_1 H \dot\cup \ldots \dot\cup x_n H$ with $x_1 = x$. Then

$$U^\sigma = (x_1 H)^\sigma \dot\cup \ldots \dot\cup (x_n H)^\sigma \subseteq x_1^\sigma H^\sigma \cup \ldots \cup x_n^\sigma H^\sigma$$

and therefore $|U^\sigma : H^\sigma| \leqslant n$. By the induction assumption, $|U^\sigma : H^\sigma| = n$. Then the $x_i^\sigma H^\sigma$ are distinct and we have $(x_i H)^\sigma = x_i^\sigma H^\sigma$ for all i. In particular, $(xH)^\sigma = x^\sigma H^\sigma$. This shows that σ induces a normed $\mathcal{S}$-isomorphism (by 1.2 even a normed affinity) in U and by 1.6,

$$(2) \qquad (g^k)^\sigma = (g^\sigma)^k \quad \textit{for all } k \in \mathbb{Z}.$$

Now assume that $\mathfrak{a}(x, y) \neq 1$ for elements $x, y \in G$ with x or y of infinite order. Since $\mathfrak{a}(x, y) \in \langle x \rangle^\sigma \cap \langle y \rangle^\sigma$, then $i(x) = i(y) = \infty$ and $\mathfrak{a}(x, t) = (x^\sigma)^r = (y^\sigma)^s$ for $r, s \in \mathbb{Z}$. By (2), $x^r = y^s$ and applying the associativity identities 1.5 to $x, x^{-r}, x^r y$ we get

$$\mathfrak{a}(x, x^{-r})^{(x^r y)^\sigma} \mathfrak{a}(x^{1-r}, x^r y) = \mathfrak{a}(x^{-r}, x^r y)\, \mathfrak{a}(x, y) .$$

Again by (2), $\mathfrak{a}(x, x^{-r}) = 1$ and

$$(y^\sigma)^s = \mathfrak{a}(x, y) \in \langle x^r y \rangle^\sigma = \langle y^{s+1} \rangle^\sigma = \langle (y^\sigma)^{s+1} \rangle .$$

This contradiction shows that $\mathfrak{a}(x, y) = 1$ and proves

$$(3) \qquad (xy)^\sigma = x^\sigma y^\sigma \quad \textit{for all } x, y \in G \textit{ with } o(x) \textit{ or } o(y) \textit{ infinite}.$$

It remains to show that σ satisfies (b) of 1.2. So let $x \in G$, $H \leqslant G$, and $h \in H$. If $o(h)$ is infinite, then $x^\sigma h^\sigma = (xh)^\sigma \in (xH)^\sigma$ by (3), and if $o(h)$ is finite, then $|(x\langle h \rangle)^\sigma| = o(h) = |x^\sigma \langle h \rangle^\sigma|$ and $x^\sigma h^\sigma \in x^\sigma \langle h \rangle^\sigma = (x\langle h \rangle)^\sigma$ by (1). Hence $(xH)^\sigma = x^\sigma H^\sigma$. Since $(hx)^\sigma = h^\sigma x^\sigma \mathfrak{a}(h, x) = h^\sigma \mathfrak{a}(h, x) x^\sigma \in H^\sigma x^\sigma$, we get in the same way that $(Hx)^\sigma = H^\sigma x^\sigma$.

An easy consequence of 1.7 is that normed affinities map abelian groups onto abelian groups.

1.8. THEOREM. *Let $\sigma\colon G \to \bar{G}$ be a normed affinity.*

(*a*) *If $x, y \in G$ such that $xy = yx$, then $x^\sigma y^\sigma = y^\sigma x^\sigma$.*

(*b*) *If $H \leqslant G$ is abelian, then H^σ is abelian.*

(*c*) $Z(G)^\sigma = Z(\bar{G})$.

PROOF. If $o(x)$ or $o(y)$ is infinite, then $x^\sigma y^\sigma = (xy)^\sigma = (yx)^\sigma = y^\sigma x^\sigma$ by 1.7. Hence assume that $o(x)$ and $o(y)$ are finite. Then $H = \langle x, y\rangle$ is a finite abelian group, hence $H = \langle h_1, \ldots, h_r\rangle$ with $\langle h_i\rangle \cap \langle h_j\rangle = 1$ for $i \neq j$. Since σ induces a projectivity, $H^\sigma = \langle h_1^\sigma, \ldots, h_r^\sigma\rangle$ and $h_i^\sigma h_j^\sigma = (h_i h_j)^\sigma = (h_j h_i)^\sigma = h_j^\sigma h_i^\sigma$, since $\mathfrak{a}(h_i, h_j) = 1$ by 1.7 if $\mathfrak{a}$ is the amorphy of σ. So H^σ is abelian and $x^\sigma y^\sigma = y^\sigma x^\sigma$. This proves (*a*). Obviously, (*a*) implies (*b*) and also $Z(G)^\sigma \leqslant Z(\bar{G})$. Since σ^{-1} also is a normed affinity, we get the other inclusion.

It is clear that a normed affinity of a group induces a normed affinity in every subgroup. Theorem 1.2 shows that it also induces a normed affinity in every factor group.

1.9. LEMMA. *Let $\sigma\colon G \to \bar{G}$ be a normed affinity. If $N \trianglelefteq G$, then $N^\sigma \trianglelefteq \bar{G}$ and the map $\bar{\sigma}\colon G/N \to \bar{G}/N^\sigma$ with $(xN)^{\bar{\sigma}} = x^\sigma N^\sigma = (xN)^\sigma$ is a normed affinity. If $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ are the amorphies of σ resp. $\bar{\sigma}$, then $\bar{\mathfrak{a}}(xN, yN) = \mathfrak{a}(x, y)N^\sigma$ for all $x, y \in G$.*

PROOF. For $x \in G$ we have $xN = Nx$ and by 1.2 therefore $x^\sigma N^\sigma = (xN)^\sigma = (Nx)^\sigma = N^\sigma x^\sigma$. Since σ is surjective, $N^\sigma \trianglelefteq \bar{G}$. Furthermore 1.2 shows that $\bar{\sigma}$ is bijective and induces a projectivity from G/N to $\bar{G}/N^\sigma$. For $x, y \in G$ we have

$$\bar{\mathfrak{a}}(xN, yN) = (y^\sigma N^\sigma)^{-1}(x^\sigma N^\sigma)^{-1}(xy)^\sigma N^\sigma = \mathfrak{a}(x, y)N^\sigma ,$$

and since $\mathfrak{a}(x, y) \in \langle x\rangle^\sigma \cap \langle y\rangle^\sigma = \langle x^\sigma\rangle \cap \langle y^\sigma\rangle$,

$$\bar{\mathfrak{a}}(xN, yN) \in \langle x^\sigma N^\sigma\rangle \cap \langle y^\sigma N^\sigma\rangle = \langle xN\rangle^{\bar{\sigma}} \cap \langle yN\rangle^{\bar{\sigma}} .$$

We shall need the following simple result; for a proof see [6], p. 293.

1.10 LEMMA. *If $\sigma\colon G \to \bar{G}$ is a normed affinity, then also $\tau\colon G \to \bar{G}$ given by $g^\tau = ((g^{-1})^\sigma)^{-1}$ for $g \in G$ is a normed affinity.*

2. The amorphy and regular elements.

In this section we define two characteristic subgroups of G which in a way measure how near to isomorphisms the normed affinities of G are.

2.1 Definition. Let G and $\bar{G}$ be groups, $\sigma\colon G \to \bar{G}$ be a map, $\mathfrak{a}$ the amorphy of σ. The element $x \in G$ is called *right* (*left*) *regular* for σ if $(xg)^\sigma = x^\sigma g^\sigma$ $\big((gx)^\sigma = g^\sigma x^\sigma\big)$ or, equivalently, $\mathfrak{a}(x, g) = 1$ $\big(\mathfrak{a}(g, x) = 1\big)$ for all $g \in G$; x is called *regular* for σ if x is left and right regular for σ. $\operatorname{Reg}^r(\sigma)$ resp. $\operatorname{Reg}^l(\sigma)$ is the set of right resp. left regular elements for σ and $\operatorname{Reg}(\sigma) = \operatorname{Reg}^r(\sigma) \cap \operatorname{Reg}^l(\sigma)$ is the set of regular elements for σ.

2.2. Lemma. *If $\sigma\colon G \to \bar{G}$ is a normed affinity, then* $\operatorname{Reg}^r(\sigma)$ *and* $\operatorname{Reg}^l(\sigma)$ *are subgroups of G.*

Proof. Since $1^\sigma = 1$, $1 \in \operatorname{Reg}^r(\sigma)$. If $x, y \in \operatorname{Reg}^r(\sigma)$, then for all $g \in G$ we have

$$(xyg)^\sigma = x^\sigma (yg)^\sigma = x^\sigma y^\sigma g^\sigma = (xy)^\sigma g^\sigma ,$$

hence $xy \in \operatorname{Reg}^r(\sigma)$. If $o(x)$ is finite, this also implies that $x^{-1} \in \operatorname{Reg}^r(\sigma)$ and if $o(x)$ is infinite, $x^{-1} \in \operatorname{Reg}^r(\sigma)$ by 1.7. Hence $\operatorname{Reg}^r(\sigma)$ is a subgroup of G. The proof that $\operatorname{Reg}^l(\sigma) \leqslant G$ is similar.

2.3. Theorem. *Let G be a group and let* $\operatorname{Reg}^r(G)$ $\big(\operatorname{Reg}^l(G)\big)$ *be the set of elements of G which are right* (*left*) *regular for every normed affinity from G to any group $\bar{G}$.*

(*a*) $\operatorname{Reg}^r(G) = \operatorname{Reg}^l(G) =: \operatorname{Reg}(G)$.

(*b*) *If $\tau\colon G \to H$ is a normed affinity, then* $\operatorname{Reg}(G)^\tau = \operatorname{Reg}(H)$.

Hence $\operatorname{Reg}(G)$ *is a characteristic subgroup of G containing all elements of infinite order and all involutions of G.*

Proof. (*a*) By 2.2, $\operatorname{Reg}^r(G)$ and $\operatorname{Reg}^l(G)$ are subgroups of G. Let $x \in \operatorname{Reg}^r(G)$ and let $\sigma\colon G \to \bar{G}$ be a normed affinity. By 1.10, also $\tau\colon G \to \bar{G}$ with $g^\tau = \big((g^{-1})^\sigma\big)^{-1}$ for all $g \in G$ is a normed affinity. Since

$x^{-1} \in \mathrm{Reg}^r(\tau)$, we have for $g \in G$

$$\big((gx)^\sigma\big)^{-1} = \Big(\big((x^{-1}g^{-1})^{-1}\big)^\sigma\Big)^{-1} = (x^{-1}g^{-1})^\tau = (x^{-1})^\tau (g^{-1})^\tau$$
$$= (x^\sigma)^{-1}(g^\sigma)^{-1} = (g^\sigma x^\sigma)^{-1},$$

i.e. $(gx)^\sigma = g^\sigma x^\sigma$. Since σ and g were arbitrary, $x \in \mathrm{Reg}^l(G)$. So $\mathrm{Reg}^r(G) \leqslant \mathrm{Reg}^l(G)$, and the other inclusion we get similarly.

(*b*) Let $x \in \mathrm{Reg}(G)$ and let $\sigma\colon H \to \bar{H}$ be a normed affinity. For $h \in H$ there is $g \in G$ such that $g^\tau = h$, and since $x \in \mathrm{Reg}(G)$ and τ and $\tau\sigma$ are normed affinities, we have

$$(x^\tau h)^\sigma = (x^\tau g^\tau)^\sigma = (xg)^{\tau\sigma} = x^{\tau\sigma} g^{\tau\sigma} = (x^\tau)^\sigma h^\sigma,$$

i.e. $x^\tau \in \mathrm{Reg}^r(H) = \mathrm{Reg}(H)$. Hence $\mathrm{Reg}(G)^\tau \leqslant \mathrm{Reg}(H)$, but since τ^{-1} also is a normed affinity, we get the other inclusion. This proves (*b*).

Since every automorphism of G is a normed affinity, $\mathrm{Reg}(G)$ is a characteristic subgroup of G which contains all elements of infinite order (by 1.7) and all involutions (by 1.3) of G.

Now we prove our first main result on the amorphy of a normed affinity.

2.4. THEOREM. *If $\sigma\colon G \to \bar{G}$ is a normed affinity and $\mathfrak{a}$ the amorphy of σ, then $\mathfrak{a}(x, y) \in Z(\bar{G})$ for all $x, y \in G$.*

PROOF. Suppose that $\mathfrak{a}(x, y) \notin Z(\bar{G})$ for elements $x, y \in G$. By 1.7, x and y have finite order and we chose x, y with $\mathfrak{a}(x, y) \notin Z(\bar{G})$ so that $o(x)$ is as small as possible. If $x = x_1x_2$ with $o(x_1)$ and $o(x_2)$ less than $o(x)$, then the associativity identities 1.5 with x_1, x_2 and y would imply

$$\mathfrak{a}(x, y) = \big(\mathfrak{a}(x_1, x_2)^{y^\sigma}\big)^{-1}\mathfrak{a}(x_2, y)\,\mathfrak{a}(x_1, x_2y) \in Z(\bar{G}),$$

a contradiction. Hence $o(x) = p^n$, p a prime, $n \in \mathbb{N}$.

Let $z \in G$ such that $\mathfrak{a}(x, y)$ does not permute with z^σ. Then $\mathfrak{a}(x, y) \notin \langle z^\sigma \rangle$. Since $o(x^\sigma) = o(x) = p^n$, the subgroups of $\langle x^\sigma \rangle$ form a chain and by 1.7, we have

$$\langle \mathfrak{a}(z, x) \rangle \leqslant \langle x^\sigma \rangle \cap \langle z^\sigma \rangle < \langle \mathfrak{a}(x, y) \rangle \leqslant \langle x^\sigma \rangle \cap \langle y^\sigma \rangle .$$

Hence $\mathfrak{a}(z, x)^{y^\sigma} = \mathfrak{a}(z, x) = \mathfrak{a}(x, y)^{kp}$ with $k \in \mathbb{Z}$. The associativity identities for z, x and y yield

$$(*)\qquad 1 = \mathfrak{a}(x, y)^{1-kp}\mathfrak{a}(z, xy)\,\mathfrak{a}(zx, y)^{-1}.$$

Since $\mathfrak{a}(x, y)$ and $\mathfrak{a}(zx, y)$ are elements of $\langle y^\sigma\rangle$, they permute with y^σ and therefore also $\mathfrak{a}(z, xy)$ permutes with y^σ. Furthermore, $\mathfrak{a}(z, xy)$ permutes with $(xy)^\sigma = x^\sigma y^\sigma \mathfrak{a}(x, y) = x^\sigma(y^\sigma)^s$ $(s \in \mathbf{Z})$ and hence also with x^σ. Since $\mathfrak{a}(x, y)$ permutes with x^σ, by $(*)$ also $\mathfrak{a}(zx, y)$ does. But $\mathfrak{a}(zx, y)$ also permutes with $(zx)^\sigma = z^\sigma(x^\sigma)^t$ $(t \in \mathbf{Z})$ and hence with z^σ. Certainly, $\mathfrak{a}(z, xy)$ permutes with z^σ, by $(*)$ also $\mathfrak{a}(x, y)^{1-kp}$ does. Since $\mathfrak{a}(x, y) \in \langle x^\sigma\rangle$ is a p-element, finally $\mathfrak{a}(x, y)$ permutes with z^σ, a contradiction.

Theorem 2.4 is fundamental to our study of affinities. We give two immediate consequences.

2.5. Corollary. *If G is a group with $Z(G) = 1$, then every normed affinity of G is an isomorphism.*

This follows from 2.4 and 1.8 whereas the next result, the simplified associativity identities, of course follow from 1.5 and 2.4.

2.6. Corollary. *If $\sigma\colon G \to \bar{G}$ is a normed affinity and $\mathfrak{a}$ the amorphy of σ, then*

$$\mathfrak{a}(x, y)\,\mathfrak{a}(xy, z) = \mathfrak{a}(y, z)\,\mathfrak{a}(x, yz) \qquad \textit{for all } x, y, z \in G.$$

We define a second characteristic subgroup connected with the normed affinities of G.

2.7. Definition. If $\sigma\colon G \to \bar{G}$ is a normed affinity with amorphy $\mathfrak{a}$, we let $\mathrm{Am}(\sigma) = \langle \mathfrak{a}(x, y)^{\sigma^{-1}} | x, y \in G\rangle$. The *amorphy* $\mathrm{Am}(G)$ of G is the subgroup of G generated by all the $\mathrm{Am}(\sigma)$ where σ ranges over the normed affinities from G to any group $\bar{G}$.

2.8. Theorem. *If G is a group, $\mathrm{Am}(G)$ is a characteristic subgroup of G contained in the centre $Z(G)$ of G.*

Proof. That $\mathrm{Am}(G) \leqslant Z(G)$ follows from 2.4 and 1.8. Let $\sigma\colon G \to \bar{G}$ be a normed affinity with amorphy $\mathfrak{a}$ and let $\alpha \in \mathrm{Aut}(G)$. Then

$\tau = \alpha^{-1}\sigma$ is a normed affinity. If $\mathfrak{b}$ is the amorphy of τ, then for $x, y \in G$ we have

$$x^\sigma y^\sigma \mathfrak{a}(x, y) = (xy)^\sigma = (x^\alpha y^\alpha)\alpha^{-1}\sigma = x^\sigma y^\sigma \mathfrak{b}(x^\alpha, y^\alpha) .$$

Hence $(\mathfrak{a}(x, y)^{\sigma^{-1}})^\alpha = \mathfrak{b}(x^\alpha, y^\alpha)^{\tau^{-1}} \in \mathrm{Am}\,(G)$ and so $\mathrm{Am}\,(G)^\alpha \leqslant \mathrm{Am}\,(G)$. Thus $\mathrm{Am}\,(G)$ is a characteristic subgroup of G.

3. The structure of $\mathrm{Am}(G)$ and $G/\mathrm{Reg}(G)$.

We show in this section that $\mathrm{Am}\,(G)$ is locally cyclic and $G/\mathrm{Reg}\,(G)$ is π-closed for every set π of primes.

3.1 Lemma. *Let $\sigma: G \to \bar{G}$ be a normed affinity, $\mathfrak{a}$ the amorphy of σ, and let $g_1, g_2 \in G$ such that $g_1 g_2 = g_2 g_1$ and $\langle g_1 \rangle \cap \langle g_2 \rangle = 1$. Then $\mathfrak{a}(g_1 g_2, x) = \mathfrak{a}(g_1, x)\mathfrak{a}(g_2, x)$ and $\mathfrak{a}(x, g_1 g_2) = \mathfrak{a}(x, g_1)\mathfrak{a}(x, g_2)$ for all $x \in G$.*

Proof. By 2.6, we have

$$\mathfrak{a}(g_1, g_2)\mathfrak{a}(g_1 g_2, x) = \mathfrak{a}(g_2, x)\mathfrak{a}(g_1, g_2 x) ,$$

and by 1.7, $\mathfrak{a}(g_1, g_2) \in \langle g_1 \rangle^\sigma \cap \langle g_2 \rangle^\sigma = 1$. Hence

$$\mathfrak{a}(g_1 g_2, x) = \mathfrak{a}(g_2, x)\mathfrak{a}(g_1, g_2 x) . \tag{1}$$

Replacing g_2 by g_1 and g_1 by g_2, we get

$$\mathfrak{a}(g_1 g_2, x) = \mathfrak{a}(g_1, x)\mathfrak{a}(g_2, g_1 x) . \tag{2}$$

Since $\mathfrak{a}(g_2, x)$, $\mathfrak{a}(g_2, g_1 x) \in \langle g_2 \rangle^\sigma \cap Z(\bar{G})$ and $\mathfrak{a}(g_1, g_2 x)$, $\mathfrak{a}(g_1, x) \in \langle g_1 \rangle^\sigma \cap$ $\cap\, Z(\bar{G})$ and the product of $\langle g_1 \rangle^\sigma \cap Z(\bar{G})$ and $\langle g_2 \rangle^\sigma \cap Z(\bar{G})$ is direct, it follows that $\mathfrak{a}(g_1, g_2 x) = \mathfrak{a}(g_1, x)$ and hence by (1), $\mathfrak{a}(g_1 g_2, x) = \mathfrak{a}(g_1, x) \cdot$ $\cdot\, \mathfrak{a}(g_2, x)$. The other equation is proved similarly.

3.2. Corollary. *Let $\sigma: G \to \bar{G}$ be a normed affinity, $\mathfrak{a}$ the amorphy of σ, and let $x, y \in G$ be of finite order, $x = \prod_{p \in \mathrm{P}} x_p$, $y = \prod_{p \in \mathrm{P}} y_p$ with*

p-primary components x_p, y_p of x resp. y. Then

$$\mathfrak{a}(x, y) = \prod_{p\in\mathbf{P}} \mathfrak{a}(x_p, y) = \prod_{p\in\mathbf{P}} \mathfrak{a}(x, y_p) = \prod_{p\in\mathbf{P}} \mathfrak{a}(x_p, y_p)\,,$$

and $\mathfrak{a}(x_p, y_p)$ is the p-primary component of $\mathfrak{a}(x, y)$.

The main tool in the description of $\mathrm{Am}\,(G)$ and $G/\mathrm{Reg}\,(G)$ is the following result.

3.3. THEOREM. *Let G be a group. If $g, h \in G$ such that $\langle g\rangle \cap \langle h\rangle = 1 = \langle gh\rangle \cap \langle h\rangle$, then $h \in \mathrm{Reg}\,(G)$.*

PROOF. Let $\sigma\colon G \to \bar{G}$ be a normed affinity, $\mathfrak{a}$ the amorphy of σ and let $x \in G$. We have to show that $\mathfrak{a}(h, x) = 1$; then $h \in \mathrm{Reg}^r\,(G) = \mathrm{Reg}\,(G)$. By 1.7 and 3.2, it suffices to do this for all $x \in G$ of prime power order. So let $x \in G$ of prime power order and assume that $\mathfrak{a}(h, x) \neq 1$. Then $\langle h\rangle^\sigma \cap \langle x\rangle^\sigma \neq 1$ and hence by our assumption, $\langle g\rangle \cap \langle x\rangle = 1$ and $\langle gh\rangle \cap \langle x\rangle = 1$. Therefore, in the associativity identity

$$\mathfrak{a}(g, h)\,\mathfrak{a}(gh, x) = \mathfrak{a}(h, x)\,\mathfrak{a}(g, hx)$$

we have $\mathfrak{a}(g, h) = 1$ and $\mathfrak{a}(gh, x) = 1$ by 1.7, and hence

$$\mathfrak{a}(h, x) = \mathfrak{a}(g, hx)^{-1} \in \langle g\rangle^\sigma \cap \langle x\rangle^\sigma = 1\,,$$

a contradiction.

3.4. COROLLARY. *Let p be a prime. If g and h are p-elements of a group G with $g \notin \mathrm{Reg}\,(G)$ and $h \notin \mathrm{Reg}\,(G)$, then $\langle g\rangle \cap \langle h\rangle \neq 1$.*

PROOF. If $\langle g\rangle \cap \langle h\rangle = 1$, then by 3.3, we would have $\langle gh\rangle \cap \langle h\rangle \neq 1$. But then $\langle gh\rangle \cap \langle g\rangle = 1$, hence $\langle hg\rangle \cap \langle g\rangle = 1$, and by 3.3, $g \in \mathrm{Reg}\,(G)$, a contradiction.

3.5. LEMMA. *Let $\sigma\colon G \to H$ and $\tau\colon G \to K$ be normed affinities with amorphies $\mathfrak{a}$ and $\mathfrak{b}$ resp. and let x, y, u, v be p-elements of G, p a prime. Then either $\langle \mathfrak{a}(x, y)^{\sigma^{-1}}\rangle \leqslant \langle \mathfrak{b}(u, v)^{\tau^{-1}}\rangle$ or $\langle \mathfrak{b}(u, v)^{\tau^{-1}}\rangle \leqslant \langle \mathfrak{a}(x, y)^{\sigma^{-1}}\rangle$.*

PROOF. Assume that this is false. Then $N = \langle \mathfrak{a}(x, y)^{\sigma^{-1}}\rangle \cap \langle \mathfrak{b}(u, v)^{\tau^{-1}}\rangle$ is a proper subgroup of these two cyclic p-groups. By

2.8, $N \leqslant Z(G)$ and hence by 1.9, σ and τ induce normed affinities $\bar{\sigma}$ and $\bar{\tau}$ in G/N with amorphies $\bar{\mathfrak{a}}$ resp. $\bar{\mathfrak{b}}$ given in 1.9. Since x and u are p-elements, the subgroup lattices of $\langle x \rangle$ and $\langle u \rangle$ are chains and hence $\langle x \rangle \cap \langle u \rangle = N$, *i.e.* $\langle xN \rangle \cap \langle uN \rangle = 1$. By 3.4, $xN \in \mathrm{Reg}\,(G/N)$ or $uN \in \mathrm{Reg}\,(G/N)$. But $xN \in \mathrm{Reg}\,(G/N)$ implies $1 = \bar{\mathfrak{a}}(xN, yN) = \mathfrak{a}(x, y)N^{\sigma}$, i.e. $\mathfrak{a}(x, y)^{\sigma^{-1}} \in N$, and $uN \in \mathrm{Reg}\,(G/N)$, similarly, implies $\mathfrak{b}(u, v)^{\tau^{-1}} \in N$, a contradiction.

3.6. THEOREM. *If G is a group, then* $\mathrm{Am}\,(G) = \prod_{p \in \mathbf{P}} \mathrm{Am}\,(G)_p$ *where* $\mathrm{Am}\,(G)_p \simeq Z_{p^n}$ *with $n \in \mathbf{N} \cup \{0, \infty\}$. Hence* $\mathrm{Am}\,(G)$ *is locally cyclic.*

PROOF. By 1.7 and 2.8, $\mathrm{Am}\,(G)$ is a periodic abelian subgroup of G. Hence $\mathrm{Am}\,(G) = \prod_{p \in \mathbf{P}} \mathrm{Am}\,(G)_p$ where $\mathrm{Am}\,(G)_p$ is the p-component of $\mathrm{Am}\,(G)$. By 3.2, $\mathrm{Am}\,(G)_p$ is generated by the $\mathfrak{a}(x, y)^{\sigma^{-1}}$ with $x, y \in G$ p-elements, σ a normed affinity with amorphy $\mathfrak{a}$ from G to some group H. By 3.5, $\mathrm{Am}\,(G)_p \simeq Z_{p^n}$ with $n \in \mathbf{N} \cup \{0, \infty\}$ (see for example [7], p. 98).

3.7. LEMMA. *If π is a set of primes, g and h are π-elements of the group G and x is the π'-component of gh, then $x \in \mathrm{Reg}\,(G)$.*

PROOF. If gh has infinite order, then $x = 1 \in \mathrm{Reg}\,(G)$. So assume that $o(gh)$ is finite and let $gh = xy = yx$ where y is a π-element. As in the proof of 3.3 we have to show that $\mathfrak{a}(x, z) = 1$ for every q-element $z \in G$, q a prime, $\mathfrak{a}$ the amorphy of a normed affinity σ from G to some group $\bar{G}$. If $q \in \pi$, this follows from 1.7. So let $q \notin \pi$. Then in

$$\mathfrak{a}(g, h)\,\mathfrak{a}(gh, z) = \mathfrak{a}(h, z)\,\mathfrak{a}(g, hz)\,,$$

$\mathfrak{a}(g, h)$, $\mathfrak{a}(h, z)$ and $\mathfrak{a}(g, hz)$ are π-elements whereas $\mathfrak{a}(gh, z)$ is a π'-element in $Z(\bar{G})$. Hence $\mathfrak{a}(gh, z) = 1$ and by 3.1,

$$\mathfrak{a}(gh, z) = \mathfrak{a}(xy, z) = \mathfrak{a}(x, z)\,\mathfrak{a}(y, z) = \mathfrak{a}(x, z)\,,$$

since y is a π-element.

3.8. THEOREM. *If G is a group, then $G/\mathrm{Reg}\,(G)$ is π-closed for every set π of primes. In particular, if $G/\mathrm{Reg}\,(G)$ is finite, it is nilpotent.*

PROOF. If $g\,\mathrm{Reg}\,(G)$ and $h\,\mathrm{Reg}\,(G)$ are nontrivial π-elements in $G/\mathrm{Reg}\,(G)$, then $o(g)$ and $o(h)$ are finite by 1.7 and so we may assume that g and h are π-elements. By 3.7, then $gh\,\mathrm{Reg}\,(G)$ is a π-element in $G/\mathrm{Reg}\,(G)$.

4. Construction of normed affinities.

Most of our results so far show that normed affinities are near to being isomorphisms and that for many groups every normed affinity is in fact an isomorphism. In this section we want to show that on the other hand there do exist normed affinities which are not isomorphisms if the group has certain structural peculiarities suggested by 3.4. These results will be used in § 6 to give a characterization of $\mathrm{Am}\,(G)$ and $\mathrm{Reg}\,(G)$ in arbitrary groups G. Furthermore we present examples of normed affinities between non-isomorphic groups. Basic for all these examples is the following construction.

4.1. LEMMA. *Let G be a group and let N and M be subgroups of G such that*

(*a*) $1 < N \leqslant M < G$ *and*

(*b*) $N \leqslant \langle x \rangle$ *for all* $x \in G \setminus M$.

Then for every $1 \neq d \in N$ *the map* $\sigma = \sigma_d \colon G \to G$ *with* $x^\sigma = x$ *for all* $x \in M$ *and* $x^\sigma = xd^{-1}$ *for all* $x \in G \setminus M$ *is a normed affinity. For* $x, y \in G \setminus M$ *and the amorphy* $\mathfrak{a}$ *of* σ *we have*

$$\mathfrak{a}(x, y) = \begin{cases} d^2 & \text{if } xy \in M \\ d & \text{if } xy \notin M\,. \end{cases}$$

So $\mathrm{Reg}\,(\sigma) = M$ *except when* $|G : M| = 2$ *and* $d^2 = 1$ *in which case* σ *is an automorphism.*

PROOF. Obviously, σ is bijective. Furthermore, if $U \leqslant G$, then $U^\sigma = U$ if $U \leqslant M$ and $U^\sigma \leqslant UN = U$ if $U \nleqslant M$. Hence $U^\sigma = U$ for all subgroups U of G. Since G is generated by the elements $g \in G \setminus M$, $N \leqslant Z(G)$. Hence $\mathfrak{a}(x, y) = 1$ if one of x and y is contained in M, and if $x, y \in G\setminus M$, then

$$\alpha(x, y) = dy^{-1}dx^{-1}(xy)^\sigma = \begin{cases} d^2 & \text{if } xy \in M \\ d & \text{if } xy \notin M\,. \end{cases}$$

So always, $\alpha(x, y) \in N \leqslant \langle x\rangle \cap \langle y\rangle = \langle x\rangle^\sigma \cap \langle y\rangle^\sigma$ if $x, y \in G\setminus M$. By 1.7, σ is a normed affinity. The last assertion of the lemma follows immediately from the formula for $\alpha(x, y)$.

It is easy to construct groups with subgroups N, M having the properties of 4.1.

4.2. Example. Let p be a prime and let $G = \langle a\rangle \times B$ be a p-group with $o(a) = p^n > p^m = \exp(B)$, $m \geqslant 1$. Then $N = \langle a^{p^{n-1}}\rangle = \mho_{n-1}(G)$ and $M = \langle a^{p^{n-m}}\rangle \times B = \Omega_m(G)$ satisfy (a) and (b) of 4.1. If $1 \neq d \in N$, then σ_d is a normed affinity of G which is an automorphism only for $p = 2$ and $n = m + 1$.

If G is a cyclic p-group, (b) of 4.1 is automatically satisfied if (a) holds, and so 4.1 produces normed affinities of G. We shall, however, also need normed affinities of G of a different kind.

4.3. Lemma. *Let $p > 2$ be a prime, $G = \langle g\rangle$ a cyclic group of order p^n, $n \geqslant 1$, and let $H = \langle g^p\rangle$. Then $\sigma\colon G \to G$ defined by $(gx)^\sigma = g^2x$, $(g^2x)^\sigma = gx$ for all $x \in H$ and $y^\sigma = y$ for all $y \in G\setminus(gH \cup g^2H)$ is a normed affinity of G with $H \leqslant \operatorname{Reg}(\sigma)$ and $\alpha(g, g) = g^{-3}$ if α is the amorphy of σ.*

Proof. Since σ only permutes the cosets gH and g^2H and fixes every other element of G, it is bijective and induces a projectivity. Let $u, v \in G$. If one of these elements, u say, is contained in H, then v and uv are in the same coset of H and $\alpha(u, v) = 1$. If $u, v \in G\setminus H$, then $\alpha(u, v) \in \langle u\rangle^\sigma \cap \langle v\rangle^\sigma = G$. By 1.7, σ is a normed affinity, $H \leqslant \operatorname{Reg}(\sigma)$ and $\alpha(g, g) = g^{-2}g^{-2}g = g^{-3}$.

If $H \trianglelefteq G$ with G/H cyclic of order n and $\bar{H}$ has the same properties in $\bar{G}$, then it is quite clear how to extend a map from H to $\bar{H}$ to a map from G to $\bar{G}$. We describe how to construct normed affinities in this way.

4.4. Lemma. *Let $G = H\langle g\rangle$, $H \trianglelefteq G$, $|G:H| = n > 1$ and $\bar{G} = \bar{H}\langle \bar{g}\rangle$, $\bar{H} \trianglelefteq \bar{G}$, $|\bar{G}:\bar{H}| = n$, let $\tau\colon H \to \bar{H}$ be an isomorphism and define $\sigma\colon G \to \bar{G}$ by $(g^ix)^\sigma = \bar{g}^ix^\tau$ for $i \in \{0, \ldots, n-1\}$, $x \in H$. Then* (*a*) *σ is an isomorphism if and only if* $(x^\tau)^{\bar g} = (x^g)^\tau$ *for all* $x \in H$ *and* $(g^n)^\tau = \bar g^n$.

(*b*) *σ is a normed affinity with* $H = \mathrm{Reg}(\sigma)$ *if and only if*

(1) $(x^\tau)^{\bar g} = (x^g)^\tau$ *for all* $x \in H$,

(2) $(g^n)^\tau = \bar g^n t$ *with* $1 \neq t \in \bigcap_{y \in \bar G \setminus \bar H} \langle y \rangle$ *and* $t^{\tau^{-1}} \in \bigcap_{z \in G \setminus H} \langle z \rangle$,

(3) $o(g)$ *and* $o(\bar g)$ *are finite.*

PROOF. We only prove (*b*) since the proof of (*a*) is quite similar and rather obvious. So let σ be a normed affinity with amorphy $\mathfrak{a}$ and $H = \mathrm{Reg}(\sigma)$. Then for $x \in H$

$$x^\sigma g^\sigma = (xg)^\sigma = (gx^g)^\sigma = g^\sigma (x^g)^\sigma\,,$$

and since $g^\sigma = \bar g$, this proves (1). Furthermore, for $i \in \{1, \dots, n-1\}$ we have

$$(g^n)^\tau x^\tau = (g^n x)^\sigma = (g^{n-i})^\sigma (g^i x)^\sigma \mathfrak{a}(g^{n-i}, g^i x) = \bar g^{n-i} \bar g^i x^\tau \mathfrak{a}(g^{n-i}, g^i x)\,.$$

Since σ is a normed affinity, $\mathfrak{a}(g^{n-i}, g^i x) \in Z(\bar G)$ and hence

$$t = \bar g^{-n}(g^n)^\tau = \mathfrak{a}(g^{n-i}, g^i x) \in \langle g^i x\rangle^\sigma$$

and $t^{\tau^{-1}} = t^{\sigma^{-1}} \in \langle g^i x \rangle$ for all $i \in \{1, \dots, n-1\}$, $x \in H$. Since σ is not an isomorphism, $t \neq 1$ by (*a*). Hence (2) holds and (3) follows from 1.7 and $H = \mathrm{Reg}(\sigma)$.

Conversely, let τ satisfy (1)-(3). Then σ is a bijective map, and an easy computation shows that for $u, v \in G$, $u = g^i x$, $v = g^j y$ $\big(i, j \in \{0, \dots, n-1\}$; $x, y \in H\big)$ we have

$$(uv)^\sigma = \begin{cases} u^\sigma v^\sigma & \text{if } i + j < n \\ u^\sigma v^\sigma t & \text{if } i + j \geqslant n\,. \end{cases}$$

This shows that $H = \mathrm{Reg}(\sigma)$ and that $\mathfrak{a}(u, v) \in \langle u^\sigma \rangle \cap \langle v^\sigma \rangle$ for all $u, v \in G$, by (2). By 1.7, it remains to show that σ induces a projectivity. So let $U \leqslant G$. If $U \leqslant H$, then $U^\sigma = U^\tau \leqslant \bar H$. If $U \nleqslant H$, then $t^{\sigma^{-1}} \in U \cap H$ and so for $u, v \in U$ we have $u^\sigma v^\sigma = (uv)^\sigma \in U^\sigma$ or $u^\sigma v^\sigma = \big(uv(t^{-1})^{\sigma^{-1}}\big)^\sigma \in U^\sigma$. Hence also $(u^\sigma)^{-1} \in U^\sigma$ if $o(u^\sigma)$ is finite, but if u^σ

has infinite order, then by (2) and (3), $u^\sigma \in \bar{H}$ and so $(u^\sigma)^{-1} = (u^{-1})^\sigma \in U^\sigma$. Hence $U^\sigma \leqslant \bar{G}$, and similarly, $V \leqslant G$ if $V^\sigma \leqslant \bar{G}$. This proves the lemma.

We saw in 4.1 that σ_d is an automorphism if $|N| = 2 = |G:M|$ there. 4.4 shows how to construct then and also in more general situations a normed affinity from G to a new group $\bar{G}$ which is not an isomorphism.

4.5. Lemma. *Let G be a group and let N and M be subgroups of G such that*

(*a*) $1 < N \leqslant M < G$,

(*b*) $N \leqslant \langle x \rangle$ *for all* $x \in G \setminus M$,

(*c*) $|G:M| = 2$,

(*d*) $o(y) \geqslant 4|N|$ *for every 2-element* $y \in G \setminus M$.

Let $g \in G \setminus M$ *and* $N = \langle t \rangle$. *Take* $\bar{G} = M\langle \bar{g} \rangle$ *with* $\bar{g}^2 = g^2 t^{-1}$ *and* $x^{\bar{g}} = x^g$ *for all* $x \in M$. *Then* $\sigma\colon G \to \bar{G}$ *defined by* $(g^i x)^\sigma = \bar{g}^i x$ *for* $i \in \{0, 1\}$ *and* $x \in M$ *is a normed affinity with amorphy* $\mathfrak{a}(g, g) = t$.

Proof. By (*b*) and (*c*), there are 2-elements in $G \setminus M$ and N is a 2-group. Since $t \in N \leqslant Z(G)$, the extension $\bar{G}$ exists, and if $\tau\colon M \to M$ is the identity, (1) and (3) of 4.4, (*b*) are satisfied. Furthermore, $t \in \bigcap_{y \in G \setminus M} \langle y \rangle$. But if $w \in \bar{G} \setminus M$, then $w = \bar{g}x$ for some $x \in M$ and so

$$w^2 = \bar{g}^2 x^{\bar{g}} x = g^2 t^{-1} x^g x = t^{-1}(gx)^2 .$$

If u is the 2-component of gx, then by (*d*), $o(u) \geqslant 4|N|$ and hence

$$t \in \langle t^{-1}(gx)^2 \rangle = \langle w^2 \rangle \leqslant \langle w \rangle .$$

Thus $t \in \bigcap_{w \in \bar{G} \setminus M} \langle w \rangle$ and by 4.4, σ is a normed affinity. Clearly, $\mathfrak{a}(g, g) = \bar{g}^{-2} g^2 = t$.

We remark that it is in general not possible to have such an affinity if there are elements of order $2|N|$ in $G \setminus M$.

4.6. Lemma. *Let G be a group and let N and M be subgroups of G such that*

(*a*) $1 < N \leqslant M < G$,

(*b*) $N \leqslant \langle x \rangle$ *for all* $x \in G \setminus M$, (*c*) $|G:M| = 2 = |N|$,

(*d*) *there exists* $y \in G\setminus M$ *with* $o(y) = 4$,

(*e*) *for every normal subgroup* M_0 *of* G *properly contained in* M *with* G/M_0 *a* 2*-group there is* $x_0 \in G\setminus M_0$ *with* $N \not\leqslant \langle x_0\rangle$.

Then every 2*-element of* G *is contained in* $\mathrm{Reg}\,(G)$.

PROOF. Assume that there are 2-elements in $G\setminus \mathrm{Reg}\,(G)$. By 3.8, $G/\mathrm{Reg}\,(G) = (S/\mathrm{Reg}\,(G)) \times (T/\mathrm{Reg}\,(G))$ where $S/\mathrm{Reg}\,(G)$ is a non-trivial 2-group and $T/\mathrm{Reg}\,(G)$ is a $2'$-group. Since M and T are proper subgroups of G, there is $g \in G$ such that $g \notin M$ and $g \notin T$, and since G/M and G/T are 2-groups, also $h \notin M$ and $h \notin T$ if h is the 2-component of g. By (*b*), $N \leqslant \langle h\rangle$. If $x \in G\setminus(M\cap T)$, then either $x \notin M$ and then $N \leqslant \langle x\rangle$ or $x \notin T$ and then also $x_2 \notin T$ if x_2 is the 2-component of x. By 3.4, $\langle x_2\rangle \cap \langle h\rangle \neq 1$ and so $N \leqslant \langle x\rangle$. By (*e*), $M \cap T = M$. This implies that $M = T$ and $|S:\mathrm{Reg}\,(G)| = 2$. Hence $y \notin \mathrm{Reg}\,(G)$ and so by 3.2, there exists a group $\bar{G}$, a normed affinity $\sigma\colon G \to \bar{G}$ with amorphy $\mathfrak{a}$ and a 2-element $z \in G$ with $\mathfrak{a}(y, z) \neq 1$. Since $|S:\mathrm{Reg}\,(G)| = 2$, $z = yw$ with $w \in \mathrm{Reg}\,(G)$ and by 2.6,

$$\mathfrak{a}(y, y) = \mathfrak{a}(y, y)\,\mathfrak{a}(y^2, w) = \mathfrak{a}(y, w)\,\mathfrak{a}(y, yw) = \mathfrak{a}(y, z) \neq 1\,.$$

But σ induces a normed affinity on $\langle y\rangle$ which is an isomorphism since $|\langle y\rangle| = 4$. This contradiction proves the lemma.

Note that the assumptions in 4.5 and 4.6 are satisfied in abelian groups of type $(2^n, 2, \ldots, 2)$ with $n \geqslant 3$ resp. $n = 2$ if one takes $N = \mho_{n-1}(G)$ and $M = \Omega_{n-1}(G)$. Hence both cases arise and lead to different situations in the characterizations of $\mathrm{Am}\,(G)$ and $\mathrm{Reg}\,(G)$ in § 6.

We now use 4.4 to give an example of a normed affinity between two non-isomorphic groups.

4.7. EXAMPLE. Let $p > 2$ be a prime,

$$H = \langle x, y, z \mid x^p = y^p = z^p = 1, [y, x] = z = z^x = z^y\rangle$$

the non-abelian group of order p^3 and exponent p and α the automorphism of order p of H given by $x^\alpha = xy^{-1}$, $y^\alpha = y$. Finally, let s be a quadratic non-residue modulo p and $G = H\langle g\rangle$, $\bar{G} = H\langle \bar{g}\rangle$ with $g^p = z$, $\bar{g}^p = z^s$, and $h^g = h^\alpha = h^{\bar{g}}$ for all $h \in H$. Then G and $\bar{G}$ are two non-isomorphic groups of order p^4 (see [4], p. 347), but since $\Omega(G) = \Omega(\bar{G}) = H$ and $\mho(G) = \mho(\bar{G}) = \langle z\rangle$, the identity τ on H satisfies (1)-(3) of 4.4, (*b*). Hence there exists a normed affinity from G to $\bar{G}$.

It is easy to see that if $G = H \times K$ is a torsion group with $(o(x), o(y)) = 1$ for all $x \in H$, $y \in K$ (so that $\mathfrak{L}(G) \simeq \mathfrak{L}(H) \times \mathfrak{L}(K)$) and if $\nu: H \to \bar{H}$ and $\mu: K \to \bar{K}$ are normed affinities, then $\sigma: H \times K \to \bar{H} \times \bar{K}$ with $(xy)^\sigma = x^\nu y^\mu$ $(x \in H, y \in K)$ is a normed affinity. We shall need a similar result in the case that the product is not direct.

4.8. Lemma. *Let $G = HK$ be a torsion group, $K \leqslant Z(G)$ a p-group, p a prime, and assume that every element in $H/H \cap K$ has order prime to p. Let ν be an automorphism of H, $\mu: K \to K$ a normed affinity with amorphy $\mathfrak{b}$ and let*

(1) *$x^\nu = x^\mu$ for all $x \in H \cap K$ and*

(2) *$H \cap K \leqslant \operatorname{Reg}(\mu)$.*

For $g = hk \in G$ $(h \in H, k \in K)$ we define $g^\sigma = h^\nu k^\mu$. Then $\sigma: G \to G$ is a normed affinity, and if $\mathfrak{a}$ is the amorphy of σ, we have $\mathfrak{a}(g, g') = \mathfrak{b}(k, k')$ for $g' = h'k'$ $(h' \in H, k' \in K)$.

Proof. Let $\tau: G \to G$ be defined in the following way. For $g = xy \in G$ with p'-component x and p-component y let $g^\tau = x^\nu y^\mu$. Clearly, since $x \in H$, $y \in K \leqslant Z(G)$, this is well-defined and bijective. If $g = hk$ $(h \in H, k \in K)$ and $h = uv$ with p'-component u and p-component v, then $v \in H \cap K$ and u is the p'-component and vk the p-component of g. Hence by (2) and (1),

$$g^\tau = u^\nu (vk)^\mu = u^\nu v^\mu k^\mu = h^\nu k^\mu ,$$

i.e. $\sigma = \tau$ is a well-defined and bijective map. Now

$$(gg')^\sigma = (hh'kk')^\sigma = h^\nu (h')^\nu k^\mu (k')^\mu \mathfrak{b}(k, k') = g^\sigma (g')^\sigma \mathfrak{b}(k, k') ,$$

and hence $\mathfrak{a}(g, g') = \mathfrak{b}(k, k')$. If U is a subgroup of G, then $U = (U \cap H)(U \cap K)$ by our assumptions, and hence $U^\sigma = (U \cap H)^\nu \cdot (U \cap K)^\mu$ is a subgroup of G; similarly, $V^{\sigma^{-1}} \leqslant G$ if $V \leqslant G$. So σ induces a projectivity of G. If $g = hk = uvk$ as above and similarly $g' = h'k' = u'v'k'$, then $g^\sigma = g^\tau = u^\nu (vk)^\mu$ and $(vk)^\mu$ is the p-component of g^σ. Hence $\langle vk \rangle^\mu \leqslant \langle g \rangle^\sigma$ and $\langle v'k' \rangle^\mu \leqslant \langle g' \rangle^\sigma$. Since $v, v' \in \operatorname{Reg}(\mu)$, $\mathfrak{b}(vk, v'k') = \mathfrak{b}(k, k')$ and so finally, $\mathfrak{a}(g, g') = \mathfrak{b}(k, k') \in \langle g \rangle^\sigma \cap \langle g' \rangle^\sigma$. By 1.7, σ is a normed affinity.

We use 4.1 to describe all groups having $\operatorname{Am}(G)_p \simeq Z_{p^\infty}$ for some prime p. For the definition of a central product with amalgamated subgroup see [4], p. 49.

4.9. THEOREM. *Let G be a group, p a prime.*

If $\mathrm{Am}(G)_p \simeq Z_{p^\infty}$, *then there exist $H \leqslant G$ and $r \in \mathrm{N} \cup \{0\}$ such that*

(1) *H is a torsion group,*

(2) *$o(x) \leqslant p^r$ for every p-element $x \in H$,*

(3) *$Z(H)$ contains an element of order p^r, and*

(4) *G is the central product of $\mathrm{Am}(G)_p$ and H where the subgroup of order p^r of $\mathrm{Am}(G)_p$ is amalgamated with a central subgroup of order p^r of H.*

Conversely, if H is a group and $r \in \mathrm{N} \cup \{0\}$ with properties (1)-(3) *and if G is the central product of Z and H where $Z \simeq Z_{p^\infty}$ and the subgroup of order p^r of Z is amalgamated with a central subgroup of order p^r of H, then* $\mathrm{Am}(G)_p = Z$.

PROOF. Let $\mathrm{Am}(G)_p \simeq Z_{p^\infty}$. We show that

(*) *if $x \in G$ is a p-element and $z \in \mathrm{Am}(G)_p$ with $o(z) = o(x)$, then $\langle x \rangle \mathrm{Reg}(G) = \langle z \rangle \mathrm{Reg}(G)$.*

For this observe that since $z \in Z(G)$, $A = \langle x, z \rangle$ is abelian of exponent $o(x) = o(z)$ and hence there exists a subgroup B of A with $A = \langle x \rangle \times \times B = \langle z \rangle \times B$. Then $B \cap \mathrm{Am}(G)_p = B \cap \langle z \rangle = 1$ and $B \leqslant \mathrm{Reg}(G)$ by 1.7. Thus $\langle x \rangle \mathrm{Reg}(G) = \langle z \rangle \mathrm{Reg}(G)$ as asserted.

Since $\mathrm{Am}(G)_p \neq 1$, there exist p-elements in $G \setminus \mathrm{Reg}(G)$ and hence by (*), there exists $z \in \mathrm{Am}(G)_p$ with $z \notin \mathrm{Reg}(G)$. Let

$$H/\mathrm{Reg}(G) = \{y \in G/\mathrm{Reg}(G) \mid (o(y), p) = 1\}$$

be the p'-component of $G/\mathrm{Reg}(G)$. By 3.8, H is a subgroup of G, and $H \cap \mathrm{Am}(G)_p = \mathrm{Reg}(G) \cap \mathrm{Am}(G)_p < \langle z \rangle$ since $\mathfrak{L}(Z_{p^\infty})$ is a chain. Put $|H \cap \mathrm{Am}(G)_p| = p^r$. We show that H and r satisfy (1)-(4).

If $g \in G$ would be of infinite order, then $\langle g, z \rangle = \langle g \rangle \times \langle z \rangle$ would be generated by elements of infinite order and therefore would be contained in $\mathrm{Reg}(G)$ by 1.7. But $z \notin \mathrm{Reg}(G)$. Hence G is a torsion group and (1) holds. If $y \in H$ would be of order p^{r+1}, then we would take $w \in \mathrm{Am}(G)_p$ with $o(w) = o(y)$. By $(*)$, $w \in \langle y \rangle \mathrm{Reg}(G) \leqslant H$, a contradiction since $|H \cap \mathrm{Am}(G)_p| = p^r$. Hence (2) holds, and since $\mathrm{Am}(G) \leqslant Z(G)$, also (3) is satisfied. Finally, $(*)$ shows that every p-element of G lies in $\mathrm{Am}(G)_p H$. Hence $G = \mathrm{Am}(G)_p H$ by definition of H, and since $\mathrm{Am}(G)_p \leqslant Z(G)$, G is a central product of $\mathrm{Am}(G)_p$ and H and (4) holds.

Conversely, assume that G is the central product of Z and H as in the theorem, let N be the subgroup of order p^n in Z ($n \in \mathbb{N}$) and let $M = HS$ where S is the subgroup of order p^{n+r} of Z. If $x \in G \setminus M$, then $x = zh$ with $z \in Z$, $h \in H$ and $o(z) > p^{n+r}$. Then $x^{p^r} = z^{p^r} h^{p^r}$ and $(o(h^{p^r}), p) = 1$. Thus $N \leqslant \langle z^{p^r} \rangle \leqslant \langle x \rangle$, and $N \leqslant \mathrm{Am}(G)$ by 4.1. This holds for every proper subgroup of Z and therefore $Z \leqslant \mathrm{Am}(G)_p$. Conversely, if $g \in \mathrm{Am}(G)_p$ take $z \in Z$ with $o(z) = o(g)$. By 3.6, $\langle g, z \rangle$ is cyclic and hence $g \in \langle z \rangle \leqslant Z$. Thus $Z = \mathrm{Am}(G)_p$ and the theorem is proved.

5. p-collecting subgroups and p-collectors.

To determine $\mathrm{Am}(G)$ and $\mathrm{Reg}(G)$ for a group G we have to consider every affinity from G to any group $\bar{G}$. We shall use 3.4 to give a characterization of $\mathrm{Am}(G)$ and $\mathrm{Reg}(G)$ within the group G. For this we define further characteristic subgroups of G.

5.1. Definition. Let p be a prime, G a group. Put

$$O^p(G) = \langle x \in G \,|\, o(x) = \infty \text{ or } (o(x), p) = 1 \rangle .$$

A subgroup S of G is a *p-collecting subgroup* of G if

(a) $O^p(G) \leqslant S < G$ and

(b) $K := \bigcap_{x \in G \setminus S} \langle x \rangle \neq 1$.

We call K the *p-collector* of S, $K = \mathcal{K}(S)$. We also call G a p-collecting subgroup of G and 1 its p-collector.

5.2. Remark. (a) If K is a p-collector, then the intersection of all p-collecting subgroups of G with p-collector K is a p-collecting subgroup of G. Hence there is a unique minimal p-collecting subgroup $\mathcal{S}(K)$ with given p-collector K.

(*b*) Certainly, K is centralized by every $x \in G \setminus S$, and since these elements generate G, we have that $K \leqslant Z(G)$ if K is the p-collector of some p-collecting subgroup S of G.

(*c*) If $O^p(G) \leqslant S < G$, then

$$K = \bigcap_{x \in G \setminus S} \langle x \rangle \leqslant \bigcap \{\langle x \rangle | x \in G \setminus S,\ x\ p\text{-element}\} = L\,.$$

But if $x \in G \setminus S$, then $o(x)$ is finite, let $x = uv$ with p-component u and p'-component v. Then $v \in O^p(G) \leqslant S$, hence $u \in G \setminus S$ and $L \leqslant \langle u \rangle \leqslant \leqslant \langle x \rangle$. Thus $L \leqslant K$ and so, finally, $L = K$. Therefore, in order to find out whether S is a p-collecting subgroup and to determine $\mathcal{K}(S)$ we only have to look at p-elements in $G \setminus S$. Furthermore, $\mathcal{K}(S)$ is a p-group, if S is a p-collecting subgroup of G.

We show that there is a minimal p-collecting subgroup in G.

5.3. Theorem. *Let p be a prime, G a group and let $T_p(G)$ be the intersection of all p-collecting subgroups of G. Then $T_p(G)$ is a p-collecting subgroup of G, and the p-collecting subgroups of G are precisely the subgroups of G containing $T_p(G)$.*

Proof. There is nothing to show if G is the only p-collecting subgroup of G. So assume $T_p(G) < G$. Then $O^p(G) \leqslant T_p(G)$ since $O^p(G)$ is contained in every p-collecting subgroup. Let x and y be p-elements in $G \setminus T_p(G)$. Then there exist p-collecting subgroups S and T of G with $x \notin S$, $y \notin T$. Since G is not the set theoretical union of two proper subgroups, there exists $g \in G$ with $g \notin S$ and $g \notin T$. Then $\langle g \rangle \cap \langle x \rangle \neq 1 \neq \langle g \rangle \cap \langle y \rangle$ since S and T are p-collecting subgroups of G, and hence $\Omega(\langle x \rangle) = \Omega(\langle y \rangle)$ is the subgroup of order p of $\langle g \rangle$. Thus

$$\Omega(\langle y \rangle) \leqslant \bigcap \{\langle x \rangle | x \in G \setminus T_p(G),\ x\ p\text{-element}\}\ ,$$

and $T_p(G)$ is a p-collecting subgroup of G by 5.2, (*c*). Clearly, all subgroups of G containing $T_p(G)$ are also p-collecting subgroups of G.

5.4. Theorem. *Let p be a prime, G a group and let $L_p(G)$ be the subgroup of G generated by all p-collectors of G. Then $L_p(G) \leqslant Z(G)$ and $L_p(G) \simeq Z_{p^n}$ for some $n \in \mathbf{N} \cup \{0, \infty\}$.*

PROOF. By 5.2, $L_p(G) \leqslant Z(G)$. If G is the only p-collecting subgroup of G, then $L_p(G) = 1 \simeq Z_{p^0}$. So assume there are proper p-collecting subgroups S_1, S_2 of G, let K_1, K_2 be their p-collectors and take $g \in G$ with $g \notin S_1$ and $g \notin S_2$. Then K_1 and K_2 are p-subgroups of $\langle g \rangle$ and hence $K_1 \leqslant K_2$ or $K_2 \leqslant K_1$. As in the proof of 3.6, $L_p(G) \simeq Z_{p^n}$ with $n \in \mathbb{N} \cup \{\infty\}$.

We shall need the following result.

5.5. LEMMA. *Let p be a prime, S a p-collecting subgroup of G and $K = \mathcal{K}(S)$ its p-collector. If $K \nleqslant S$, then $|K: K \cap S| = p$ and every element in $S/K \cap S$ has finite order prime to p.*

PROOF. K is a p-group, and for every $x \in K \setminus S$ we have $K = \langle x \rangle$ by the definition of a p-collector. Hence $|K: K \cap S| = p$. If G would contain an element g of infinite order, then $K\langle g \rangle = K \times \langle g \rangle$ since $K \leqslant Z(G)$. But $K \times \langle g \rangle$ would be generated by elements of infinite order and so $K \leqslant O^p(G) \leqslant S$, a contradiction. Hence G is a torsion group. If $S/K \cap S$ would contain a subgroup $U/K \cap S$ of order p, then $UK/K \cap S$ would be elementary abelian of order p^2 and UK would contain an element g with $g \notin K$ and $g \notin U = UK \cap S$. But then also $K \nleqslant \langle g \rangle$, a contradiction since $K = \mathcal{K}(S)$. This proves the lemma.

6. A characterization of $\mathrm{Am}(G)$ and $\mathrm{Reg}(G)$.

Theorems 3.6 and 5.4 look very similar and there is a good reason for this. We show in fact that $L_p(G)$ and $\mathrm{Am}\,(G)_p$ are nearly the same subgroup of G.

6.1. THEOREM. *If p is a prime and G a group, then* $\mathrm{Am}\,(G)_p = L_p(G)$ *except in the following cases:*

(*a*) *$p = 3$, $L := L_3(G) \neq 1$ is finite, and $G = LT$ where $T = \mathcal{S}(L)$ is the minimal 3-collecting subgroup of G with 3-collector L; here* $\mathrm{Am}\,(G)_3$ *is the maximal subgroup of $L_3(G)$.*

(*b*) *$p = 2$, $L := L_2(G) \neq 1$ is finite, let $T = \mathcal{S}(L)$ be the minimal 2-collecting subgroup of G with 2-collector L, and*

(*b*1) $G = LT$, $|L| \geqslant 4$,

(*b*2) $G = LT$, $|L| = 2$,

(*b*3) $|G: LT| = 2,\ L \not\leqslant T$,

(*b*4) $|G: LT| = 2,\ L \leqslant T$, *and there exists* $y \in G \setminus T$ *with* $o(y) = 2|L|$.

Here $\mathrm{Am}(G)_2$ *is the subgroup of* $L_2(G)$ *of index* 4 *in case* (*b*1) *resp. index* 2 *in the other cases.*

PROOF. We first show that always

(I) $\mathrm{Am}(G)_p \leqslant L_p(G)$.

For this we take a generator of $\mathrm{Am}(G)_p$, i.e. by 3.2, we take a normed affinity σ from G to some group $\bar{G}$, $\mathfrak{a}$ the amorphy of σ, p-elements $x, y \in G$ and let $K = \langle \mathfrak{a}(x, y)^{\sigma^{-1}} \rangle$. We have to show that $K \leqslant L_p(G)$. If $K = 1$, this is clear, so assume $K \neq 1$. We define

$$S = \{g \in G | z \in G,\ \langle \mathfrak{a}(x, z) \rangle = K^\sigma \Rightarrow \langle a(x, zg) \rangle = K^\sigma\}$$

and show that

(1) *if* $g \in G$ *with* $K \not\leqslant \langle g \rangle$, *then* $g \in S$.

For this take $z \in G$ with $\langle \mathfrak{a}(x, z) \rangle = K^\sigma$. Since x is a p-element, the subgroup lattice of $\langle x \rangle$ is a chain and hence $\langle x \rangle \cap \langle g \rangle < K$ since $K \not\leqslant \langle g \rangle$ but $K^\sigma \leqslant \langle x \rangle^\sigma$. The associativity identities 2.6 for x, z, g together with 2.4 and 1.7 give

$$\mathfrak{a}(x, z)\,\mathfrak{a}(x, zg)^{-1} = \mathfrak{a}(z, g)\,\mathfrak{a}(xz, g)^{-1} \in \langle x \rangle^\sigma \cap \langle g \rangle^\sigma < K^\sigma$$

and therefore with $\mathfrak{a}(x, z)$ also $\mathfrak{a}(x, zg)$ generates K^σ. Thus $g \in S$.

Now (1) shows that S contains all elements of infinite order and all $g \in G$ with $(o(g), p) = 1$, since $1 \neq K \leqslant \langle x \rangle$ is a p-group. So if we can show that S is a subgroup of G, then certainly

(2) $O^p(G) \leqslant S \leqslant G$.

But $1 \in S$, and if $g, h \in S$, then for all $z \in G$, $\langle \mathfrak{a}(x, z) \rangle = K^\sigma$ implies $\langle \mathfrak{a}(x, zg) \rangle = K^\sigma$, this implies that $\langle \mathfrak{a}(x, zgh) \rangle = K^\sigma$, and hence $gh \in S$. Since S contains all elements of infinite order, S is a subgroup of G.

Since $\langle \mathfrak{a}(x, y) \rangle = K^\sigma$ and $\langle \mathfrak{a}(x, yy^{-1}) \rangle = \langle \mathfrak{a}(x, 1) \rangle = 1 \neq K^\sigma$, we have that $y^{-1} \notin S$. Hence $S < G$ and by (1), $K \leqslant \bigcap_{g \in G \setminus S} \langle g \rangle$. Thus S is a p-collecting subgroup of G and $K \leqslant \mathcal{K}(S) \leqslant L_p(G)$. This proves (I).

(II) *If* $\mathrm{Am}\,(G)_p \neq L_p(G) =: L$, *then* $p = 3$ *or* 2, $L \neq 1$ *is finite, and* $G = LT$ *or* $|G : LT| = 2$, *where* $T = \mathcal{S}(L)$ *is the minimal p-collecting subgroup of G with p-collector L.*

To show this we take a p-collector $K \neq 1$ in G and a p-collecting subgroup S of G with p-collector K.

(3) *Let* $KS < G$. If $|G : KS| \neq 2$, *then* $K \leqslant \mathrm{Am}\,(G)_p$. *If* $|G : KS| = 2$, *then the maximal subgroup of K is contained in* $\mathrm{Am}\,(G)_2$.

For this let $M = KS$, $N = K$, and $\sigma = \sigma_d$ given by 4.1 with $1 \neq d \in N$. If $|G : M| \neq 2$, there are $x, y \in G \setminus M$ with $xy \notin M$. Then $d = \mathfrak{a}(x, y)^{\sigma^{-1}} \in \mathrm{Am}\,(G)_p$, i.e. $K \leqslant \mathrm{Am}\,(G)_p$. If $|G : M| = 2$, then for $x, y \in G \setminus M$, $d^2 = \mathfrak{a}(x, y)^{\sigma^{-1}} \in \mathrm{Am}\,(G)_p$, and so the maximal subgroup of K is contained in $\mathrm{Am}\,(G)_2$.

(4) *Let* $KS = G$. *If* $p \neq 2$ *and* 3, *then* $K \leqslant \mathrm{Am}\,(G)_p$. *If* $p = 3$, *then* $\mathrm{Am}\,(G)_3$ *is the maximal subgroup of K. If* $p = 2$, *then* $\mathrm{Am}\,(G)_2$ *is the subgroup of index* 4 *in K if* $|K| > 4$ *and* $\mathrm{Am}\,(G)_2 = 1$ *if* $|K| = 2$.

To prove this observe that $K \nleqslant S$ since $S < G$. By 5.5, $|K : K \cap S| = p$ and every element in $S/K \cap S$ has finite order prime to p. Let $K = \langle g \rangle$, ν be the identity on S and $\mu : K \to K$ for $p > 2$ be the normed affinity given by $(gx)^\mu = g^2 x$, $(g^2 x)^\mu = gx$ for all $x \in K \cap S$ and $y^\mu = y$ for all other $y \in K$. By 4.3 and 4.8, $\sigma : G \to G$ with $(sk)^\sigma = s^\nu k^\mu$ $(s \in S, k \in K)$ is a normed affinity of G and $g^{-3} \in \mathrm{Am}\,(G)_p$. This shows that $K \leqslant \mathrm{Am}\,(G)_p$ if $p > 3$ and that $K \cap S \leqslant \mathrm{Am}\,(G)_p$ if $p = 3$. But in this case by 1.9, every normed affinity of G induces a normed affinity in the cyclic group G/S of order 3 which of course is an isomorphism. Hence $\mathrm{Am}\,(G)_3 \leqslant S$ and so $\mathrm{Am}\,(G)_3 = K \cap S$. If $p = 2$ and $|K| \geqslant 4$, we take $\mu = \sigma_d$ with $d = g^2 \in N = M = K \cap S$ in the notation of 4.1 and construct σ as above. Then $d^2 = g^4 \in \mathrm{Am}\,(G)_2$. By 1.9, every normed affinity of G induces a normed affinity in $K/\langle g^4 \rangle$ which again is an isomorphism. Together with 3.2 this shows that $\mathrm{Am}\,(G)_2 = \langle g^4 \rangle$ and, similarly, $\mathrm{Am}\,(G)_2 = 1$ if $|K| = 2$.

Now (II) follows immediately from (3) and (4). For if K is any generator of $L_p(G)$, then (3) and (4) show that the second maximal subgroup of K is contained in $\mathrm{Am}(G)_p$. By 5.4, $L_p(G)$ is finite if $\mathrm{Am}(G)_p < L_p(G)$. Then we may take $K = L_p(G) = L$ and $S = T$ in (3) and (4) and get that $p = 2$ or 3 and $G = LT$ or $|G: LT| = 2$.

By (4), also $\mathrm{Am}(G)_p$ is as stated in the theorem if $G = LT$. So it remains to consider the case that $p = 2$ and $|G:LT| = 2$. If $L \nleqslant T$, then by 5.5, $|L:L \cap T| = 2$ and hence $|G:T| = 4$. Again every normed affinity of G induces a normed affinity in G/T which is an isomorphism and so $\mathrm{Am}(G)_2 \leqslant T$. By (3), $\mathrm{Am}(G)_2 = L \cap T$ is the maximal subgroup of L. Finally, let $L \leqslant T$ and hence $|G:T| = 2$. If ($b4$) does not hold, then $L \leqslant \mathrm{Am}(G)$ by 4.5. If ($b4$) holds and L_0 is the maximal subgroup of L, then (a)-(d) of 4.6 are satisfied in G/L_0 with $M = T/L_0$ and $N = L/L_0$. And if $M_0 = S/L_0$ is as in (e) of 4.6, then there exists $x \in G \setminus S$ with $L \nleqslant \langle x \rangle$ since $T = \mathcal{S}(L)$ is the minimal 2-collecting subgroup with 2-collector L. Then $\langle x \rangle L_0 \cap L \leqslant L_0$ and so also (e) of 4.6 holds. Hence every 2-element in G/L_0 is regular and therefore $\mathrm{Am}(G)_2 \leqslant L_0$ by 1.9 and 3.2. By (3), $\mathrm{Am}(G)_2 = L_0$ what we had to show.

We now characterize $\mathrm{Reg}(G)$ in terms of the minimal p-collecting subgroups $T_p(G)$ defined in 5.3.

6.2. Definition. If p is a prime and G a group, let $\mathrm{Reg}(G)^p$ be the subgroup of G with $\mathrm{Reg}(G)^p/\mathrm{Reg}(G) = O^p(G/\mathrm{Reg}(G))$.

By 3.8, $\mathrm{Reg}(G) = \bigcap_{p \in \mathbf{P}} \mathrm{Reg}(G)^p$. So as in 6.1 the following theorem not only characterizes $\mathrm{Reg}(G)^p$ but also $\mathrm{Reg}(G)$.

6.3. Theorem. *If p is a prime and G a group, then* $\mathrm{Reg}(G)^p = T_p(G)$ *except in the following cases:*

(a) $p = 3$, $G = H \times K$ *is a torsion group,* $|K| = 3$, *and* $(o(x), 3) = 1$ *for all* $x \in H$; *here* $\mathrm{Reg}(G)^3 = G$ *and* $T_3(G) = H$.

(b) $p = 2$, *let* $T := T_2(G)$ *and* $L = \mathcal{K}(T)$ *be the 2-collector of* T,

and

($b1$) $L \cap T = 1$, $|G:LT| > 4$,

($b2$) $L \cap T = 1$, $|G:LT| \leqslant 2$, $T \neq G$,

($b3$) $|L \cap T| = 2$, $|G:T| = 2$, $L \nleqslant T$,

($b4$) $|L \cap T| = 2$, $|G:T| = 2$, $L \leqslant T$, *and there exists an element of order* 4 *in* $G \setminus T$.

Here $\operatorname{Reg}(G)^2 = LT$ *in case* (*b*1) *and* $\operatorname{Reg}(G)^2 = G$ *in the other three cases.*

PROOF. By 2.3, every element of G of infinite order is contained in $\operatorname{Reg}(G)^p$, and so $O^p(G) \leqslant \operatorname{Reg}(G)^p$. By 3.4,

$$\cap \{\langle g\rangle | g \in G \setminus \operatorname{Reg}(G)^p,\ g\ p\text{-element}\} \neq 1\,,$$

and remark 5.2, (*c*) shows that $\operatorname{Reg}(G)^p$ is a p-collecting subgroup of G. Hence

(1) $T_p(G) \leqslant \operatorname{Reg}(G)^p$.

Let $T = T_p(G)$ and $L = \mathcal{K}(T)$ be its p-collector and assume first that $L \cap T \neq 1$. Then $T \neq G$ and 4.1 shows that $x \notin \operatorname{Reg}(G)$ for $x \in G \setminus T$ and therefore $T = \operatorname{Reg}(G)^p$, except possibly when $|L \cap T| = 2 = |G : T|$. But in this case if $L \nleqslant T$, then $|L| = 4$, $LT = G$ and L contains every 2-element of G by 5.5. On L every normed affinity of G is an isomorphism and therefore $L \leqslant \operatorname{Reg}(G)$ by 3.2. So $\operatorname{Reg}(G)^2 = G \neq T$ by (1). If $L \leqslant T$, then $|L| = 2 = |G : T|$ and 4.5 and 4.6 show that $\operatorname{Reg}(G) \nleqslant T$ if and only if there exists an element of order 4 in $G \setminus T$. By (1), $\operatorname{Reg}(G)^2 = G$ in this case and $\operatorname{Reg}(G)^2 = T$ if there is no element of order 4 in $G \setminus T$.

Now assume that $L \cap T = 1$. If $T = G$, then $\operatorname{Reg}(G)^p = T$ by (1). So let $T \neq G$ and therefore $L \neq 1$. Then $L \nleqslant T$ and 5.5 shows that $|L| = p$, $LT = L \times T$, and T is a p'-group. We show

(2) *If* $\nu : L \to L$ *is any bijective map with* $1^\nu = 1$, *then* $\sigma : G \to G$ *defined by* $x^\sigma = x$ *for* $x \in G \setminus (L \times T)$ *and* $(lt)^\sigma = l^\nu t$ *for* $l \in L$, $t \in T$ *is a normed affinity.*

Clearly, σ is bijective and since every subgroup of $L \times T$ not contained in T contains L, σ fixes every subgroup of $L \times T$ and hence also of G. Thus σ induces the trivial autoprojectivity on G and if α is the amorphy of σ, $\alpha(x, y) \in L$ for all $x, y \in G$. This is contained in $\langle x\rangle = \langle x\rangle^\sigma$ if $p | o(x)$. But if x or y is a p'-element, it is contained in T and then $\alpha(x, y) = 1$. By 1.7, σ is a normed affinity.

Now (2) shows that every normed affinity $\nu : L \to L$ can be extended to a normed affinity of G. Thus $L \nleqslant \operatorname{Reg}(G)$ if $|L| = p \geqslant 5$ and since $L = \mathcal{K}(T)$, $\operatorname{Reg}(G) \leqslant T$ in this case. By (1), $T = \operatorname{Reg}(G)^p$. For $p = 3$, we take the automorphism ν of order 2 of L and define σ as in (2). If G contains an element x of order 9, then $L = \langle x^3 \rangle$ and

$$(xx^3)^\sigma = x^4 \neq x^7 = x^\sigma (x^3)^\sigma .$$

Again, $L \nleqslant \operatorname{Reg}(G)$ and $T = \operatorname{Reg}(G)^3$. But if G does not contain an element of order 9, then $G = L \times T$ and (*a*) holds. In this case $L \leqslant \operatorname{Reg}(G)$ by 3.2 and hence $\operatorname{Reg}(G)^3 = G$. Finally, let $p = 2$. If $|G:LT| \leqslant 2$, then $|G:T| \leqslant 4$ and every normed affinity $\sigma\colon G \to \bar{G}$ induces an isomorphism in G/T. For $x, y \in G$ and the amorphy α of σ therefore $\alpha(x, y) \in T^\sigma \cap \langle x \rangle^\sigma = 1$ if x is a 2-element. So every 2-element of G is contained in $\operatorname{Reg}(G)$ and $\operatorname{Reg}(G)^2 = G$. If $|G:LT| \geqslant 4$, then 4.1 with $N = L$ and $M = LT$ shows that $\operatorname{Reg}(G) \leqslant LT$. Hence $\operatorname{Reg}(G)^2 \leqslant LT$, but $L \leqslant \operatorname{Reg}(G)$ by 1.3 and so $\operatorname{Reg}(G)^2 = LT$ in this case.

We remark that 6.1 and 6.3 nearly give lattice theoretic characterizations of $\operatorname{Am}(G)$ and $\operatorname{Reg}(G)$. At least an index-preserving projectivity from G to a group H maps $O^p(G)$ to $O^p(H)$, hence $T_p(G)$ to $T_p(H)$ and $L_p(G)$ to $L_p(H)$ for every prime p and preserves the other conditions in 6.1 and 6.3.

6.4. Corollary. *If σ is an index-preserving projectivity from the group G to the group H, then* $\operatorname{Am}(G)^\sigma = \operatorname{Am}(H)$ *and* $\operatorname{Reg}(G)^\sigma = \operatorname{Reg}(H)$.

So, finally, $\operatorname{Am}(G)$ is not only a characteristic subgroup of G but also satisfies $\operatorname{Am}(G)^\sigma = \operatorname{Am}(G^\sigma)$ for every normed affinity σ of G. For $\operatorname{Reg}(G)$ the corresponding statement was proved in 2.3.

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