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Butler Groups, Valuated Vector Spaces, and Duality.

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0. Introduction.

A Butler group is a torsion-free abelian group that is generated by a finite number of rank-one subgroups [BTLR1], [ARNL]. In [BTLR2] it was shown that the study of Butler groups under quasi-homomorphism is essentially the same as the study of valuated finite-dimensional vector spaces over the rationals, the value set of each space being a finite subset of the lattice of types. This was proved independently in [LADY] for a subclass of these valuated vector spaces, which by [BTLR2] is in fact all of them.

In this paper we study the general properties of finite-dimensional valuated vector spaces with values in a finite distributive lattice. We develop a duality theory for these spaces and show that every such space is an image, and a subspace, of a direct sum of one-dimensional spaces. The duality theory yields a duality theory for Butler groups.

More precisely, if T is a finite lattice of types we consider the category whose objects are Butler groups with type sets contained in T, and whose morphisms are the quasi-homomorphisms, that is, elements of $Q \otimes \operatorname{Hom}(A, B)$. We show, as in [BTLR2], that this category is isomorphic to the category $C_Q(T)$ of finite-dimensional T-valuated vector spaces over the rational numbers Q. Moreover we show that if T' is a lattice that is anti-isomorphic to an arbitrary finite distributive lattice T, then the categories $C_F(T)$ and $C_F(T')$ are dual for any field F. As any finite distributive lattice is isomorphic

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to a sublattice of the lattice of types, this gives a duality under quasi-homomorphism between Butler groups with type sets in a finite lattice T of types, and Butler groups with type sets in some other lattice T'.

1. T-valuated vector spaces.

Let T be a finite distributive lattice and V a vector space over a field F. A *T-valuation* on V is a function v from V to $T \cup \{\infty\}$ such that

1) $vx = \infty$ if and only if $x = 0$;

2) $v(x + y) \geqslant \inf(vx, vy)$;

3) $v(rx) = vx$ if $r \neq 0$.

The prototype T-valuated vector space for our purposes is constructed from a finite-rank torsion-free group G with finite type set by setting $V = Q \otimes G$ and setting $vx = \text{type}(nx)$ for any nonzero n such that $nx \in G$. Maps between T-valuated vector spaces V and W are vector space maps f such that $vf(x) \geqslant vx$ for all x. We shall be interested in T-valuated finite-dimensional vector spaces over F, which we shall call simply *T-spaces over F*, dropping the reference to F in general.

Let T be a finite distributive lattice. A lattice homomorphism from T to the two element lattice $\{0, 1\}$ will be called a *localization* of T. As T is a subdirect product of two element lattices [CD; 11.2], we have $\tau \geqslant \tau'$ in T if and only if $\theta(\tau) \geqslant \theta(\tau')$ for every localization θ. If θ is a localization of T, then θ induces a localization functor from the category of T-spaces to the category of $\{0, 1\}$-spaces by simply changing the valuation from v to θv. It will be convenient to have the following duality right away.

THEOREM 1.1. *Let T be a finite distributive lattice and let T' be anti-isomorphic to T. Then the category of T-spaces is dual to the category of T'-spaces.*

PROOF. Let $\varphi\colon T \to T'$ be an anti-isomorphism. For V a T-space we let V' be the vector space of homomorphisms from V to Q valuated by setting $v'\lambda = \inf\{\varphi(vx)\colon \lambda(x) \neq 0\}$. To see that v' is a valuation we compute $v'(\lambda + \mu) = \inf\{\varphi(vx)\colon \lambda(x) + \mu(x) \neq 0\} \geqslant \inf\{\varphi(vx)\colon \lambda(x) \neq 0$ or $\mu(x) \neq 0\} = \inf(v'\lambda, v'\mu)$.

If f takes V to W is a map of T-spaces, we must show that the induced map f' taking W' to V' is a map of T'-spaces. For λ in W' we have

$$v'f'(\lambda) = v'(f\lambda) = \inf\{\varphi(vx) \colon f(\lambda(x)) \neq 0\} \geqslant \inf\{\varphi(vx) \colon \lambda(x) \neq 0\} = v'\lambda.$$

Finally we want to show that W'' is isomorphic to W, as a T-space under the natural isomorphism of vector spaces. If w is in W, then the value it gets as an element of W'' is given by $v''w = \inf\limits_{\lambda} \{\varphi^{-1}(v'\lambda) \colon \lambda(w) \neq 0\}$ where λ ranges over W'. Now $v'\lambda = \inf\limits_{x} \{\varphi(vx) \colon \lambda(x) \neq 0\}$ where x ranges over W so we have $v''w = \inf\limits_{\lambda \colon \lambda(w) \neq 0} \sup\limits_{x \colon \lambda(x) \neq 0} vx$. We must show that $v''w = vw$. By setting $x = w$ we see that $v''w \geqslant vw$. To show that $v''w \leqslant vw$ it suffices to show that $\theta v''w \leqslant \theta vw$ for all localizations θ. Suppose $\theta vw = 0$ for some localization θ. Then there is λ in W' such that $\lambda(w) \neq 0$ and $\lambda x = 0$ if $\theta vx \geqslant 1$. Hence $\sup\limits_{x \colon \lambda(x) \neq 0} \theta vx = 0$ so $v''w = 0 = \theta vw$. If $\theta vw = 1$ then of course $\theta v''w \leqslant \theta vw$. □

As we will be concerned with finite lattices of types, it is of interest to know that we can realize the lattice T' in Theorem 1.1 as a lattice of types.

THEOREM 1.2. Any finite distributive lattice is isomorphic to a lattice of types of subrings of Q.

PROOF. As any finite distributive lattice can be embedded in a finite Boolean algebra [CD; 11.3], it suffices to show that any finite Boolean algebra is isomorphic to a lattice of types of subrings of Q. A finite Boolean algebra is isomorphic to the algebra of all subsets of its atoms. If P is a finite set of primes, then to each subset I of P we can associate the ring $R_I = \{m/n \colon n$ is not divisible by any prime outside $P\}$. Then the types of the rings R_I form a lattice isomorphic to the Boolean algebras of all subsets of P. □

The category of T-spaces is not abelian but it is additive and has kernels and cokernels, that is, it is pre-abelian (see Theorem 1.3). In a pre-abelian category we can still speak of short exact sequences, but this class need not be closed under pushouts and pullbacks as it is in the abelian case. Those short exact sequences which remain exact under pushout and pullback are called *stable* [RW].

THEOREM 1.3. The category of T-spaces is pre-abelian and every short exact sequence is stable.

PROOF. If f is a map from V to W, then the kernel of f is $\{x \in V: f(x) = 0\}$ with the valuation inherited from V. The cokernel is the quotient vector space $W/f(V)$ with the value of $w + f(V)$ equal to the supremum of the values of $w + f(x)$ as x ranges over V. To verify that this latter is a valuation on $W/f(V)$ use the fact that the lattice T is distributive.

We now show that every exact sequence is stable. By duality it suffices to show that the pushout of every exact sequence is exact. Suppose

$$\begin{array}{ccc} A & \subset & B \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ C & \rightarrow & D \end{array}$$

is a pushout diagram. Then D is equal to $C \oplus B$ modulo $\{(\varphi(a), -a): a \in A\}$. We must show that the map from C to D is an embedding. If $c \in C$, then c goes to $(c, 0)$ in D whose value is sup $\{v(c - \varphi(a), a): a \in A\}$ so it suffices to show that $vc \geqslant \inf\{v(c - \varphi(a)), va\}$ for all a in A. But $vc \geqslant \inf\{v(c - \varphi(a)), v\varphi(a)\}$. $\square$

We say that a T-space is *completely decomposable* if it can be written as the direct sum of one-dimensional T-spaces.

THEOREM 1.4. Let T be a finite distributive lattice and let V be a T-space. Then

1) V is a quotient space of a completely decomposable T-space.

2) V is a subspace of a completely decomposable T-space.

PROOF. By duality we need only establish 1). As 1) clearly holds if V is one-dimensional, it suffices, by induction on dimension, to show that an extension of a T-space satisfying 1) by a one-dimensional T-space satisfies 1). Let $V \subset W$ be T-space with W/V one-dimensional. Let $V_1, \ldots, V_m$ be one-dimension subspaces of V such that the natural map from $V_1 \oplus \ldots \oplus V_m$ to V is a cokernel. Let $W_1, \ldots, W_n$ be one-dimensional subspaces of W mapping onto W/V such that the type of W/V is the supremum of the types of the W_j. Then the V_i together with the W_j generate W. We want to show that the natural map from the direct sum of the V_i and W_j to W induces the valuation on W. By localizing we may assume that $T = \{0, 1\}$. Suppose $vw = 1$ for some w in W. If $w \in V$ then we are all right. If $w \notin V$ then $v(w + V) = 1$ so we can find w_j in W, such that $vw_j = 1$ and $w - w_j \in V$. As $vw = 1$ we have $v(w - w_j) = 1$ so $w - w_j$ is the image of an element in the direct sum of the V_i of value 1. $\square$

2. Butler T-groups.

We say that a finite-rank torsion-free group is a *T-group* if its type set is contained in T. The functor $Q\otimes$ from the category of T-groups to the category of T-spaces over Q is easily seen to be left exact. In fact it is exact because of the following property of epimorphisms between Butler groups:

THEOREM 2.1. Let G be a Butler group and f a map from G onto a torsion-free group H. Then every pure rank-one subgroup of H is the image of the sum of finitely many rank-one subgroups of G.

PROOF. Let K be a pure rank-one subgroup of H. As pure subgroups of Butler groups are again Butler groups [ARNL], [BTLR1], the pre-image of K in G is a Butler group and so is generated by finitely many rank-one subgroups. $\square$

COROLLARY 2.2. The functor $Q\otimes$ from the category of T-groups to T-spaces is exact on the category of Butler T-groups.

PROOF. We need only show that if G is a Butler group mapping onto H, then the map $Q\otimes G$ onto $Q\otimes H$ is a cokernel of T-spaces. But Theorem 2.1 implies that if $y \in H$, then some nonzero multiple of y has a finite set of pre-images, the supremum of whose types is the type of y. $\square$

Theorem 2.1 suggests a class of short exact sequences more restricted than the balanced sequences. A short exact sequence $A \to B \to C$ of torsion-free groups is balanced if every pure rank-one subgroup of C is the image of a rank-one subgroup of B. We shall call such a sequence *semi-balanced* if every pure rank-one subgroup of C is the image of a sum of finitely many rank-one subgroups of B. Theorem 2.1 says that any short exact sequence of Butler groups is semi-balanced. The next theorem shows that Butler groups are closed under semi-balanced extensions. Thus the Butler groups form the smallest class of torsion-free groups that contains the rank-one groups and is closed under semi-balanced extensions.

THEOREM 2.3. Let $A \to B \to C$ be a semi-balanced short exact sequence of abelian groups. If A and C are Butler groups, then so is B.

PROOF. Let $\{A_i\}$ and $\{C_j\}$ be finite sets of rank-one subgroups of A and C that generate A and C. Because the sequence is semi-balanced, there exist, for each j, a finite number of rank-one subgroups of B whose sum maps onto C_j. These subgroups, together with the A_i generate B. $\square$

We now show that if T is a finite lattice of types, then every T-space over Q comes from some Butler group (see [BTLR; Theorem 4]).

THEOREM 2.4. Let T be a finite lattice of types and let V be a T-space over Q. Then $V \cong Q \otimes G$ for some Butler T-group G.

PROOF. If V is a quotient space of the completely decomposable T-space $Q_1 \oplus \ldots \oplus Q_n$, then choose subgroups A_i of Q_i such that the type of A_i is the value of a nonzero element of Q_i. Let G be the image of $A_1 \oplus \ldots \oplus A_n$ in V. By Theorem 2.1 the type of an element of G is equal to its value in V, so $V \cong Q \otimes G$. $\square$

To show that the category of Butler T-groups and quasi-homomorphisms is isomorphic to the category of T-spaces over Q, it remains to show that the groups of morphisms are isomorphic. This result, which can be found in [LADY; Theorem 1.5], is essentially a special case of [BTLR2; Theorem 1], and the proof follows [BTLR2; Proposition 1].

THEOREM 2.5. If G and H are T-groups, then there is a natural embedding $Q \otimes \mathrm{Hom}\,(G, H) \to \mathrm{Hom}_T\,(Q \otimes G, Q \otimes H)$. If G is a Butler group, then this map is an isomorphism. Hence the category of T-spaces over Q is isomorphic to the category of Butler T-groups under quasi-homomorphism.

PROOF. The only problem is showing that the natural embedding is an isomorphism. If G is a rank-one group of type τ, then a map $f: Q \otimes G \to Q \otimes H$ is a map of T-spaces if and only if type $(f(x)) \geqslant \tau$ for nonzero x in G. Thus there is $n \neq 0$ such that $nf(G) \subset H$, so

$nf \in \mathrm{Hom}(G, H)$ whence $f \in Q \otimes \mathrm{Hom}(G, H)$. Thus the theorem is true when G is rank-one, and hence when G is a finite direct sum of rank-one groups. For an arbitrary Butler group G, let F be a finite direct sum of rank-one groups mapping onto G with kernel K and consider the diagram

$$\begin{array}{ccccc} 0 \to Q \otimes \mathrm{Hom}(G, H) & \to & Q \otimes \mathrm{Hom}(F, H) & \to & Q \otimes \mathrm{Hom}(K, H) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}_T(Q \otimes G, Q \otimes H) & \to & \mathrm{Hom}_T(Q \otimes F, Q \otimes H) & \to & \mathrm{Hom}_T(Q \otimes K, Q \otimes H) \end{array}$$

The top tow is exact, the bottom row is a zero-sequence whose first map is one-to-one, the vertical maps are injective, and the middle vertical map is an isomorphism as F is a finite direct sum of rank-one groups. A simple diagram chase shows that the first vertical map is onto. $\square$

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