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# An Existence Theorem for a Stochastic Partial Differential Equation Arising from Filtering Theory.

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## 1. Introduction.

In this paper we consider the following stochastic partial differential problem:

$$
\begin{cases} du(t, x) = u_{xx}(t, x)\, dt + h(x) u(t, x)\, dW(t) \\ u(0, x) = u_0(x) \end{cases} \tag{1.1}
$$

where $h$ is any polynomial of degree $n$ and $W(t)$ is a real Wiener process.

Our method consists in performing a transformation of the problem so to get a deterministic equation w.p.1. In fact, putting

$$
v(t, x) = \exp[-h(x) W(t)] u(t, x) \tag{1.2}
$$

it is easy to see that $v$ formally satisfies the following problem w.p.1:

$$
\begin{cases} v_q = v_{xx} + \beta(t, x) v_x + \gamma(t, x) v \\ v(0, x) = u_0(x) \end{cases} \tag{1.3}
$$

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here

$$\beta(t, x) = 2W(t)h_x(x) \tag{1.4}$$

$$\gamma(t, x) = W(t)h_{xx}(x) + W(t)^2 h_x^2(x) - \tfrac{1}{2}h^2(x) \tag{1.5}$$

In the next section we will solve problem (2) by semigroups methods in order to get a solution to problem (1.1) by performing the « inverse » transformation

$$u = \exp[hW(t)]v .$$

We remark that the same procedure adopted for problem (1.1) allows to treat the more general problem

$$\begin{cases} du = (au_{xx} + bu_x + cu)\,dt + (gu_x + hu)\,dW(t) \\ u(0, x) = u_0(x); \end{cases} \tag{1.6}$$

a few details about it will be given at the end of section 3.

Problem (1.6) has been studied by Fleming-Mitter ([4]) using methods of dynamic programming. In a previous paper [1] we have studied a general method which applies to problem (1.6) assuming that $h$ is bounded.

Part of the results of the present paper have been reported in [2].

We are grateful to prof. Bove for useful discussions.

**2.** Here we solve problem (1.3). It can be written as an abstract Cauchy problem in the space $H = L^2(\mathbf{R})$

$$\frac{dv}{dt} = C(t)v , \quad v(0) = u_0 . \tag{2.1}$$

where $C(t)\colon D_{C(t)} \subset H \to H$ is an operator family with constant domain

$$Y = H^2(\mathbf{R}) \cap L^2(\mathbf{R}; x^{4n}\,dx) \ (^1) \tag{2.2}$$

(1) $H^2(\mathbf{R})$ is the usual Sobolev space and $L^2(\mathbf{R}, x^{4n}\,dx)$ denotes the space of square integrable functions with respect to the measure $x^{4n}\,dx$; here $n$ is the degree of the polynomial $h$.

defined by putting

$$C(t)v = v_{xx} + \beta(t, x)v_x + \gamma(t, x)v \qquad \forall v \in Y \tag{2.3}$$

In order to proceed for any $t \in [0, T]$ we consider $C(t)$ as the sum of the following two operators

$$C_1(t) \equiv \begin{cases} D_{C_1(t)} = Y \\ C_1(t)v = v_{xx} + \gamma(t, x)v \end{cases} \tag{2.4}$$

$$C_2(t) \equiv \begin{cases} D_{C_2(t)} = \{v \in H^1(\mathbf{R}), \beta(t, x)v \in L^2(\mathbf{R})\} \\ C_2(t)v = \beta(t, x)v_x \end{cases} \tag{2.5}$$

We have:

LEMMA 1. *For any $t \in [0, T]$ $C_1(t)$ is the infinitesimal generator of an analytic semigroup on $H$.*

PROOF. The proof can be found in [5] pag. 274. In fact here $\gamma(t, x)$ is bounded from above with respect to $x$ as it is polynomial of even order and the leading coefficient is negative (2).

REMARK 2. We remark that the graph norm induced in $Y$ by the operator $C_1(t)$ is equivalent to the norm:

$$|v|_Y^2 = \int_{-\infty}^{+\infty} v_{xx}^2\, dx + \int_{-\infty}^{+\infty} (1 + x^{4n}) v^2\, dx\,, \qquad \forall v \in Y\,.$$

LEMMA 3. *For any fixed $t \in [0, T]$ and $\varepsilon > 0$ there exists $K_{\varepsilon,t} > 0$ such that*

$$|C_2(t)v|_H^2 \leqslant K_{\varepsilon,t}|v|_H^2 + \varepsilon|C_1(t)v|_H^2 \qquad \text{[w.p.1]} \tag{2.6}$$

PROOF. First we note

$$|C_2(t)v|_H^2 = 4W^2(t) \int_{-\infty}^{+\infty} h_x^2 u_x^2\, dx \tag{2.7}$$

(2) We actually remark that $C_1(t)$ is a self-adjoint operator.

Integrating by parts we have:

$$\int_{-\infty}^{+\infty} h_x^2 u_x^2\, dx = -\int_{-\infty}^{+\infty} 2h_x h_{xx} u_x u\, dx - \int_{-\infty}^{+\infty} h_x^2 u_{xx} u\, dx$$

Now it is

$$\int_{-\infty}^{+\infty} 2h_x h_{xx} u u_x\, dx \leqslant \frac{1}{2}\int_{-\infty}^{+\infty} h_x^2 u_x^2\, dx + 2\int_{-\infty}^{+\infty} h_{xx}^2 u^2\, dx$$

$$\int_{-\infty}^{+\infty} h_x^2 u_{xx} u dx \leq \frac{1}{4\varepsilon}\int_{-\infty}^{+\infty} h_x^4 u^2\, dx + \varepsilon\int_{-\infty}^{+\infty} u_{xx}^2\, dx$$

so that

$$\int_{-\infty}^{+\infty} h_x^2 u_x^2\, dx \leqslant 4\int_{-\infty}^{+\infty} h_{xx}^2 u^2\, dx + 2\varepsilon\int_{-\infty}^{+\infty} u_{xx}^2\, dx + \frac{1}{2\varepsilon}\int_{-\infty}^{+\infty} h_x^4 u^2\, dx$$

Denote by $a(\varepsilon)$ a suitable constant such that

$$h_{xx}^2 \leqslant a(\varepsilon) + \varepsilon x^{4n} \qquad h_x^4 \leqslant a(\varepsilon) + 4\varepsilon^2 x^{4n}$$

hence

$$\int_{-\infty}^{+\infty} h_x^2 u_x^2\, dx \leq 6\varepsilon\left[\int_{-\infty}^{+\infty} u_{xx}^2\, dx + \int_{-\infty}^{+\infty} x^{4n} u^2\, dx\right] + (1 + \tfrac{1}{2}) a(\varepsilon)\int_{-\infty}^{+\infty} u^2\, dx$$

so that (2.6) follows from (2.7) and Remark 2.

We further remark that, though not necessary for the sequel, it is possible to prove that the constant $K_{\varepsilon,t}$ can be chosen independently of $t$.

LEMMA 4. *For any* $t \in [0, T]$, $C(t)$ *is the infnitesimal generator of an analytic semigroup. Moreover for any* $\alpha \in ]0, \frac{1}{2}[$ *there exists a constant* $K$ *such that*

(2.8) $$|C(t)v - C(s)v|_H \leqslant K|t - s|^\alpha |v|_Y \quad \text{[w.p.1]}$$

PROOF. The first statement follows by observing that $C_2(t)$ works as a perturbation of $C_1(t)$ (see for instance Kato [5], pag. 500). Finally (2.8) can be easily checked, taking in account that the Wiener process $W(t)$ is w.p.1 pathwise hölder-continuous with any exponent $\alpha \in ]0, \frac{1}{2}[$.

The previous results show that the assumptions of theorem 4.2 of [3] (3) for the existence of a solution to problem (2.1) are verified. Hence we can state the following result:

THEOREM 5. *For any* $u_0 \in H$ *there exists a unique classical solution to problem* (2.1). *That is there exists a unique function*

$$v \in \mathbf{C}([0, T]; H) \cap \mathbf{C}^1(]0, T]; H) \cap \mathbf{C}(]0, T]; Y)$$

*such that* (2.1) *is verified. If moreover* $u_0 \in Y$ *then* $v \in \mathbf{C}([0, T]; Y) \cap \cap \mathbf{C}^1([0, T]; H)$.

**3.** Now we are ready to prove the following result on the equation (1.1).

THEOREM 6. *For any* $u_0 \in H = L^2(\mathbf{R})$ *there exists a process* $u$ *which solves* (1.1) *in the following sense*:

i) $u \in \mathbf{C}([0, T]; L^2_{\text{loc}}(\mathbf{R})) \cap \mathbf{C}(]0, T]; H^2_{\text{loc}})$ [w.p.1]

ii) *for any* $\varphi \in C_0^\infty(\mathbf{R})$ *it is*

$$d(u, \varphi) = (u_{xx}, \varphi)\, dt + (hu, \varphi)\, dW(t) \qquad \textit{for } t > 0;$$

*if moreover* $u_0 \in H$ *then*

$$u_0 \in \mathbf{C}([0, T]; H^2_{\text{loc}})$$

*and* ii) *is verified also for* $t = 0$.

PROOF. To show the existence of a solution take $v$, the solution to problem (2.1), and put

$$u(t, x) = \exp[h(x)\, W(t)]v(t, x)$$

(3) The theorem is an improved version of the well-known result of Tanabe.

It is straightforward to check property i). For ii) consider $(u(t), \varphi)$, $\varphi$ being in $C_0^\infty(\mathbf{R})$; remark that

$$(u(t), \varphi) = (v(t), \exp[h(\cdot)\, W(t)]\varphi)_H; \tag{3.1}$$

by applying Itô formula at the right hand side of (3.1) it is easy to verify iii).

Concerning the more general problem (1.6) we consider the following assumptions:

$$\begin{cases} a \in C_b^1(\mathbf{R}); \quad b, c \in C_b(\mathbf{R}), \quad g \in C_b^2(\mathbf{R}) \\ h \text{ any polynomial of order } n \\ 2a - g^2 \geqslant \varepsilon > 0 \end{cases}$$

Then (1.6) can be solved with the same procedure for problem (1.1) by using the following transform

$$v(t, x) = u\big(t, \varphi(W(t), x)\big) \exp\left[\int_0^{W(t)} h(\varphi(\xi), x)\, d\xi\right]$$

where $\varphi$ is the solution of the following problem

$$\frac{\partial \varphi}{\partial t} = g(\varphi) \qquad \varphi(0, x) = x\,.$$

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