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Straightening of a Noncylindrical Region and Evolution Equations (*).

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Sunto - In questo lavoro viene presentato un metodo geometrico per trattare equazioni d'evoluzione in una regione non cilindrica che consiste nell'uso di un opportuno diffeomorfismo che trasforma la regione non cilindrica in una regione cilindrica.

1. - Introduction.

In this paper, we shall exhibit a geometrical method of attacking evolution equations in a noncylindrical region, which consists in the use of a suitable time-preserving diffeomorphism mapping the noncylindrical domain into a cylindrical one.

We note that one can find in the literature very few results of some generality on existence and regularity of solutions of parabolic or hyperbolic mixed problems in non-cylindrical domains (a remarkable exception is the big paper by Solonnikov [10], where parabolic systems in spaces of Hölder continuous functions are considered).

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Our device allows to bypass the remarkable technical difficulties arising in the study of such problems.

In Section 2 we shall construct the diffeomorphism (having the same regularity as the lateral boundary Σ), supposing that Σ is nowhere tangent to the hyperplanes $t = \text{constant}$. In Section 3 we give some applications to various mixed boundary value problems for parabolic and hyperbolic equations (obviously, no effort of being exhaustive has been made). In view of the properties of our transformation, we are able to extend to the non-cylindrical case the classical results in the cylindrical domains. To avoid a cumbersome list of formal theorems which would have considerably lengthened the paper, we have preferred a more informal style, which however clearly points out the relevant aspects of our method.

2. – The straightening diffeomorphism.

Let T be a fixed positive real number and let E be a bounded, connected and relatively open set in $[0, T]\times R^n$. Put

$$E_t = E \cap \{(t, y);\ y \in R^n\}\ ,$$

Σ_t the boundary of E_t relative to $\{(t, y);\ y \in R^n\}$,

$$\Sigma = \bigcup_{t\in[0,T]} \Sigma_t\ .$$

In the sequel, we shall always suppose that:

I) Σ is an n-manifold of class $C^{(k)}$ ($k\geq 2$) with boundary $\partial\Sigma = \Sigma_0 \cup \Sigma_T$;

II) E is locally on one side of Σ;

III) The tangent space to Σ is never $t = 0$.

We note that Σ is the boundary of E relative to $[0, T]\times R^n$.

Whenever there is no way to misunderstanding, we shall identify E_t and Σ_t with their projections onto R^n.

Let us now state our main result.

THEOREM. *Let hypotheses* I), II), III) *be satisfied. Then there exists a* $C^{(k)}$ *diffeomorphism* r *from* $\bar{E}$ *onto* $[0, T]\times\bar{E}_0$, *such that*:

i) $r(\Sigma) = [0, T]\times\Sigma_0$;

ii) $r(\bar{E}_t) = \{t\}\times\bar{E}_0$.

Here and in what follows, by a $C^{(k)}$ function defined in the closure of an open set, we mean a function having uniformly continuous partial derivatives up to the k-th order in the interior of the domain. Furthermore, a $C^{(k)}$ diffeomorphism will be a homeomorphism which is a $C^{(k)}$ function together with its inverse.

PROOF. Let $(t_0, x_0)\in\Sigma$, $0 < t_0 < T$; then by hypotheses I), II), III), there exist a neighbourhood $U_{(t_0,x_0)}$ of (t_0, x_0) in $\bar{E}$ and a $C^{(k)}$ diffeomorphism $\varphi_{(t_0,x_0)}$: $U_{(t_0,x_0)}\to R\times\bar{R}^n_+$, such that

$$\varphi_{(t_0,x_0)}(\Sigma\cap U_{(t_0,x_0)})\subseteq R\times R^{n-1}\times\{0\}\,, \tag{2.a}$$

$$\varphi_{(t_0,x_0)}(t, x) = \big(t, \psi_{(t_0,x_0)}(t, x)\big)\,. \tag{2.b}$$

Slightly modifying the preceding formulas, we can handle the cases $t_0 = 0$ and $t_0 = T$, too.

Let $\{U_1, \ldots, U_m\}$ be a finite subcovering of $\{U_{(t_0,x_0)};\ (t_0, x_0)\in\Sigma\}$ and $\varphi_1, \ldots, \varphi_m$ be the corresponding diffeomorphisms. Let us define the following vector fields:

$$X_j = \varphi_{j*}(1, 0)\,,\qquad j = 1, \ldots, m\,,$$

where φ_{j*} is the pull-back of φ_j. It turns out that X_j is a $C^{(k-1)}$ vector field tangent to Σ in $U_j\cap\Sigma$; furthermore, by (2.b), the t-component of X_j always equals 1.

Let $\{\omega_j;\ j = 1, \ldots, m\}$ be a C^∞ partition of unity subordinated to the covering $\{U_j;\ j = 1, \ldots, m\}$; we agree to think of each ω_j as continued by zero to all of $\bar{E}$. Put $\omega_0 = 1-\sum_{j=1}^{m}\omega_j$ and

$$X = \omega_0(1, 0) + \sum_{j=1}^{m}\omega_j X_j\,.$$

It turns out that:

(*a*) X is a $C^{(k-1)}$ vector field tangent to Σ on Σ;

(*b*) the t-component of X always equals 1.

Hence the integral curves of X can be parametrized by time and are defined on all of $[0, T]$.

Now, through every point in $\bar{E}$ there is a unique integral curve of X, so we can define a bijection ϱ from $\bar{E}$ onto $[0, T]\times\bar{E}_0$ in the following way:

$$\varrho(t, x) = \big(t, \eta(-t, (t, x))\big),$$

where $(s, (t, x)) \to \big(s + t, \eta(s, (t, x))\big)$ is the flow of X. From a geometrical point of view, the function ϱ maps the point (t, x) into the point (t, y), where $(0, y)$ is the point on the bottom of $\bar{E}$ lying on the integral curve of X issued from (t, x).

Property i) required by our diffeomorphism follows by (*a*), while ii) follows by the very definition of X. As $\varrho^{-1}(t, y) = \big(t, \eta(t, (0, y))\big)$, by the regularity theorem of the flow, it follows that ϱ is a $C^{(k-1)}$ diffeomorphism. Furthermore, $\partial\varrho/\partial t$ and $\partial\varrho^{-1}/\partial t$ are $C^{(k-1)}$ functions.

If $k = \infty$ the theorem is proved; otherwise we must modify the diffeomorphism in order to get a more regular one.

Roughly speaking, this will be accomplished first by showing that the set of diffeomorphisms is an open subset of the set of all $C^{(k-1)}$ functions having the same structure as ϱ, and then by showing that ϱ is a limit point of $C^{(k)}$ functions.

Let us now proceed more formally; put, if $l \geqslant 1$,

$$\Phi_l = \{f \in C^{(l)}(\bar{E}; [0, T]\times\bar{E}_0); f(t, x) = (t, \psi(t, x)), \text{ where } \psi(t, \cdot)(\Sigma_t) \subseteq \Sigma_0\}$$

equipped with the norm

$$\|f\|_l = \sum_{|\alpha|\leqslant l} \sup_E |D^\alpha f|.$$

LEMMA 1. *The set of diffeomorphisms is an open subset of* Φ_l.

PROOF. Let us first note that the set of diffeomorphisms in Φ_l

may be described as the set Ψ_l of functions f in Φ_l, such that:

α) f is bijective;

β) f^{-1} is continuous;

γ) $|\det \mathfrak{J}_f(t, x)| \geqslant c_f > 0$, $\forall (t, x) \in E$.

Let $f \in \Psi_l$ be fixed and let $\varepsilon > 0$, suitable. If $g \in \Phi_l$, $\|g - f\|_l < \varepsilon$, it is obvious that g satisfies γ).

Let us now show that, if ε is sufficiently small, then g is injective. Indeed, suppose the contrary; then there exist a sequence $(g_n)_{n\in N}$ in Φ_l, $\|g_n - f\|_l \to 0$ and two sequences $(a_n)_{n\in N}$, $(b_n)_{n\in N}$ in $\bar{E}$, such that $g_n(a_n) = g_n(b_n)$, $a_n \neq b_n$. By the compactness of $\bar{E}$, we may suppose $a_n \to a$, $b_n \to b$ and, since f is injective, it turns out that $a = b$. Suppose $a \in E$; the case $a \in \Sigma$ can be handled analogously by local charts. Put $h_n = (dg_n(a))^{-1} \circ g_n - I$; since $dh_n(a) = 0$, if n is large enough we have

$$|a_n - b_n| = |h_n(a_n) - h_n(b_n)| \leqslant \tfrac{1}{2}|a_n - b_n| ,$$

which is impossible.

Let us now show that g is surjective, too. By the definition of Φ_l we have only to show that the spatial component $\chi(t, \cdot)$ of $g(t, \cdot)$ is onto $\bar{E}_0$, $\forall t \in [0, T]$. As $\Psi_1 \neq \emptyset$ and $\bar{E}$ is connected, then $[0, T] \times \bar{E}_0$ is connected; so $\bar{E}_0$ is connected, too. Hence, $\bar{E}_t$ is connected. Since $\chi(t, \cdot)(\Sigma_t) \subseteq \Sigma_0$, by γ), $\chi(t, \cdot)$ is a submersion, hence an open map (see, e.g., [9], 3.1.8). So, $\chi(t, \cdot)(\bar{E}_t)$ is an open and compact subset of $\bar{E}_0$, thus coinciding with it, by connectedness.

Finally, β) follows by the compactness of $\bar{E}$.

To complete the proof of the Theorem, we have only to establish the possibility of approximating, in the Φ_{k-1} topology, the diffeomorphism ϱ, constructed at the beginning of the proof, by functions in Φ_k. To this end, we need the following local approximation result.

LEMMA 2. *Let X be anyone of the following sets:* i) R^{n+1}; ii) $R \times R^n_+$; iii) $\bar{R}_+ \times \overline{R^n_+}$. *Let U, V be open subsets of X and let*

$$K\colon U \to V\ , \qquad K(t, x) = (t, \varphi(t, x)) ,$$

be such that:

a) $\partial_t^l \partial_x^\alpha \varphi$ *is uniformly continuous in* $\operatorname{int}(U)$ *if* $l + |\alpha| \leqslant k$, $|\alpha| \leqslant k - 1$;

b) *in the cases* ii), iii), $\varphi(t,\cdot)$ *maps the hyperplane* $x_n = 0$ *into itself.*

Then, if $U' \subset \overline{U}' \subset U$, *for every* $\varepsilon > 0$, *there exists* $\varphi_\varepsilon \in C^{(k)}(\overline{U}'; R^n)$, *enjoying property* b), *such that, if*

$$K_\varepsilon(t, x) = (t, \varphi_\varepsilon(t, x)) ,$$

then $\sum_{|\beta| \leqslant k-1} \sup_{U'} |\partial^\beta K_\varepsilon - \partial^\beta K| < \varepsilon.$

This Lemma is proved by a machinery analogous to the one in [3], Chapter 2, Lemma 3.1, but regularizing only with respect the space variables.

The proof of the claimed approximation result can now be accomplished by arguments analogous to those in [3], Theorem 2.6, applying Lemma 2.

3. – Applications.

3.1. Since the diffeomorphism exhibited in the preceding section is time-preserving, parabolic systems of every type are transformed in systems of the same type in a cylindrical region; furthermore, the transformed boundary operators satisfy the complementary condition if this was satisfied by the original boundary operators. The compatibility conditions (cc) for the data need a more careful treatment. Indeed, in a noncylindrical region, these conditions must be given via time preserving local charts; since cc are invariant under a time preserving diffeomorphism in $R_+ \times R^{n-1} \times R_+$, if they are satisfied in an atlas, they are satisfied in any other one, too. An analogous argument shows that compatible original data are mapped into compatible ones by the diffeomorphism built in Section 2.

Obviously, Sobolev, Besov and Hölder spaces of parabolic type do not change. So, existence, uniqueness and regularity results for parabolic problems in noncylindrical regions satisfying hypotheses I), II), III) can be obtained directly by the corresponding cylindrical results. In particular, for Petrovskiĭ parabolic systems, Solonnikov's results ([10], Theorem 5.4, [6] Theorems 10.1 and 10.4) hold true even in noncylindrical regions. Analogously, Grisvard's results for parabolic equations of higher order in t ([2], Theorem 8.9) can be extended to the noncylindrical situation.

This approach may be also useful in several situations in nonlinear problems.

3.2. Quite analogously, strict hyperbolicity of systems and equations is preserved by the diffeomorphism r, built in Section 2. Hence, we shall be able to consider mixed boundary value problems for these operators if Σ satisfies the further hypothesis

IV) Σ is never characteristic for the operator.

This hypothesis is clearly invariant under our diffeomorphism. Furthermore, boundary conditions satisfying the uniform Lopatinskiĭ conditions go over to boundary conditions with the same property. As far as compatible data are concerned, we may repeat what has been said in the preceding subsection.

So existence, uniqueness and regularity results for hyperbolic problems in noncylindrical regions may be obtained from the analogous ones in cylindrical regions (see e.g., [5], [8], [4], [1] Chapter VII, § 7, [7] Chapter V).

Added in proof. Some recent results by T. Miyakawa and Y. Teramoto ([11], [12]) show that our technique can be used also for Navier-Stokes equations with moving boundary.

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