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On a Synonymy Relation for Extensional 1st Order Theories.

PART III

A Necessary and Sufficient Condition for Synonymy.

C. BONOTTO - A. BRESSAN (*)

10. Intuitive description of the theory \mathcal{F} . Semiotics for \mathcal{F} ⁽¹⁾.

We want to associate theory \mathcal{T} based on the language \mathcal{L} —see [1], § 6—with an interpreted theory \mathcal{F} in compliance with the following intuitive requirements, which refer to any model $\mathcal{I} = (\mathcal{D}, \mathcal{I}, \alpha)$ and any v -valuation V for \mathcal{F} , and hold for n , $i = 1, 2, \dots$

(a) *The variable x_i of \mathcal{F} acts, as far as its sense is concerned, as a proper name, i.e. as primitive constant.*

(b) *The relator R_i^n and connectives \sim and \supset are associated with three individual constants \hat{R}_i^n , $\hat{\sim}$, and $\hat{\supset}$ of \mathcal{F} , that designate (\mathcal{I}, V) -senses of R_i^n , \sim , and \supset respectively—i.e. the senses of R_i^n to \supset with respect to \mathcal{I} and V .*

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⁽¹⁾ The present paper is the third part of a work whose first and second part are [1] and [2] respectively. Therefore the numbering of its sections follows those for [1] and [2].

(c) The functor f_i^n is associated with the individual constant f_i^n of \mathcal{F} , that denotes the (\mathcal{F}, V) -sense of f_i^n .

(d) [(e)] $A_n [V_n]$ is an $(n + 1)$ -ary functor of \mathcal{F} , and wffs having the 1st $[2^{nd}]$ of the forms

$$(10.1) \quad \eta = A_n(\varrho, \xi_1, \dots, \xi_n), \quad \zeta = V_n(\sigma, \xi_1, \dots, \xi_n)$$

express that η [ζ] is the (\mathcal{F}, V) -sense of the application [value] of the n -ary attribute R [functor f], of (\mathcal{F}, V) -sense ϱ [σ], to the terms Δ_1 to Δ_n of the respective (\mathcal{F}, V) -senses ξ_1 to ξ_n .

(f) \mathcal{F} has a predicate \mathcal{V} such that any wff of \mathcal{F} , of the form $\mathcal{V}(\xi)$ expresses that ξ is the (\mathcal{F}, V) -sense of a wff \mathcal{A} of \mathcal{F} , and that \mathcal{A} is true in \mathcal{F} at V , i.e. $\text{des}_{\mathcal{F}, V}(\mathcal{A}) = 0$.

(g) The variable x_i is associated with an operator (Ωx_i) of \mathcal{F} to be denoted with (\dot{x}_i) , such that $(\dot{x}_i)\Delta$ denotes the (\mathcal{F}, V) -sense of the wff $(x_i)\Delta$ of \mathcal{F} .

Now in order to define the theory \mathcal{F} rigorously, we stipulate, first, that, as well as \mathcal{F} , it is based on the language \mathcal{L} —see [1], §§ 2, 3—, so that \mathcal{F} has the same variables as \mathcal{F} .

It is not restrictive to assume \mathcal{F} to have only constants c_i , R_i^n , or f_i^n with an odd value of i ; for should this situation not occur, we can render it holding by performing the replacement $i \rightarrow 2i - 1$ ($i = 1, 2, \dots$) of all constants of \mathcal{F} . Thus denoting by $\mathcal{P}\mathcal{C}$ (primitive constants and connectives) the set formed by the constants c_i , R_i^n , and f_i^n of \mathcal{F} and by the connectives \sim and \supset , we can choose

(i) an injection χ of $\mathcal{P}\mathcal{C}$ into the new individual constants, i.e. c_{2e} ($e = 1, 2, \dots$), and

(ii) a predicate \mathcal{V} and two $(n + 1)$ -ary functors A_n and V_n in \mathcal{L} outside the counterdomain of χ ($n = 1, 2, \dots$).

We shall denote $\chi(c_i)$ to $\chi(\supset)$ by \dot{c}_i to $\dot{\supset}$ respectively.

We also stipulate that the constants of \mathcal{F} are \mathcal{V} , A_n , V_n ($n = 1, 2, \dots$), the equality sign R_1^2 , the χ -transformed of the elements in $\mathcal{P}\mathcal{C}$, and a term-term operator sign Ω ⁽²⁾.

⁽²⁾ \mathcal{F} need not have any inexistent object.

We now associate every wff Δ of \mathcal{F} with the *corresponding* wff $\dot{\Delta}$ of $\dot{\mathcal{F}}$ by means of the following recursive rules:

Rule	If Δ is	then $\dot{\Delta}$ is
s_1	x_i or c_i	\dot{x}_i or \dot{c}_i respectively
s_2	$f_i^n(\Delta_1, \dots, \Delta_n)$	$V_n(\dot{f}_i^n, \dot{\Delta}_1, \dots, \dot{\Delta}_n)$
s_3	$R_i^n(\Delta_1, \dots, \Delta_n)$	$A_n(\dot{R}_i^n, \dot{\Delta}_1, \dots, \dot{\Delta}_n)$
s_{4-5}	$\sim \Delta_1$ [$\Delta_1 \supset \Delta_2$]	$A_1(\sim, \dot{\Delta}_1)$ [$A_2(\dot{\supset}, \dot{\Delta}_1, \dot{\Delta}_2)$]
s_6	$(x_i)\Delta$	$(\Omega x_i)\dot{\Delta}$.

Remark that if \mathcal{A} is a wff of \mathcal{F} , then $\dot{\mathcal{A}}$ is a term of $\dot{\mathcal{F}}$.

11. Axioms for $\dot{\mathcal{F}}$. Some theorems relating \mathcal{F} and $\dot{\mathcal{F}}$.

As proper axioms or axiom schemes of $\dot{\mathcal{F}}$, we take those on identity—see [4]—, i.e.

$$\text{AA1-2} \quad x = x, \quad x = y \supset (\mathcal{A}(x) \supset \mathcal{A}(y)),$$

the special axioms AA3-6 below on \mathcal{V} , where \mathcal{B} , \mathcal{B}_1 , and \mathcal{B}_2 are arbitrary wffs of \mathcal{F} ,

$$\text{A3} \quad \mathcal{V}[A_1(\sim, \dot{\mathcal{B}})] \equiv \sim \mathcal{V}(\dot{\mathcal{B}}),$$

$$\text{A4} \quad \mathcal{V}[A_2(\dot{\supset}, \dot{\mathcal{B}}_1, \dot{\mathcal{B}}_2)] \equiv \mathcal{V}(\dot{\mathcal{B}}_1) \supset \mathcal{V}(\dot{\mathcal{B}}_2),$$

$$\text{A5} \quad \mathcal{V}[(\Omega x_i)\dot{\mathcal{B}}] \equiv (x_i)\mathcal{V}(\dot{\mathcal{B}}),$$

$$\text{A6} \quad \sim \mathcal{V}(\tau) \text{ if } \tau \neq \dot{\Delta} \text{ for every wff } \Delta \text{ of } \mathcal{F},$$

the following three axioms, connected with the synonymy conditions C_6 , C_1 , and C_7) in [1], § 6—see Def. 3.1.

$$\text{AA7-8} \quad (x_i)(\dot{p} = \dot{p}') \supset (\Omega x_i)\dot{p} = (\Omega x_i)\dot{p}', \quad \dot{D}'_\nu = \dot{D}''_\nu \quad (0 < \nu < \omega),$$

$$\text{A9} \quad (\Omega x_i)\dot{\mathcal{B}} = (\Omega x_i)\dot{\mathcal{C}} \text{ if } \mathcal{B} \text{ and } \mathcal{C} \text{ are } (x_i, x_j)\text{-similar wffs of } \mathcal{F},$$

and the following counterparts of \mathcal{T} 's axioms

A10 $\mathcal{V}(\mathcal{A})$ whenever \mathcal{A} is an axiom of \mathcal{T} .

Since \mathcal{F} is based on \mathcal{L} , its inference rules are MP and Gen. By AA1-2 theorems (11.1-4) below hold. They are connected with the synonymy conditions C_2 to C_5 in [1], and in them p, q, p' , and q' are arbitrary wffs of \mathcal{T} , f and f' [R and R'] are arbitrary n -ary functor [relators] of \mathcal{T} and $\Delta_1, \Delta'_1, \dots, \Delta_n, \Delta'_n$ are arbitrary terms of \mathcal{T} ; n runs over \mathbb{Z}^+ and $\Delta_i^n p_i$ means $p_1 \wedge \dots \wedge p_n$.

$$(11.1) \quad \vdash_{\mathcal{F}} f = f' \wedge \Delta_i^n \Delta_i = \Delta'_i \supset V_n(f, \Delta_1, \dots, \Delta_n) = V_n(f', \Delta'_1, \dots, \Delta'_n),$$

$$(11.2) \quad \vdash_{\mathcal{F}} R = R' \wedge \Delta_i^n \Delta_i = \Delta'_i \supset A_n(R, \Delta_1, \dots, \Delta_n) = \\ = A_n(R', \Delta'_1, \dots, \Delta'_n),$$

$$(11.3) \quad \vdash_{\mathcal{F}} \dot{p} = \dot{p}' \supset A_1(\dot{\sim}, \dot{p}) = A_1(\dot{\sim}, \dot{p}'),$$

$$(11.4) \quad \vdash_{\mathcal{F}} \dot{p} = \dot{p}' \wedge \dot{q} = \dot{q}' \supset A_2(\dot{\supset}, \dot{p}, \dot{q}) = A_2(\dot{\supset}, \dot{p}', \dot{q}').$$

THEOR. 11.5. *If p is a wff of \mathcal{T} and $\vdash_{\mathcal{F}} p$, then $\vdash_{\mathcal{F}} \mathcal{V}(\dot{p})$.*

Indeed let \mathcal{B}_1 to \mathcal{B}_n be a proof of p in \mathcal{T} . We assume

$$(11.6) \quad \vdash_{\mathcal{F}} \mathcal{V}(\dot{\mathcal{B}}_j) \quad \text{for } j = 1, 2, \dots, i-1, \text{ where } i \leq n$$

(which holds vacuously for $i = 1$) as the hypothesis of our (complete) induction. Then one of alternatives a) to c) below holds.

a) \mathcal{B}_i is an axiom of \mathcal{T} . Then $\vdash_{\mathcal{F}} \mathcal{V}(\dot{\mathcal{B}}_i)$ by A10.

b) For some r and s smaller than i , \mathcal{B}_s is $\mathcal{B}_r \supset \mathcal{B}_i$ (MP). Then by rule s_5) in § 10 and A4 $\vdash_{\mathcal{F}} \mathcal{V}(\dot{\mathcal{B}}_s) \equiv [\mathcal{V}(\dot{\mathcal{B}}_r) \supset \mathcal{V}(\dot{\mathcal{B}}_i)]$. Hence, by (11.6) and MP $\vdash_{\mathcal{F}} \mathcal{V}(\dot{\mathcal{B}}_i)$.

c) For some $r < i$ and some k , \mathcal{B}_i is $(x_k)\mathcal{B}_r$. By (11.6) $\vdash_{\mathcal{F}} \mathcal{V}(\dot{\mathcal{B}}_r)$. Then, by Gen, $\vdash_{\mathcal{F}} (x_k)\mathcal{V}(\dot{\mathcal{B}}_r)$. Furthermore, by rule s_6) in § 10 and A5 $\vdash_{\mathcal{F}} (x_k)\mathcal{V}(\dot{\mathcal{B}}_r) \equiv \mathcal{V}(\dot{\mathcal{B}}_i)$. Hence $\vdash_{\mathcal{F}} \mathcal{V}(\dot{\mathcal{B}}_i)$.

Then by the principle of complete induction, (11.6) holds for $j = n$ (and $\mathcal{B}_n = p$). q.e.d.

THEOR. 11.6. *If a and b are wfs of \mathcal{T} and $a \succ b$ then $\vdash_{\mathcal{F}} \dot{a} = \dot{b}$.*

Indeed let \mathcal{R} be the relation such that $a\mathcal{R}b$ iff a and b are wfs of \mathcal{T} and $\vdash_{\mathcal{F}} \dot{a} = \dot{b}$. By AA1, 2, \mathcal{R} is an equivalence relation. Furthermore by A8, theorems (11.1-4), A7, and A9, relation \mathcal{R} satisfies the synonymy conditions C_1 to C_7 in [1], § 6.

To check the assertion above is obvious, except in connection with condition C_6). Therefore we now assume that $p \mathcal{R} p'$ where p and p' are wfs of \mathcal{T} . Then $\vdash_{\mathcal{F}} \dot{p} = \dot{p}'$ by the definition of \mathcal{R} . Hence $\vdash_{\mathcal{F}} (x_i)\dot{p} = \dot{p}'$ by Gen. Then by A7 we deduce $\vdash_{\mathcal{F}} (\Omega x_i)\dot{p} = (\Omega x_i)\dot{p}'$, which by rule s_6 in § 10 is $\vdash_{\mathcal{F}} \dot{a} = \dot{b}$, where a is $(x_i)p$ and b is $(x_i)p'$. Hence, by the definition of \mathcal{R} , $(x_i)p \mathcal{R} (x_i)p'$. We conclude that also condition C_7 in [1] is satisfied by \mathcal{R} , and the italicized assertion above is completely proved.

Since the synonymy relation \succ is the least equivalence relation that satisfies conditions C_{1-7} in [1], $\succ \subseteq \mathcal{R}$. Then our thesis holds. q.e.d.

THEOR. 11.7. $\vdash_{\mathcal{F}} (x_i)\dot{p} = \dot{p}' \supset (\Omega x_i)\dot{p} = (\Omega x_i)\dot{p}''$ where p' and p'' are (x_i, x_j) -similar.

PROOF. By the completeness of \mathcal{F} it suffices to show that the wff $(x_i)\dot{p} = \dot{p}' \supset (\Omega x_i)\dot{p} = (\Omega x_i)\dot{p}''$ is true in every normal model of \mathcal{F} . Let \mathbf{M} be such a model. Note that

- (i) V is an \mathbf{M} -valuation,
- (ii) $\text{des}_{\mathbf{M},V} [(x_i)\dot{p} = \dot{p}'] = 0$,
- (iii) p' and p'' are (x_i, x_j) -similar.

By (ii), A7 yields $\text{des}_{\mathbf{M},V} ((\Omega x_i)\dot{p} = (\Omega x_i)\dot{p}') = 0$ and hence (since \mathbf{M} is normal)

$$(11.7) \quad \text{des}_{\mathbf{M},V} (\Omega x_i)\dot{p} = \text{des}_{\mathbf{M},V} (\Omega x_i)\dot{p}' .$$

By (iii) and A9 we have

$$(11.8) \quad \text{des}_{\mathbf{M},V} (\Omega x_i)\dot{p}' = \text{des}_{\mathbf{M},V} (\Omega x_i)\dot{p}'' .$$

Hence by (11.7) and (11.8) we deduce

$$(11.9) \quad \text{des}_{\mathbf{M},V} (\Omega x_i)\dot{p} = \text{des}_{\mathbf{M},V} (\Omega x_i)\dot{p}'' ,$$

i.e.

$$(11.10) \quad \text{des}_{\mathbf{M}, \mathcal{V}}[(\Omega x_i)\dot{p} = (\Omega x_i)\dot{p}'] = 0.$$

Hence the wff $(x_i)\dot{p} = \dot{p}' \supset (\Omega x_i)\dot{p} = (\Omega x_i)\dot{p}'$ is true in \mathbf{M} . Since \mathbf{M} is arbitrary, the thesis holds.

12. Statement of a necessary and sufficient condition for synonymy.

DEFINITION 12.1. *We say that a and b are $\dot{\mathcal{F}}$ -equivalent, briefly $a \mathcal{E} b$, if a and b are wfes of \mathcal{F} and*

$$(12.1) \quad \text{des}_{\mathbf{M}, \mathcal{W}} \dot{a} = \text{des}_{\mathbf{M}, \mathcal{W}} \dot{b}$$

for every normal model \mathbf{M} of $\dot{\mathcal{F}}$ and every \mathbf{M} -valuation \mathcal{W} ⁽³⁾.

The main aim of Part 3 is the following equivalence theorem, in that it allows the inversion of Theor. 11.6.

THEOR. 12.1. *If a and b are wfes of \mathcal{F} , $a \succ b \Leftrightarrow a \mathcal{E} b$.*

PROOF OF THE \Rightarrow -PART. Let $a \succ b$. Then by Theor. 11.6,

$$\vdash_{\dot{\mathcal{F}}} \dot{a} = \dot{b}.$$

Since $\dot{\mathcal{F}}$ is a theory based on \mathcal{L} , by Theor. 3.4 in [1] the wff $\dot{a} = \dot{b}$ is true in every (normal) model \mathbf{M} of $\dot{\mathcal{F}}$ and at any \mathbf{M} -valuation \mathcal{W} . Then by the definition of normal model, (12.1) holds for arbitrary such \mathbf{M} and \mathcal{W} . Then, by Def. 12.1, $a \mathcal{E} b$.

PROOF OF THE \Leftarrow -PART. Let $a \mathcal{E} b$. In order to construct a suitable model $\mathcal{F} = (\mathbf{D}, \mathcal{I})$ of the theory $\dot{\mathcal{F}}$ (with a general operator Ω), associated to the one $\mathcal{F} = (\mathbf{D}, \mathcal{I}, \alpha)$ for the theory \mathcal{F} (without general operators) we consider the set B formed by the wfes of \mathcal{F} , the functional and predicative letters of \mathcal{F} , and the connectives \sim and \supset .

⁽³⁾ Obviously $a \mathcal{E} b$ iff a and b are wfes of \mathcal{F} and $\text{des}_{\mathbf{M}, \mathcal{W}}(\dot{a} = \dot{b}) = 0$ for every model \mathbf{M} of \mathcal{F} and every \mathbf{M} -valuation \mathcal{W} —i.e. iff, for all such \mathbf{M} and \mathcal{W} , we have $\text{des}_{\mathbf{M}, \mathcal{W}} a E_{\mathbf{M}} \text{des}_{\mathbf{M}, \mathcal{W}} b$, where $E_{\mathbf{M}}$ is the equivalence denoted in \mathbf{M} by the identity sign of $\dot{\mathcal{F}}$.

Furthermore we set

$$(12.2) \quad \dot{B} = \{\Delta \mid \Delta \in B\}, \quad [\Delta] = \{\Delta_1 \mid \Delta_1 \in B \ \& \ \Delta_1 \succ \Delta\} \cup \{\Delta\} \quad (*).$$

Now we specify the domain \mathbf{D} of \mathcal{F} :

$$(12.3) \quad \mathbf{D} = \{[\Delta] \mid \Delta \in B\} \cup \{\emptyset\} \quad (\emptyset = \text{the empty set}).$$

Note that \mathbf{D} is *denumerable (hence infinite)*. Indeed such is B ($x_i \in B$ for $i = 1, 2, \dots$). Furthermore, by Theor. 8.1 in [2] $x_i \not\asymp x_j$ for $i \neq j$ and $i, j = 1, 2, \dots$; and $(12.2)_2$ and (12.3) hold.

Let $\Delta = \hat{\Delta}(x_1, \dots, x_n)$ be an expression of \mathcal{F} in B , whose free variables, if any, occur among the n (distinct) variables x_1 to x_n ; and let us consider the condition

COND. 12.1. For some wfes Δ_1 to Δ_n of \mathcal{F} , $\hat{\Delta}(\Delta_1, \dots, \Delta_n)$ too is a wfe of \mathcal{F} and $\Phi_i = [\Delta_i]$ ($\in \mathbf{D}$) ($i = 1, \dots, n$)—cf. Convention 2.2 in [1].

STEP 1 (*in the proof of \Leftarrow*). There is a mapping $g_{\Delta; x_1, \dots, x_n}$ of \mathbf{D}^n into \mathbf{D} for which

$$(12.4) \quad g_{\Delta; x_1, \dots, x_n}(\Phi_1, \dots, \Phi_n) = \begin{cases} [\hat{\Delta}(\Delta_1, \dots, \Delta_n)] & \text{under Cond. 12.1,} \\ \emptyset & \text{otherwise (for } \Phi_1, \dots, \Phi_n \in \mathbf{D}). \end{cases}$$

To prove Step 1 it suffices to consider the case when Cond. 12.1 holds; (in the remaining case the proof is trivial). Therefore we assume that for $i = 1$ to n , $\Phi_i = [\Delta_i] = [\Delta'_i]$ with Δ_i and Δ'_i wfes of \mathcal{F} ; hence $\Delta_i \succ \Delta'_i$. Hence, by Theor. 6.1 in [1], $\hat{\Delta}(\Delta_1, \dots, \Delta_n) \succ \hat{\Delta}(\Delta'_1, \dots, \Delta'_n)$, which by (12.2) yields our thesis.

STEP 2. There is an interpretation $\xi^* = \mathcal{F}(\xi)$ of \mathcal{F} that, under the definition

$$(12.5) \quad W_1(x_i) = [x_i] \quad (i = 1, 2, \dots)$$

(*) The simplification of $(12.2)_2$ into $[\Delta] = \{\Delta_1 \mid \Delta_1 \in B \ \& \ \Delta_1 \succ \Delta\}$ would imply $[\sim] = [\subset] = \emptyset$, since the synonymy relation is defined between wfes of \mathcal{F} , and \sim, \supset are not wfes of \mathcal{F} .

of the v -valuation W_1 for \mathcal{F} , fulfils the (interpretation) conditions (i₁) to (i₆) below for all $\Phi_0, \dots, \Phi_n \in \mathbf{D}$, for every element Δ of B that has at most x_i as a free variable, and for every v -valuation W .

$$(i_1) \quad \dot{c}_i^* = \mathcal{I}(\dot{c}_i) = [c_i].$$

(i₂) [(i₃)] $A_n^*(\Phi_0, \dots, \Phi_n) [V_n^*(\Phi_0, \dots, \Phi_n)]$ is $[\Delta]$, in case for some elements Δ_0 to Δ_n of B , $\Phi_i = [\Delta_i]$ ($i = 0, \dots, n$) and $\Delta_0(\Delta_1, \dots, \Delta_n)$ is a wff [term] Δ of \mathcal{F} respectively; it is \emptyset otherwise.

$$(i_4) \quad \Omega^*(g_{\Delta; x_i}) \text{ is } [(x_i)\Delta] \text{ if } \Delta \text{ is a wff of } \mathcal{F}; \text{ it is } \emptyset \text{ otherwise.}$$

$$(i_5) \quad \mathcal{V}^* = \{\xi \in \mathbf{D} \mid \text{for some wff } \Delta \text{ of } \mathcal{F} \text{ des}_{\mathcal{F}, W}(\Delta) = 0 \text{ and } [\Delta] = \xi\}.$$

$$(i_6) \quad =^* = \text{the identity relation in } \mathbf{D}.$$

To prove Step 2 it suffices to show that conditions (i₂) to (i₄) are good definitions of functions. To reach this aim in connection with (i₂) [(i₃)] we assume that (i) $\Delta_0, \Delta'_0, \dots, \Delta_n, \Delta'_n$ are elements of B , (ii) $\Delta_0(\Delta_1, \dots, \Delta_n)$ and $\Delta'_0(\Delta'_1, \dots, \Delta'_n)$ are two wffs [terms], say Δ and Δ' respectively, and (iii) $[\Delta_i] = [\Delta'_i]$ ($i = 0, \dots, n$). Then by (12.2)₂ $\Delta_i \asymp \Delta'_i$ ($i = 0, \dots, n$), so that by Theor. 6.1 in [1] $\Delta \asymp \Delta'$. Then $[\Delta] = [\Delta']$ by (12.2)₂.

Thus our goal is reached in the first case considered in (i₂) [(i₃)]. In the remaining case this thesis is trivial.

To prove the acceptability of definition in (i₄) we assume that $\Delta_1 = \hat{\Delta}_1(x_i)$, $\Delta_2 = \hat{\Delta}_2(x_j)$ and

$$(12.6) \quad g_{\Delta_1; x_i} = g_{\Delta_2; x_j}, \quad \text{hence } g_{\Delta_1; x_i}([\mathbf{x}_r]) = g_{\Delta_2; x_j}([\mathbf{x}_r]) \\ (r = 1, 2, \dots).$$

Then, by (12.4) $[\hat{\Delta}_1(x_r)] = [\hat{\Delta}_2(x_r)]$ and hence, by (12.2)₂, $\hat{\Delta}_1(x_r) \asymp \hat{\Delta}_2(x_r)$ ($r = 1, 2, \dots$). By condition C₆) in [1], this yields

$$(12.7) \quad (x_r)\hat{\Delta}_1(x_r) \asymp (x_r)\hat{\Delta}_2(x_r) \quad (r = 1, 2, \dots).$$

Now let x_h fail to occur in $\hat{\Delta}_1(x_i) \wedge \hat{\Delta}_2(x_j)$. Then, for $s = 1, 2$, $\hat{\Delta}_s(x_i)$ $[\hat{\Delta}_s(x_j)]$ and $\hat{\Delta}_s(x_h)$ are (x_i, x_h) -similar $[(x_j, x_h)$ -similar], so that

by condition C₇) in [1],

$$(x_i) \Delta_1(x_i) \asymp (x_n) \Delta_1(x_n) \quad \text{and} \quad (x_j) \Delta_2(x_j) \asymp (x_h) \Delta_2(x_h).$$

These results and (12.7) for $r = h$ yield

$$(x_i) \Delta_1(x_i) \asymp (x_j) \Delta_2(x_j). \quad \text{q.e.d.}$$

13. Completion of the proof of Theor. 12.1. A consequence of it.

STEP 3. Assume that (i) $\hat{\Delta} = \hat{\Delta}(x_1, \dots, x_n)$ is a wfe of \mathcal{F} whose free variables, if any, are some among x_1 to x_n , (ii) Δ_1 to Δ_n are terms of \mathcal{F} and (iii) W is a v -valuation for \mathcal{F} , for which

$$(13.1) \quad W(x_s) = [\Delta_s] \quad (s = 1, \dots, n).$$

(Note that $\hat{\Delta}^* = \text{des}_{\mathcal{F}, W}(\hat{\Delta})$ is independent of a such a choice of W). Then

$$(13.2) \quad \hat{\Delta}^* = [\hat{\Delta}(\Delta_1, \dots, \Delta_n)].$$

For the proof we use induction on the length l of Δ . For $l = 1$, Δ is c_i or x_i , so that by (i₁) in § 12, or (13.1), $\hat{\Delta}^* = [c_i]$ or $\hat{\Delta}^* = W(x_i) = [\Delta_i]$ respectively. Hence (13.2) holds for $l = 1$.

As inductive hypothesis, assume that (13.2) holds for $l < \nu$; furthermore let l be ν . Only Cases 1 to 5 below can hold.

Case 1 [2]. Δ is $\varphi[\hat{\Delta}_1(x_1, \dots, x_n), \dots, \hat{\Delta}_m(x_1, \dots, x_n)]$ where φ is R_i^m [f_i^m]. Then, for $\Psi = A_m$ [$\Psi = V_m$], $\hat{\Delta}$ is

$$\Psi[\hat{\varphi}, \hat{\Delta}_1(x_1, \dots, x_n), \dots, \hat{\Delta}_m(x_1, \dots, x_n)]$$

by s₃) [s₂] in § 10, and $\hat{\Delta}^*$ is

$$\Psi^*[\hat{\varphi}^*, \hat{\Delta}_1(x_1, \dots, x_n)^*, \dots, \hat{\Delta}_m(x_1, \dots, x_n)^*].$$

By the inductive hypothesis—see (13.2)—,

$$\hat{\Delta}_s(x_1, \dots, x_n)^* = [\hat{\Delta}_s(\Delta_1, \dots, \Delta_n)] \quad (s = 1, \dots, m).$$

Then by condition (i₂) [(i₃)] in § 12,

$$\Delta^* = [\varphi[\hat{\Delta}_1(\Delta_1, \dots, \Delta_n), \dots, \hat{\Delta}_m(\Delta_1, \dots, \Delta_n)]] .$$

Thus (13.2) holds in this case.

Case 3 [4]: Δ is $\sim \hat{\Delta}_1(x_1, \dots, x_n)$ [$\hat{\Delta}_1(x_1, \dots, x_n) \supset \hat{\Delta}_2(x_1, \dots, x_n)$]. Then (13.2) follows as a particular case of Case 1 in that \sim and \supset are regarded as predicates of \mathcal{F} .

Case 5. Δ is $(y_0)\mathcal{A}(y_0, \dots, y_n)$ where y_0 to y_n are $n+1$ variables. We set

$$(13.3) \quad \begin{cases} \mathcal{A}_1 \equiv_D \mathcal{A}_1(y_0) \equiv_D \mathcal{A}(y_0, \Delta_1, \dots, \Delta_n) , \\ \mathcal{A}' \equiv_D (y_0)\mathcal{A}_1, \quad W_1 = \left(\begin{array}{c} y_0 \\ [\Delta_0] \end{array} \right) W , \end{cases}$$

where Δ_0 is any term of \mathcal{F} . By the inductive hypothesis

$$\text{des}_{\mathcal{F}, W_1} (\mathcal{A}(y_0, \dots, y_n)') = [\mathcal{A}(\Delta_0, \dots, \Delta_n)]$$

hence—see (13.1) and (13.3)₄—

$$(13.4) \quad \varphi_{\mathcal{A}(y_0, \dots, y_n)'; y_0; \mathcal{F}, W}([\Delta_0]) = [\mathcal{A}(\Delta_0, \dots, \Delta_n)] .$$

By (12.4)

$$(13.5) \quad g_{\mathcal{A}_1; y_0}([\Delta_0]) = [\mathcal{A}_1(\Delta_0)] = [\mathcal{A}(\Delta_0, \dots, \Delta_n)] .$$

Since Δ_0 is an arbitrary term of \mathcal{F} , by (13.4) and (13.5) the restriction of the functions $\varphi_{\mathcal{A}(y_0, \dots, y_n)'; y_0; \mathcal{F}, W}$ and $g_{\mathcal{A}_1; y_0}$ to $\mathbf{D} - \emptyset$ coincide. Furthermore, at \emptyset both of them take the value \emptyset . Hence they coincide in \mathbf{D} , so that—see (i₄)—

$$\begin{aligned} \text{des}_{\mathcal{F}, W} [(\Omega y_0)(\mathcal{A}(y_0, \dots, y_n))'] &= \Omega^*(\varphi_{\mathcal{A}(y_0, \dots, y_n)'; y_0; \mathcal{F}, W}) = \\ &= \Omega^*(g_{\mathcal{A}_1; y_0}) = [(y_0)\mathcal{A}(y_0, \Delta_1, \dots, \Delta_n)] = [\hat{\Delta}_1(\Delta_1, \dots, \Delta_n)] . \end{aligned}$$

Thus (13.2) holds in this case; and Step 3 is proved.

STEP 4. $\text{des}_{\mathcal{J}, W_1}(\Delta) = [\Delta]$ if $W_1(x_i) = [x_i]$ ($i = 1, 2, \dots$). This is a simple corollary of Step 3.

One easily verifies that $\mathcal{J} = (\mathbf{D}, \mathcal{J})$ is a model for \mathcal{F} .

Now, in order to complete the proof of the \leftarrow -part of Theor. 12.1 we assume $a \mathcal{E} b$, so that (12.1) holds for every normal model \mathbf{M} of \mathcal{F} and every \mathbf{M} -valuation.

Hence, for \mathbf{M} equal to the above normal model \mathcal{J} of \mathcal{F} and for $W = W_1$ —see Step 4—, $\dot{a}^* = \dot{b}^*$. Furthermore, by Step 4, (13.2) holds for $\Delta = a$ and for $\Delta = b$: $\dot{a}^* = [a]$ and $\dot{b}^* = [b]$. Hence $[a] = [b]$ which by (12.2)₂ yield $a \succ b$.

Thus the above completion has been performed, and also Theor. 12.1 is now completely proved. q.e.d.

Equality (12.1) is true in every normal model \mathbf{M} of \mathcal{F} only if it is so every model \mathbf{M}' (because \mathbf{M}' can be contracted in an equivalent normal model); and by the completeness theorem for \mathcal{F} —see [1], Theor. 3.4—this occurs iff $\vdash_{\mathcal{F}} \dot{a} = \dot{b}$. Therefore Theor. 12.1 yields the following

THEOR. 13.1. *If a and b are wfes of \mathcal{F} , then*

$$\alpha) a \succ b \Leftrightarrow \Vdash_{\mathcal{F}} \dot{a} = \dot{b} \text{ } ^{(5)} \text{ and}$$

$$\beta) a \succ b \Leftrightarrow \vdash_{\mathcal{F}} \dot{a} = \dot{b}.$$

14 Proof of the assertion $\sim p \not\sim \sim \sim p$ made in [2], § 8.

We now prove assertion (8.2)₃ which, together with (8.2)₂—see [2]— shows that *the rather simple condition asserted by Theor. 8.1 in [2] to be sufficient for non-synonymy, is not necessary for this.*

We set $\mathcal{D}_1 = \mathbf{N}$ and consider a sequence $\{\mathcal{J}_\beta\}_{0 < \beta < \omega}$ of functions that are defined (only) on the constants of \mathcal{F}_β —see (ii) in [1], § 6—, have counterdomains in \mathcal{D}_1 , and satisfy conditions i_1) to i_{16}) below. By these, if $0 < \beta < \gamma < \omega$ and a is a wfe of \mathcal{F}_β (so that $\text{des}_{\mathcal{J}_\beta, V}(a)$ is meaningful for every mapping V of \mathcal{F} 's variables into \mathbf{N}), then $\text{des}_{\mathcal{J}_\gamma, V}(a) = \text{des}_{\mathcal{J}_\beta, V}(a)$, where $\mathcal{J}_\varrho = (\mathcal{D}_1, \mathcal{J}_\varrho)$ ($0 < \varrho < \omega$). Therefore in conditions i_1) to i_{16}) \mathcal{J} is written instead of \mathcal{J}_β . Furthermore it is

⁽⁵⁾ $\Vdash_{\mathcal{F}} p$ means $\text{des}_{\mathbf{M}', W'} p = 0$ at every model \mathbf{M}' of \mathcal{F} and every \mathbf{M}' -valuation W' .

assumed that $\xi^* = \mathcal{J}(\xi)$, that p_i is the i -th prime number, and that m_1 to m_r , run over \mathbb{N} while β , n , m , r , and i run over $\mathbb{N} - \{0\}$.

- i₁) $\dot{c}_i^* = 3^i$ if c_i is a primitive constant of \mathcal{T} .
- i₂) $\dot{c}_i^* = 5^\beta$ if c_i is the constant of \mathcal{T} defined by its β -th definition \mathcal{D}_β , i.e. $c_i \in S_{\mathcal{T}_\beta}$ and $c_i \notin S_{\mathcal{T}_\gamma}$ for $\gamma < \beta$.
- i₃₋₄) $\dot{\sim}^* = 2 \cdot 7$, $\dot{\succ}^* = 2^2 \cdot 7^2$.
- i₅) [i₆] $\dot{R}_i^{n*} = 2^n \cdot 7^{i+2}$ [$\dot{f}_i^{n*} = 2^n \cdot 3^i$] if R_i^n [f_i^n] is a primitive predicative [functional] constant of \mathcal{T} .
- i₇) [i₈] $\dot{R}_i^{n*} = 2^n \cdot 11^\beta$ [$\dot{f}_i^{n*} = 2^n \cdot 5^\beta$] if R_i^n [f_i^n] is in $S_{\mathcal{T}_\beta}$ and is not in $S_{\mathcal{T}_\gamma}$ for $\gamma < \beta$ —cf. i₂).
- i₉) $V_n^*(2^n \cdot 3^i, m_1, \dots, m_n) = 2^n \cdot 3^i \cdot 5^{m_1} \dots p_{n+2}^{m_n}$.
- i₁₀) $V_n^*(2^n \cdot 5^\beta, m_1, \dots, m_n) = 2^n \cdot 5^\beta \cdot 7^{m_1} \dots p_{n+3}^{m_n}$.
- i₁₁) $A_n^*(2^n \cdot 7^i, m_1, \dots, m_n) = 2^n \cdot 7^i \cdot 11^{m_1} \dots p_{n+4}^{m_n}$.
- i₁₂) $A_n^*(2^n \cdot 11^i, m_1, \dots, m_n) = \text{des}_{\mathcal{J}, V} D_\beta''$ where $V(x_i) = m_i$ for $1 \leq i \leq n$.
- i₁₃) $A_2^*(2^2 \cdot 7^3, n, n_0) = \begin{cases} \text{des}_{\mathcal{J}, V} D_\beta'' \text{ where } V(x_i) = m_i \\ (i = 1, \dots, r) \text{ provided } n_0 = 2^m \cdot 5^\beta \cdot 7^{m_1} \dots \\ \dots p_{r+3}^{m_r} \text{ for some } m, \beta, r \text{ and } m_1 \text{ to } m_r; \\ 2^2 \cdot 7^3 \cdot 11^n \cdot 13^{n_0} \text{ otherwise;} \end{cases}$
- i₁₄) $=^*$ is the identity on \mathbb{N} .
- i₁₅) $(\Omega x_i)^*(\varphi) = k_\varphi$ where k is a function of φ (independent of i).
- i₁₆) \mathcal{V}^* is any subset \mathcal{D}_0 of \mathcal{D}_1 .

Note that $2^2 \cdot 7^3$ is $\dot{=}^*$, because $=$ is the primitive predicative constant R_1^2 . For the interpretation $\mathcal{J} = (\mathcal{J}, \mathcal{D}_1)$ of $\dot{\mathcal{T}}$ we have the following

THEOR. 14.1. *If $a \succ b$, then $\text{des}_{\mathcal{J}, V} \dot{a} = \text{des}_{\mathcal{J}, V} \dot{b}$ for every \mathcal{J} -valuation V .*

Indeed let T be the equivalence relation among wffs of \mathcal{T} , for which $\Delta_1 T \Delta_2$ iff, for all \mathcal{J} -valuation V , $\text{des}_{\mathcal{J}, V} \Delta_1 = \text{des}_{\mathcal{J}, V} \Delta_2$. Then T fulfils the conditions C_1 to C_7 in \succ , written in [1] and used there

to define \succ . Hence, for the minimality property of \succ , we have $\succ \subseteq T$, and hence also the thesis. q.e.d.

COROLLARY 14.2. $\sim p \not\prec \sim \sim \sim p$, for every wff p of \mathcal{F} .

Indeed assume $\sim p \succ \sim \sim \sim p$, as an hypothesis for reduction ad absurdum. Then by Theor. 14.1, we have that

$$\text{des}_{\mathcal{S},V} A_1(\sim, \dot{p}) = \text{des}_{\mathcal{S},V} A_1(\sim, A_1(\sim, A_1(\sim, \dot{p})))$$

hence, by i_3) and i_{11}), for $n = \text{des}_{\mathcal{S},V} \dot{p}$ we arrive at the absurd result.

$$2 \cdot 7 \cdot 11^n = 2 \cdot 7 \cdot (11^{2 \cdot 7 \cdot (11^{2 \cdot 7 \cdot 11^n})}). \quad \text{q.e.d}$$

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