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Nonautonomous Functional Equations and Nonlinear Evolution Operators.

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RIASSUNTO - Lo studio dell'equazione funzionale nonlineare nonautonoma $x(t) = F(t, x_t)$, $x_t = \varphi$, dove x_t è la «storia» di $x(t)$ al tempo t e il dato iniziale φ appartiene a $L^1(-r, 0; X)$, X spazio di Banach, viene ricondotto a quello di un'equazione di evoluzione nello spazio dei dati iniziali. Se ne deducono risultati sulla regolarità e il comportamento asintotico delle soluzioni.

1. Introduction.

In this paper our purpose is to associate a nonlinear evolution operator with the nonlinear nonautonomous functional equation

$$x(t) = F(t, x_t)\,, \qquad s \leqslant t \leqslant T\,, \qquad x_s = \varphi \in X\,. \tag{1.1}$$

The notation in (1.1) is as follows: $x\colon (s-r, T] \to Y$, where $0 < r \leqslant +\infty$ and Y is a Banach space, $s \geqslant 0$, $F\colon [0, T] \times X \to Y$, where $X = L^1(-r, 0; Y)$, and $x_t \in X$ is defined by $x_t(\theta) = x(t+\theta)$ for a.e. $\theta \in (-r, 0)$. The general formulation in problem (1.1) allows applications to many classes of functional equations and in particular to Volterra integral equations of non-convolution type.

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It is possible to solve equation (1.1) by direct methods using classical iterative techniques. In the approach we take here, however, we will view the solutions of (1.1) as an evolution operator in the Banach space X. We are able, therefore, to incorporate the theory of functional equations into the framework of the general theory of abstract differential equations in Banach spaces. There are a number of recent studies devoted to the connection of functional and functional differential equations to abstract differential equations in Banach spaces.

The correspondence between nonlinear autonomous functional equations and nonlinear strongly continuous semigroups was first treated in [10]. The correspondence between nonautonomous functional differential equations and nonlinear evolution operators was first treated in [2]. The connection between nonlinear autonomous functional equations and nonlinear strongly continuous semigroups was studied in [3], [8] and [9].

The results presented here, as far as the authors know, are the first to develop the connection between nonlinear nonautonomous functional equations and nonlinear evolution operators in Banach spaces.

The abstract differential equation associated with (1.1) is

$$(1.2) \qquad du/dt = -A(t)u(t)\,, \quad 0 \leqslant s \leqslant t \leqslant T\,, \quad u(s) = \varphi \in X\,.$$

The notation of (1.2) is as follows: $u\colon [s, T] \to X$ and for each $t \in [0, T]$, $A(t)\colon X \to X$ is defined by

$$(1.3) \qquad A(t)\varphi = -\varphi',\ D_{A(t)} = \{\varphi \in X,\ \varphi \text{ is absolutely continuous in } [-r, 0],\ \varphi' \text{ exists a.e. on } (-r, 0),\ \varphi' \in X, \text{ and } \varphi(0) = F(t, \varphi)\}\,.$$

We denote by $\|\cdot\|$ and $\|\cdot\|_1$ the norm in Y and X respectively and suppose the following hypotheses on F:

(1.4) There is a bounded function $h\colon [0, T] \to \mathbf{R}$ such that for all $t \in [0, T]$, $\varphi,\ \psi \in X$

$$\|F(t, \varphi) - F(t, \psi)\| \leqslant h(t)\|\varphi - \psi\|_1\,.$$

(1.5) There exist a continuous function $k\colon [0, T] \to Y$ and an increasing function $L\colon [0, \infty) \to [0, \infty)$ such that for all $t_1, t_2 \in [0, T]$, $\varphi \in X$

$$\|F(t_1, \varphi) - F(t_2, \varphi)\| \leqslant \|k(t_1) - k(t_2)\| L(\|\varphi\|_1) .$$

(1.6) There exist constants c_1, c_2 such that for all $t \in [0, T]$, $\varphi \in X$,

$$\|F(t, \varphi)\| \leqslant c_1 \|\varphi\|_1 + c_2 .$$

In our development we will first prove that under the hypothesis (1.4) the operator $A(t)$ defined in (1.3) is densely defined in X and $A(t) + h(t)I$ is m-accretive in X. This proof will follow directly from the results in [9].

Our next step will be to prove that under the additional hypothesis (1.5) the family of operators $\{A(t), t \in [0, T]\}$ generates a nonlinear evolution operator $U(t, s)$, $0 \leqslant s \leqslant t \leqslant T$ in X in the following sense

(1.7) If $\varphi \in X$, $0 \leqslant s \leqslant t \leqslant T$, then

$$U(t, s)\varphi = \lim_{n\to\infty} \prod_{i=1}^{n} \left(I + \frac{t-s}{n} A\left(s + i\frac{t-s}{n}\right)\right)^{-1} \varphi$$

exists and the convergence is uniform in s and t;

moreover the evolution operator $U(t, s)$ has the following properties

(1.8) $$U(t, r)\,U(r, s) = U(t, s) \quad \text{for } 0 \leqslant s \leqslant r \leqslant t \leqslant T .$$

(1.9) For all $\varphi \in X$, $U(t, s)\varphi$ is continuous on the triangle

$$\Delta = \{(t, s);\ 0 \leqslant s \leqslant t \leqslant T\} .$$

(1.10) $$U(t, t) = I \quad \text{for all } t \in [0, T] .$$

(1.11) For all $0 \leqslant s \leqslant t \leqslant T$, $\varphi, \psi \in X$,

$$\|U(t, s)\varphi - U(t, s)\psi\|_1 \leqslant \exp[h(t - s)]\|\varphi - \psi\|_1 ,$$

where $h = \sup_{0 \leqslant t \leqslant T} h(t)$.

The existence of the evolution operator $U(t, s)$, $0 \leqslant s \leqslant t \leqslant T$, will follow from general results in the theory of nonlinear evolution operators due to M. Crandall and A. Pazy. In order to apply the Crandall-Pazy results it is natural to choose the space $X = L^1(-r, 0; Y)$, rather then other function spaces for the setting of the problem. We will discuss in detail this point in Section 5.

Once the evolution operator is defined, the next goal is to associate it with the solutions of (1.1). To accomplish this we will prove that the evolution operator has the following property:

(1.12) For all $\varphi \in X$, $0 \leqslant s \leqslant t \leqslant T$,

$$U(t, s)\varphi(\theta) = \begin{cases} F(t+\theta, U(t+\theta, s)\varphi) & \text{for a.e. } \theta \in (s-t, 0)\,, \\ \varphi(t+\theta-s) & \text{for a.e. } \theta \in (-r, s-t)\,. \end{cases}$$

The property (1.12) is the so called « translation property » of the evolution operator. This property has been studied for various generators defined by the first derivative operator with accompanying boundary conditions by H. Flaschka and M. Leitmann [6] and A. Plant [7]. The proof we give here for the translation property in our problem will make use of some basic results in probability theory.

In the last step of our development we will define $x\colon (s-r, T] \to Y$ by the formula

$$x(t) = \begin{cases} (U(t, s)\varphi)(0) & s < t \leqslant T \\ F(s, \varphi) & t = s \\ \varphi(t-s) & s-r < t < s \end{cases} \tag{1.13}$$

and, using the translation property we will prove that $x(t)$ is the unique solution of (1.1). From the properties of the evolution operators we can then deduce asymptotic results for the solutions.

2. The nonlinear evolution operator.

We assume henceforth that $F\colon [0, T] \times X \to Y$ satisfies (1.4), (1.5), (1.6) and for each $t \in [0, T]$, $A(t)$ is defined as in (1.3). The proof of the following proposition is given in [9] (Theorem 1 and Proposition 1):

PROPOSITION 2.1. *For each* $t \in [0, T]$, $A(t) + h(t)I$ *is m-accretive in* X (*that is, for* $0 < \lambda < 1/h(t)$ *the mapping* $I + \lambda A(t)$ *is one-to-one, has range all of* X, *and satisfies* $\|(I + \lambda A(t))\varphi - (I + \lambda A(t))\psi\|_1 \geqslant (1 - \lambda h(t))\|\varphi - \psi\|_1$ *for all* $\varphi, \psi \in D_{A(t)}$. *Furthermore, for each* $t \in [0, T]$, $D_{A(t)}$ *is dense in* X.

The proof of the following proposition requires an application of a theorem of M. Crandall and A. Pazy [1].

PROPOSITION 2.2. *The family of nonlinear operators* $\{A(t) \colon 0 \leqslant t \leqslant T\}$ *generates a nonlinear evolution operator* $U(t, s)$, $0 \leqslant s \leqslant t \leqslant T$ *as in* (1.7)-(1.11).

PROOF. By virtue of Proposition 2.1 above and Theorem 2.1 and Corollary 2.1 in [1] it suffices to show that

(2.1) for all λ positive and sufficiently small,

$$\|(I + \lambda A(t))^{-1}\varphi - (I + \lambda A(\tau))^{-1}\varphi\|_1 \leqslant \lambda \|b(t) - b(\tau)\|_1 M(\|\varphi\|_1),$$

for $0 \leqslant t \leqslant \tau \leqslant T$, $\varphi \in X$, where $b \colon [0, T] \to X$ is continuous and $M \colon [0, \infty) \mapsto [0, \infty)$ is increasing.

In [9] it is shown that for $0 < \lambda < 1/h(t)$,

$$\big((I + \lambda A(t))^{-1}\varphi\big)(\theta) = \exp[\theta/\lambda] F\big(t, (I + \lambda A(t))^{-1}\varphi\big) + \int_\theta^0 (\exp[(\theta - u)/\lambda]/\lambda)\varphi(u)\, du, \quad \theta \in (-r, 0). \tag{2.2}$$

Therefore, for $0 \leqslant t \leqslant \tau \leqslant T$, $\varphi \in X$, (1.4) and (1.5) imply that

$$\begin{aligned}
&\|(I + \lambda A(t))^{-1}\varphi - (I + \lambda A(\tau))^{-1}\varphi\|_1 = \\
&= \Big\|\exp[\theta/\lambda]\Big(F\big(t, (I + \lambda A(t))^{-1}\varphi\big) - F\big(\tau, (I + \lambda A(\tau))^{-1}\varphi\big)\Big)\Big\|_1 \leqslant \\
&\leqslant h(t)\lambda(1 - \exp[-r/\lambda])\|(I + \lambda A(t))^{-1}\varphi - (I + \lambda A(\tau))^{-1}\varphi\|_1 + \\
&\quad + \|k(t) - k(\tau)\|\lambda(1 - \exp[-r/\lambda]) L\big(\|(I + \lambda A(\tau))^{-1}\varphi\|_1\big).
\end{aligned}$$

If $0 < \lambda < 1/\sup\{h(t): t \in [0, T]\}$, then

$$\|(I + \lambda A(t))^{-1}\varphi - (I + \lambda A(\tau))^{-1}\varphi\|_1 \leqslant$$
$$\leqslant \big(1/(1 - \lambda \sup\{h(t): t \in [0, T]\})\big)\|k(t) - k(\tau)\|\lambda L(\|I + \lambda A(\tau))^{-1}\varphi\|_1\big) .$$

From (1.6) and (2.2) we obtain that for λ positive and sufficiently small, $t \in [0, T]$,

$$\|(I + \lambda A(t))^{-1}O\|_1 =$$
$$= \int_{-r}^{0} \exp[\theta/\lambda]\Big|\Big|F\big(t, (I + \lambda A(t))^{-1}O\big)\Big|\Big|\, d\theta \leqslant \lambda\big(c_1\|(I + \lambda A(t))^{-1}O\|_1 + c_2\big)$$

so that

$$\|(I + \lambda A(t))^{-1}O\|_1 \leqslant \lambda c_2/(1 - \lambda c_1) .$$

Thus, for λ positive and sufficiently small,

$$\|(I + \lambda A(\tau))^{-1}\varphi\|_1 \leqslant \|(I + \lambda A(\tau))^{-1}O\|_1 +$$
$$+ \big(1/(1 - \lambda h(\tau))\big)\|\varphi - O\|_1 \leqslant \lambda c_2/(1 - \lambda c_1) + \big(1/(1 - \lambda h(\tau))\big)\|\varphi\|_1 .$$

The inequality (2.1) then follows from these estimates.

3. The translation property.

In order to prove (1.12) we first prove three lemmas.

LEMMA 3.1. *Let $\varphi \in X$, $0 \leqslant s < t$, $n = 1, 2, \ldots$, and $\lambda = (t - s)/n$. For almost everywhere $\theta \in (-r, 0)$,*

$$(3.1) \quad \left\{ \begin{aligned} &\prod_{i=1}^{n} (I + \lambda A(s + i\lambda))^{-1}\varphi(\theta) = \\ &\qquad = \sum_{k=0}^{n-1} (-\theta/\lambda)^k (\exp[\theta/\lambda]/k!) F\big(s + (n - k)\lambda, \\ &\prod_{j=1}^{n-k}\big(I + \lambda A(s + j\lambda))^{-1}\varphi\big) + \\ &\qquad + 1/\lambda^n (1/(n-1)!) \int_{\theta}^{0} (v - \theta)^{n-1} \exp[(\theta - v)/\lambda]\varphi(v)\, dv . \end{aligned} \right.$$

PROOF. The proof is by induction. From (2.2) we obtain

$$\prod_{i=1}^{1}(I+\lambda A(s+i\lambda))^{-1}\varphi(\theta)=$$
$$=\exp[\theta/\lambda]F\big(s+\lambda,(I+\lambda A(s+\lambda))^{-1}\varphi\big)+$$
$$+(1/\lambda)\int_{\theta}^{0}\exp[(\theta-v)/\lambda]\varphi(v)\,dv\,.$$

Assume that for $1\leqslant m<n$,

$$\prod_{i=1}^{m}(I+\lambda A(s+i\lambda))^{-1}\varphi(\theta)= \tag{3.2}$$
$$=\sum_{k=1}^{m}(-\theta/\lambda)^{m-k}\big(\exp[\theta/\lambda]/(m-k)!\big)F\Big(s+k\lambda,\prod_{j=1}^{k}(I+\lambda A(s+j\lambda))^{-1}\varphi\Big)+$$
$$+(1/\lambda^{m})\big(1/(m-1)!\big)\int_{\theta}^{0}(v-\theta)^{m-1}\exp[(\theta-v)/\lambda]\varphi(v)\,dv\,.$$

Then from (2.2) and (3.2) we obtain

$$\prod_{i=1}^{m+1}(I+\lambda A(s+i\lambda))^{-1}\varphi(\theta)= \tag{3.3}$$
$$=\big(I+\lambda A(s+(m+1)\lambda)\big)^{-1}\prod_{i=1}^{m}(I+\lambda A(s+i\lambda))^{-1}\varphi(\theta)=$$
$$=\exp[\theta/\lambda]F\Big(s+(m+1)\lambda,\prod_{i=1}^{m+1}(I+\lambda A(s+i\lambda))^{-1}\varphi\Big)+$$
$$+(1/\lambda)\int_{\theta}^{0}\exp[(\theta-u)/\lambda]\cdot\Big(\sum_{k=1}^{m}(-u/\lambda)^{m-k}\big(\exp[u/\lambda]/(m-k)!\big)F\Big(s+k\lambda,$$
$$\prod_{j=1}^{k}(I+\lambda A(s+j\lambda))^{-1}\varphi\Big)+$$
$$+(1/\lambda^{m})\big(1/(m-1)!\big)\int_{u}^{0}(v-u)^{m-1}\exp[(u-v)/\lambda]\varphi(v)\,dv\Big)\,du=$$
$$=\exp[\theta/\lambda]F\Big(s+(m+1)\lambda,\prod_{i=1}^{m+1}(I+\lambda A(s+i\lambda))^{-1}\varphi\Big)+$$

$$+ (1/\lambda)\exp[\theta/\lambda]\sum_{k=1}^{m}\big(\lambda(-\theta/\lambda)^{m-k+1}/(m-k+1)(m-k)!\big)F\Big(s+k\lambda,$$

$$\prod_{j=1}^{k}(I+\lambda A(s+j\lambda))^{-1}\varphi\Big) +$$

$$+ (1/\lambda^{m+1})\int_{\theta}^{0}\int_{\theta}^{v}(1/(m-1)!)\exp[(\theta-v)/\lambda](v-u)^{m-1}\varphi(v)\,du\,dv =$$

$$= \sum_{k=1}^{m+1}(-\theta/\lambda)^{m+1-k}\big(\exp[\theta/\lambda]/(m+1-k)!\big)F\Big(s+\lambda k,$$

$$\prod_{j=1}^{k}(I+\lambda A(s+j\lambda))^{-1}\varphi\Big) +$$

$$+ (1/\lambda^{m+1})\int_{\theta}^{0}(1/m!)\exp[(\theta-v)/\lambda](v-\theta)^{m}\varphi(v)\,dv\,.$$

Now (3.1) follows immediately from (3.3) with $m = n-1$.

LEMMA 3.2. *Let $\varphi \in X$, $0 \leqslant s < t$, $\theta \in [-r, 0]$, $t+\theta \neq s$. Then*

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}(-n\theta/(t-s))^{k}\big(\exp[n\theta/(t-s)]/k!\big)F\Big(s+(n-k)(t-s)/n,\ U\big(s+(n-k)(t-s)/n,\, s\big)\varphi\Big) = \begin{cases} F(t+\theta,\ U(t+\theta, s)\varphi) & \text{if } s-t<\theta\leqslant 0\,,\\ 0 & \text{if } -r\leqslant\theta<s-t\,.\end{cases} \tag{3.4}$$

PROOF. If $u\colon [0,\infty)\to Y$ is bounded and continuous, then the Poisson probability distribution satisfies

$$\lim_{n\to\infty}\exp[-n\sigma]\sum_{k=0}^{\infty}u(k/n)(n\sigma)^{k}/k! = u(\sigma)\,,\qquad \sigma\geqslant 0 \tag{3.5}$$

(see [5], pag. 220).
If we define

$$u(\sigma) = \begin{cases} F\big(t-\sigma(t-s),\ U(t-\sigma(t-s), s)\varphi\big) & \text{if } 0\leqslant\sigma<1\,,\\ 0 & \text{if } \sigma\geqslant 1\,,\end{cases}$$

then (3.5) holds also for this function u (which has a single jump discontinuity), as may be seen from the proof in [5]. Then (3.4) follows immediately from (3.5) by setting $\sigma = -\theta/(t-s)$.

LEMMA 3.3. *Let* $\varphi\colon [-r, 0] \to Y$ *such that* φ *is continuous, let* $0 \leqslant s < t$, *and let* $\theta \in (-r, 0)$ *such that* $t + \theta \neq s$. *Then,*

$$(3.6)\quad \lim_{n\to\infty} (n/(t-s))^n (1/(n-1)!) \int_\theta^0 (v-\theta)^{n-1} \exp[n(\theta - v)/(t-s)]\varphi(v)\,dv = \begin{cases} 0 & \text{if } s-t < \theta \leqslant 0\,, \\ \varphi(t+\theta-s) & \text{if } -r \leqslant \theta < s-t\,. \end{cases}$$

PROOF. If $u\colon[0, \infty) \to Y$ is bounded and continuous, then the gamma probability distribution satisfies

$$(3.7)\quad \lim_{n\to\infty} (u/\sigma)^n (1/(n-1)!) \int_0^\infty u(x) x^{n-1} \exp[-nx/\sigma]\,dx = u(\sigma)\,, \qquad \sigma > 0$$

(see [5], p. 220). If we define

$$u(x) = \begin{cases} \varphi(x+\theta) & \text{if } 0 \leqslant x < -\theta \\ 0 & \text{if } x \geqslant -\theta \end{cases}$$

then (3.7) holds also for this function u (which has a single jump discontinuity), as may be seen from the proof in [5]. Then, (3.6) follows immediately from (3.7) with $\sigma = t - s$.

PROPOSITION 3.1. *If* $\varphi \in X$ *and* $0 \leqslant s \leqslant t \leqslant T$, *then* (1.12) *holds.*

PROOF. Let $\varphi \in X$ such that φ is continuous, let $0 \leqslant s < t$, let $\theta \in (-r, 0)$ such that $t + \theta \neq s$, and let $\lambda_n = (t-s)/n$ for $n = 1, 2, \ldots$. To prove (1.12) for this φ it suffices by Lemmas 3.1, 3.2, and 3.3 to prove that

$$(3.8)\quad \lim_{n\to\infty} \Big\| \sum_{k=0}^{n-1} (-\theta/\lambda_n)^k (\exp[\theta/\lambda_n]/k!) \Big(F(s + (n-k)\lambda_n, U(s + (n-k)\lambda_n, s)\varphi\Big) - F\Big(s + (n-k)\lambda_n, \prod_{j=1}^{n-k} \big(I + \lambda_n A(s + j\lambda_n)\big)^{-1}\varphi\Big) \Big\| = 0\,.$$

Let $\varepsilon > 0$. By the uniform convergence of (1.7), there exists m such that if $p \geqslant m$ and $s \leqslant \tau \leqslant t$, then

$$(3.9) \qquad \left\| \prod_{j=1}^{p} \big(I + ((\tau - s)/p)A(s + s(\tau - s)/p)\big)^{-1}\varphi - U(\tau, s)\varphi \right\|_1 < \varepsilon .$$

Thus, if $n > m$, then

$$(3.10) \qquad \Big\| \sum_{k=0}^{n-m} (-\theta/\lambda_n)^k (\exp[\theta/\lambda_n]/k!)\Big(F\big(s + (n-k)\lambda_n,$$
$$U(s + (n-k)\lambda_n, s)\varphi\big) - F\big(s + (n-k)\lambda_n,$$
$$\prod_{j=1}^{n-k} (I + \lambda_n A(s + j\lambda_n))^{-1}\varphi\big)\Big)\Big\| \leqslant \big(\sup_{s\leqslant\tau\leqslant t} h(\tau)\big) \sup_{0\leqslant k\leqslant n-m} \cdot$$
$$\cdot \Big\| U(s + (n-k)\lambda_n, s)\varphi - \prod_{j=1}^{n-k}\Big(I + ((n-s)/n)((t-s)/(n-k)) \cdot$$
$$\cdot A\big(s + j((n-k)/n)((t-s)/(n-k))\big)\Big)^{-1}\varphi \Big\|_1 \leqslant \big(\sup_{s\leqslant\tau\leqslant t} h(\tau)\big)\varepsilon$$

$\big($since $n - k \geqslant m$ and $s \leqslant s + (n - k/n)(t - s) \leqslant t\big)$.

Next, we claim that there exists $m_1 > m$ such that if $n > m_1$, then

$$(3.11) \qquad \sum_{k=n-m+1}^{n-1} (-\theta/\lambda_n)^k \exp[\theta/\lambda_n]/k! < \varepsilon .$$

To establish (3.11) set $x = -\theta/(t-s)$ and observe that $x^k \leqslant \max\{1, x^{1-m}\}x^n$ for $1 \leqslant n - m + 1 \leqslant k \leqslant n$ and $n^k/k! \leqslant n^n/n!$ for $1 \leqslant k \leqslant n$. Thus,

$$(3.12) \qquad \exp[-nx] \sum_{k=n-m+1}^{n} (nx)^k/k! \leqslant$$
$$\leqslant \exp[-nx](m-1)(n^n/n!) \max\{1, x^{1-m}\}x^n =$$
$$= (m-1)\max\{1, x^{1-m}\}(n^n/\exp[n]n!)\exp[n(\log x - x + 1)],$$

Now (3.11) follows from (3.12), since $\log x - x + 1 \leqslant 0$ and $n^n/$ $/\exp[n]n! \sim 1/\sqrt{2\pi n}$ by Stirling's formula.

Now use (3.10) and (3.11) to show (3.8), and hence, that (1.12) holds in the case that φ is continuous. If $\varphi \in X$ and φ is not continuous, let $\{\varphi_n\}$ be a sequence of continuous functions in X such that $\varphi_n \to \varphi$ in X. Then, by (1.11)

$$\int_{s-t}^{0} \| U(t, s)\varphi(\theta) - F(t+\theta, U(t+\theta, s)\varphi)\| \, d\theta + \\ + \int_{-r}^{s-t} \| U(t, s)\varphi(\theta) - \varphi(t+\theta-s)\| \, d\theta \leqslant \\ \leqslant \liminf_n \Big(\| U(t, s)\varphi - U(t, s)\varphi_n\|_1 + \\ + \int_{s-t}^{0} \| U(t, s)\varphi_n(\theta) - F(t+\theta, U(t+\theta, s)\varphi_n)\| \, d\theta + \\ + \int_{-r}^{s-t} \| U(t, s)\varphi_n(\theta) - \varphi_n(t+\theta-s)\| \, d\theta + \\ + \int_{s-t}^{0} h(t+\theta) \| U(t+\theta, s)\varphi - U(t+\theta, s)\varphi_n\|_1 \, d\theta + \\ + \int_{-r}^{s-t} \|\varphi_n(t+\theta-s) - \varphi(t+\theta-s)\| \, d\theta \Big) = 0 \Big) .$$

It now follows that (1.12) holds for arbitrary $\varphi \in X$.

4. The functional equation.

Let $\varphi \in X$, $s \in [0, T)$. By Proposition 3.1 there exists for each $t > s$, a representative $\hat{U}(t, s)\varphi$ of the equivalence class of $U(t, s)\varphi$ in X such that $\hat{U}(t, s)\varphi(\theta)$ is continuous for $\theta \in]-(t-s), 0]$.

Define a function $x\colon (s-r, T] \to Y$ by

$$x(t) = \begin{cases} \hat{U}(t, s)\varphi(0) & \text{if } t > s \\ F(s, \varphi) & \text{if } t = s \\ \varphi(t-s) & \text{if } s-r < t < s\,. \end{cases} \tag{4.1}$$

PROPOSITION 4.1. *For* $\varphi \in X$, $s \in [0, T)$,

$$x(t) \text{ is continuous in } t \text{ for } t > s \tag{4.2}$$

$$x_t = U(t, s)\varphi \qquad \text{for } t \geqslant s\,. \tag{4.3}$$

PROOF. (4.2) follows immediately from (1.4), (1.5), (1.9), and (1.12), since for $t \geqslant s$, $x(t) = \hat{U}(t, s)\varphi(0) = F(t, U(t, s)\varphi)$. (4.3) follows immediately from (1.12), since for $\theta \in (s - t, 0)$ we have

$$\begin{aligned} x_t(\theta) = x(t + \theta) &= \hat{U}(t + \theta, s)\varphi(0) = \\ &= F(t + \theta, \hat{U}(t + \theta, s)\varphi) = U(t, s)\varphi(\theta)\,, \end{aligned}$$

and for almost everywhere $\theta \in (-r, s - t)$ we have

$$x_t(\theta) = x(t + \theta) = \varphi(t + \theta - s) = U(t, s)\varphi(\theta)\,.$$

PROPOSITION 4.2. *Let* $\varphi \in X$, $s \in [0, T)$. *There exists at most one function* $v\colon [s, T] \to X$ *such that*

$$v(t)(\theta) = \begin{cases} F(t + \theta, v(t + \theta)) & \text{for a.e. } \theta \in (s - t, 0)\,, \\ \varphi(t + \theta - s) & \text{for a.e. } \theta \in (-r, s - t)\,. \end{cases} \tag{4.4}$$

PROOF. Suppose that v_1 and v_2 satisfy (4.4). Then, for $t \geqslant s$,

$$\begin{aligned} \|v_1(t) - v_2(t)\|_1 &= \int_{-r}^{0} \|v_1(t)(\theta) - v_2(t)(\theta)\|\, d\theta = \\ &= \int_{s-t}^{0} \|F(t + \theta, v_1(t + \theta)) - F(t + \theta, v_2(t + \theta))\|\, d\theta = \\ &= \int_{s}^{t} \|F(\tau, v_1(\tau)) - F(\tau, v_2(\tau))\|\, d\tau \leqslant \\ &\leqslant \Big(\sup_{0 \leqslant \tau \leqslant T} h(\tau)\Big) \int_{s}^{t} \|v_1(\tau) - v_2(\tau)\|_1\, d\tau\,. \end{aligned}$$

Then $v_1(t) = v_2(t)$ for all $t \geqslant s$ by Gronwall's Lemma.

PROPOSITION 4.3. *Let $\varphi \in X$, $s \in [0, T)$. Let x: $(s-r, T]$ be defined as in* (4.1). *Then,*

$$x(t) = \begin{cases} F(t, x_t) & \text{for} \quad t \geqslant s\,, \\ \varphi(t-s) & \text{for a.e. } t \in (s-r, s)\,. \end{cases} \tag{4.5}$$

Further, x is the unique function satisfying (4.5) *except possibly for a set of measure zero in $(s-r, s)$.*

PROOF. (4.5) follows directly from (1.12), (4.1), and (4.3). Define v: $[s, T] \to Y$ by $v(t) = x_t$. Then, v satisfies (4.4), and so the uniqueness assertion follows from Proposition 4.2.

5. L^1 is the natural space for the problem. Flow-invariant sets.

In order to satisfy the hypotheses of the theorem of Crandall-Pazy the family of operators $A(t)$ must satisfy condition (2.1). This condition implies that the sets $D_{A(t),1}$

$$D_{A(t),1} = \left\{\varphi \in \bar{D}_{A(t)},\ \overline{\lim_{h\to 0}}\ \|\varphi - (I + \lambda A(t))^{-1}\varphi\|_1/\lambda < +\infty\right\} \tag{5.1}$$

are independent of t (see [1] and [4]). And, in fact in [9] it is proved that

$$D_{A(t),1} = \left\{\varphi \in L^1, \int_{-r}^{-\tau} \|\varphi(\theta+\tau) - \varphi(\tau)\|\, d\theta + \int_{-\tau}^{0} \|\varphi(\theta)\|\, d\theta \leqslant K_\varphi \tau,\ -r \leqslant \tau \leqslant 0\right\}.$$

In [3] and [8] the autonomous version of (1.1) was studied in the space $C(-r, 0; Y)$ as the nonlinear semigroup generated by the operator

$$B\varphi = -\varphi'\,, \qquad D_B = \{\varphi \in C^1(-r, 0; Y), \varphi(0) = F(\varphi)\}\,.$$

It was pointed out that

$$D_{B,1} = \left\{\varphi \in \bar{D}_B,\ \lim_{\lambda \to 0}\ |||\varphi - (I + \lambda B)^{-1}\varphi|||/\lambda < +\infty\right\}$$

(where $|||\cdot|||$ is the sup norm in $C(-r, 0; Y)$), can be characterized as

$$D_{B,1} = \{\varphi \in C(-r, 0; Y),\ \varphi \text{ is Lipschitz continuous},\ \varphi(0) = F(\varphi)\}. \tag{5.2}$$

It follows that it is not possible to use the Crandall-Pazy theorem to study the nonautonomous equation (1.1) as an evolution equation in $C(-r, 0; Y)$. In fact, this equation would be

$$du/dt + B(t)u(t) = 0\,, \quad u(0) = \varphi$$

where

$$B(t)\varphi = -\varphi'\,, \quad D_{B(t)} = \{\varphi \in C^1(-r, 0; Y), \varphi(0) = F(t, \varphi)\}\,,$$

and from (5.2) we have that

$$D_{B(t),1} = \{\varphi \in C(-r, 0; Y),\ \varphi \text{ is Lipschitz continuous and } \varphi(0) = F(t, \varphi)\}.$$

Thus, the sets $D_{B(t),1}$ vary with t.

An analogous remark can be made about L^p spaces with $p > 1$.
Suppose that Y is a Hilbert space and associate with (1.1) the family of operators in $L^p(-r, 0; Y)$, $p > 1$,

$$A_p(t)\varphi = -\varphi'\,, \quad D_{A_p(t)} = \{\varphi,\ \varphi \text{ is absolutly continuous on } (-r, 0),\ \varphi' \text{ exists a.e. on } (-r, 0),\ \varphi' \in L^p(-r, 0; Y) \text{ and } \varphi(0) = F(t, \varphi)\}.$$

As $L^p(-r, 0; Y)$ is reflexive, $D_{A_p(t),1}$ coincides with $D_{A_p(t)}$, and so, again, $A_p(t)$ cannot satisfy condition (2.1).

Hence L^1 seems the natural space for studying the nonautonomous equation (1.1) as an evolution operator.

The set $D_1 = D_{A(t),1}$ is flow-invariant, that is $U(t, s)D_1 \subseteq D_1$, for all $t \geqslant s \geqslant 0$, as is proved in [1]. From Proposition 4.1 it follows that also the set

$$E = \{\varphi\colon [-r, 0] \to Y,\ \varphi \text{ is piecewise continuous}\}$$

is flow-invariant. Hence the restriction of the family $U(t, s)$ to D_1 and to E are evolution operators in the sense that they satisfy conditions (1.8)-(1.11). However D_1 and E are not closed subsets of L^1.

From Proposition 4.1 it follows also that if the initial data φ in (1.1) is continuous and satisfies $\varphi(0) = F(s, \varphi)$, then the solution is continuous. More precisely, let $\bar{D}_{B(t)}$ be the closure of $D_{B(t)}$ in $C(-r, 0; Y)$, that is,

$$\bar{D}_{B(t)} = \{\varphi \in C(-r, 0; Y),\ \varphi(0) = F(t, \varphi)\};$$

then

$$U(t, s)\bar{D}_{B(s)} \subseteq \bar{D}_{B(t)};$$

in fact, $\varphi \in \bar{D}_{B(s)}$ implies that $U(t, s)\varphi \in C(-r, 0; Y)$ for all $t \geqslant s$ and $U(t, s)\varphi(0) = F\big(U(t, s)\varphi\big)$; hence $U(t, s)\varphi \in \bar{D}_{B(t)}$.

6. Asymptotic results.

Let L^1_σ denote the space of functions $\varphi \in L^1(-r, 0; Y)$ endowed with the norm $\|\varphi\|_{1,\sigma} = \int_{-r}^{0} \exp[-\sigma\theta]\|\varphi(\theta)\|\, d\theta$. Suppose $r < +\infty$, so that the norms $\|\cdot\|_{1,\sigma}$, $\sigma \in \mathbb{R}$, are equivalent. Let $h = \sup_{t \geqslant 0} h(t)$. Using results in [9] it is easy to prove that

PROPOSITION 6.1. *Suppose that* $h\cdot r \leqslant \exp[-1]$ *and set* $\omega = (1 + \log h\cdot r)/r$ *and* $\sigma = \omega - 1/r$. *Then* $U(t, s)$ *is an evolution operator of type* $\omega(\leqslant 0)$ *in* L^1_σ.

PROOF. In [9] it is proved that $A(t) + \omega I$ is m-accretive in L^1_σ. As the norms L^1_σ are equivalent, from (2.1) we have also

$$(2.1)' \qquad \|(I + \lambda A(t))^{-1}\varphi - (I + \lambda A(\tau))^{-1}\varphi\|_{1,\sigma} \leqslant \lambda\|b(t) - b(\tau)\| M\text{-}\big(\|\varphi\|_{1,\sigma}\big),$$

where M_σ is monotone increasing; hence $A(t)$ generates an evolution operator, $U_\sigma(t, s)$, of type ω in L^1_σ; and we have $U_\sigma(t, s) = U(t, s)$.

It follows that if $x(t)$ and $y(t)$ are the solutions of (1.1) with initial data φ and ψ respectively, we have

$$\begin{aligned}\|x(t) - y(t)\| &= \|F(t, x_t) - F(t, y_t)\| \leqslant \\ &\leqslant K_\sigma h(t)\|x_t - y_t\|_{1,\sigma} = K_\sigma h(t)\|U(t, s)\varphi - U(t, s)\psi\|_{1,\sigma} \leqslant \\ &\leqslant K_\sigma h(t)\exp[\omega(t - s)\|\varphi - \psi\|_{1,\sigma} \leqslant M\exp[\omega(t - s)]\|\varphi - \psi\|_{1,\sigma},\end{aligned}$$

where $K_\sigma = \max\{1, \exp[-r\sigma]\}$ and $M = K_{\sigma'}h$. If $hr < \exp[-1]$, solutions are asymptotically exponentially stable.

7. An example.

We now apply the results we have developed to the integral equation

$$(7.1)\qquad \begin{cases} x(t) = f(t) + \displaystyle\int_{t-r}^{t} K(t, \tau, x(\tau))\, d\tau\,, & 0 \leqslant s \leqslant t \leqslant T \\ x(t) = \varphi(t-s)\,, & \text{a.e. } t \in (s-r, s) \end{cases}$$

where $f\colon [0, T] \to Y$, $K\colon [0, T]\times[-r, T]\times Y \to Y$, and $\varphi \in X$. We place the following hypotheses on f and K:

(7.2) f is continuous on $[0, T]$.

(7.3) There exists a continuous function $k_1\colon [0, T] \to Y$ such that for all $t_1, t_2 \in [0, T]$, $\tau \in [-r, T]$, $x \in Y$, $\|K(t_1, \tau, x) - K(t_2, \tau, x)\| \leqslant \|k_1(t_1) - k_1(t_2)\| \|x\|$.

(7.4) There exists a constant L_1 such that for all $t \in [0, T]$, $\tau_1, \tau_2 \in [-r, T]$, $x \in Y$, $\|K(t, \tau_1, x) - K(t, \tau_2, x)\| \leqslant |\tau_1 - \tau_2| L_1 \|x\|$.

(7.5) There exists a bounded function $h_1\colon [0, T] \to \mathbb{R}$ such that for all $t \in [0, T]$, $\tau \in [-r, T]$, $x_1, x_2 \in Y$, $\|K(t, \tau, x_1) - K(t, \tau, x_2)\| \leqslant h_1(t)\|x_1 - x_2\|$.

(7.6) There exists a constant C_1, such that for all $t \in [0, T]$, $\tau \in [-r, T]$, $x \in Y$, $\|K(t, \tau, x)\| \leqslant C_1\|x\|$.

Define $F\colon [0, T]\times X \to Y$ by

$$(7.7)\qquad F(t, \varphi) = f(t) + \int_{t-r}^{t} K(t, \tau, \varphi(\tau - t))\, d\tau\,, \qquad \text{for all } t \in [0, T],\ \varphi \in X\,.$$

We verify that F satisfies (1.4), (1.5) and (1.6).

By (7.5) we have for all $t \in [0, T]$, $\varphi, \psi \in X$,

$$\|F(t, \varphi) - F(t, \psi)\| \leqslant \int_{t-r}^{t} h_1(t)\|\varphi(\tau - t) - \psi(\tau - t)\| d\tau = h_1(t)\|\varphi - \psi\|_1$$

so that (1.4) holds. By (7.3) and (7.4) we have that for all $t_1, t_2 \in [0, T]$, $\varphi \in X$,

$$\begin{aligned}
&\|F(t_1, \varphi) - F(t_2, \varphi)\| \leqslant \|f(t_1) - f(t_2)\| + \\
&\quad + \left\| \int_{t_1-r}^{t_1} K(t_1, \tau, \varphi(\tau - t_1))\, d\tau - \int_{t_1-r}^{t_1} K(t_2, \tau + t_2 - t_1, \varphi(\tau - t_1))\, d\tau \right\| \leqslant \\
&\quad \leqslant \|f(t_1) - f(t_2)\| + \int_{t_1-r}^{t_1} \|K(t_1, \tau, \varphi(\tau - t_1)) - K(t_2, \tau, \varphi(\tau - t_1))\|\, d\tau + \\
&\quad + \int_{t_1-r}^{t_1} \|K(t_2, \tau, \varphi(\tau - t_1)) - K(t_2, \tau + t_2 - t_1, \varphi(\tau - t_1))\|\, d\tau \leqslant \\
&\quad \leqslant \|f(t_1) - f(t_2)\| + \|k_1(t_1) - k_1(t_2)\| \int_{t_1-r}^{t_1} \|\varphi(\tau - t_1)\|\, d\tau + \\
&\quad + |t_1 - t_2| L_1 \int_{t_1-r}^{t_1} \|\varphi(\tau - t_1)\|\, d\tau \leqslant \\
&\qquad\qquad \leqslant \|f(t_1) - f(t_2)\| + \big(\|k_1(t_1) - k_1(t_2)\| + |t_1 - t_2| L_1\big)\|\varphi\|_1
\end{aligned}$$

so that (1.5) holds. By (7.6) we have that for $t \in [0, T]$, $\varphi \in X$,

$$\|F(t, \varphi)\| \leqslant \sup_{0 \leqslant \tau \leqslant T} \|f(t)\| + C_1\|\varphi\|_1$$

so that (1.6) holds.

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