

RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

DANIEL SIMSON

Right pure semisimple l -hereditary PI-rings

Rendiconti del Seminario Matematico della Università di Padova,
tome 71 (1984), p. 141-175

http://www.numdam.org/item?id=RSMUP_1984__71__141_0

© Rendiconti del Seminario Matematico della Università di Padova, 1984, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (<http://rendiconti.math.unipd.it/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

*Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques*
<http://www.numdam.org/>

Right Pure Semisimple l -Hereditary PI -Rings.

DANIEL SIMSON (*) (**)

We recall from [14] that an artinian ring is said to be *l-hereditary* if any local one-sided ideal of R is projective (l for local ideals). A ring R is *right pure semisimple* if every right R -module is a direct sum of finitely generated modules (see [24, 25, 28]). We say that a ring R is a *PI-ring* if R satisfies a polynomial identity. It is well-known that an artinian ring R is a *PI-ring* if and only if the endomorphism ring of any simple R -module is finite dimensional over its center.

There is an open problem if any right pure semisimple ring is of finite representation type (see [27, 28]). In [28] the problem was solved for a class of hereditary rings including hereditary *PI*-rings.

In the present paper we give a positive solution of the problem for *l-hereditary PI*-rings. In particular we show that the Bautista's diagrammatic characterization of *l-hereditary artin algebras* of finite representation type [3, 4] remains also true for *l-hereditary PI*-rings. A complete list of indecomposable modules is given for any non-homogeneous *l-hereditary PI*-ring of finite representation type.

We recall from [5, 7] that a module M is said to be *l-hereditary* if all local submodules of M are projective. It is easy to see that if R is an *l-hereditary right QF-2 artinian ring* then an R -module M is *l-hereditary* if and only if $\text{soc}(M)$ is projective.

We recall that an artinian ring R is said to be a *right QF-2 ring* if every indecomposable projective right ideal in R has a simple socle.

(*) The author was partially supported by Consiglio Nazionale delle Ricerche, Comitato Nazionale per le Scienze Matematiche.

(**) Indirizzo dell'A.: Institute of Mathematics, Nicholas Copernicus University ul. Chopina 12/18, 87-100 Toruń, Poland.

The main tool we use in the study of right pure semisimple l -hereditary PI -rings are special Schurian vector space categories \mathbf{K}_F and the functor

$$\Phi: \mathcal{U}(\mathbf{K}_F) \rightarrow l\text{ her}(\mathbf{R}_K)$$

defined in [31] where \mathbf{R}_K is an l -hereditary right QF -2 semiperfect ring associated to \mathbf{K}_F and $l\text{ her}(\mathbf{R}_K)$ is the category of finitely generated right \mathbf{R}_K -modules (see § 1). The general idea of the proof is similar to that one in [20] and is the following. For any right pure semisimple l -hereditary PI -ring R we construct two l -hereditary PI -rings R' and R'' such that

- 1) R' is a proper factor ring of R ;
- 2) R'' is a right QF -2 artinian ring of the form \mathbf{R}_K where \mathbf{K}_F is a special Schurian vector space category associated to R ;
- 3) $\# \text{ mod}(R) = \# \text{ mod}(R') + \# l\text{ her}(R'')$ where $\# \mathcal{B}$ denotes the number of pairwise nonisomorphic indecomposable objects in \mathcal{B} .

Since by an inductive assumption we can suppose that $\# \text{ mod}(R')$ is finite then the problem is reduced to the question if $\# l\text{ her}(R'')$ is finite. Fortunately there is only few types of l -hereditary right QF -2 artinian PI -rings which appear as the rings R'' above and for any such a ring we are able to determine its indecomposable l -hereditary modules. We do it again by applying to R'' the above reduction method and by using some of the results of Dlab and Ringel [8, 9].

The paper essentially depends on the results of Bautista [3, 4], Bautista and Martinez [5], Dlab and Ringel [8, 9], Dowbor and Simson [10], Loupias [15, 16] and of the author [24, 25, 28].

The results were announced in [30] and a part of them was presented during the Trento Meeting on Abelian Groups and Modules Theory in May 1980.

Throughout this paper R denotes an indecomposable basic l -hereditary artinian ring of the triangular matrix form

$$R = \begin{bmatrix} F_1 & {}_1M_2 & \dots & {}_1M_{n-1} & {}_1M_n \\ & F_2 & \dots & {}_2M_{n-1} & {}_2M_n \\ & & \ddots & \vdots & \vdots \\ 0 & & & F_{n-1} & {}_{n-1}M_n \\ & & & & F_n \end{bmatrix}$$

where F_1, \dots, F_n are division rings, ${}_iM_j$ are F_i - F_j -bimodules finite dimensional over F_i and over F_j , and the multiplication in R is given by F_i - F_j -bimodule maps

$$c_{ijk}: {}_iM_j \otimes_j M_k \rightarrow {}_iM_k$$

with the property that $c_{ijk}(a \otimes b) = 0$ if and only if either $a = 0$ or $b = 0$. We know from [14, Lemma 1] that every l -hereditary basic artinian ring R has such a triangular matrix form.

We associate with R a *valued poset* (I_R, \mathbf{d}) where $I_R = \{1, \dots, n\}$, $i < j \Leftrightarrow {}_iM_j \neq 0$ and $\mathbf{d} = (d_{ij})$ is a matrix with

$$d_{ij} = \dim ({}_iM_j)_{F_j}, \quad d_{ji} = \dim_{F_i} ({}_iM_j) \quad \text{for } i \neq j.$$

We will write

$$i \xrightarrow{(a_{ij}, a_{ji})} j$$

if $i < j$ and there is no k in I_R such that $i < k < j$; if $d_{ij} = d_{ji} = 1$ we write simply $i \rightarrow j$. The valued poset (I_R, \mathbf{d}) is said to be *homogeneous* if $d_{ij} \neq 0$ implies $d_{ij} = 1$.

It is easy to see that an indecomposable l -hereditary artinian ring R is a right QF-2 ring if and only if the valued poset (I_R, \mathbf{d}) has a unique maximal element m and $d_{jm} = 1$ for every j .

Throughout this paper $E(X)$ denotes the injective envelope of the module X and $P(X)$ denotes the projective cover of X . We denote by $\text{mod}(R)$ the category of finitely generated right R -modules and by $l\text{her}(R)$ the category of finitely generated l -hereditary right R -modules. The reader is referred to [5, 7] for basic properties of l -hereditary modules.

1. Preliminaries.

We recall from [31] that an additive category \mathbf{K} together with an additive faithful functor

$$|-|: \mathbf{K} \rightarrow \text{mod}(F),$$

where F is a division ring, is denoted by \mathbf{K}_F and it is called *special*

Schurian vector space category if \mathbf{K} is a Krull-Schmidt category, \mathbf{K} has only a finite number of pairwise nonisomorphic indecomposable objects and $\dim |X|_F = 1$ as well as $\text{End}(X)$ is a division ring for any indecomposable object X in \mathbf{K} .

We recall from [20, 21] that the *subspace category* $\mathfrak{U}(\mathbf{K}_F)$ of \mathbf{K}_F is defined as follows. The objects of $\mathfrak{U}(\mathbf{K}_F)$ are triples (U, X, φ) where U is a finite dimensional vector space over F , X is an object in \mathbf{K} and $\varphi: U \rightarrow |X|_F$ is an F -linear map. The map from (U, X, φ) into (U', X', φ') in $\mathfrak{U}(\mathbf{K}_F)$ is a pair (u, h) where $u \in \text{Hom}_F(U, U')$ and $h: X \rightarrow X'$ is a map in \mathbf{K} such that $|h|\varphi = \varphi'u$. It is clear that $\mathfrak{U}(\mathbf{K}_F)$ is an additive Krull-Schmidt category.

Throughout we suppose that \mathbf{K}_F is a special Schurian vector space category, we fix a complete set X_1, \dots, X_n of pairwise nonisomorphic indecomposable objects in \mathbf{K} and we put

$$F_{n+1} = F \quad \text{and} \quad F_i = \text{End}(X_i) \quad \text{for } i = 1, \dots, n.$$

For any $i, j \leq n + 1$ we consider F_i - F_j -bimodules defined by formula

$$\begin{aligned} {}_iN_j &= \text{Hom}(X_j, X_i) \quad \text{for } i, j \leq n, \\ &= {}_{F_i}|X_i|_F \quad \text{for } j = n + 1. \end{aligned}$$

Since $\dim |X_i|_F = 1$ and F_i are division rings for all i then ${}_iN_j \neq 0$ implies ${}_jN_i = 0$. Therefore without loss of generality we can suppose that $i \leq j$ whenever ${}_iN_j \neq 0$.

In [31] we have associated to \mathbf{K}_F the triangular matrix ring

$$R_{\mathbf{K}} = \begin{bmatrix} F_1 & {}_1N_2 & \dots & {}_1N_n & {}_1N_{n+1} \\ & F_2 & \dots & {}_2N_n & {}_2N_{n+1} \\ & & \ddots & \vdots & \vdots \\ 0 & & & F_n & {}_nN_{n+1} \\ & & & & F_{n+1} \end{bmatrix}$$

where the multiplication is given by F_i - F_k -bilinear maps

$$c_{ijk}: {}_iN_j \otimes {}_jN_k \rightarrow {}_iN_k (\otimes = \otimes_{F_i})$$

defined by the formula

$$\begin{aligned} c_{ijk}(f \otimes g) &= fg \quad \text{for } k < n, \\ &= |f|(g) \quad \text{for } k = n + 1. \end{aligned}$$

We denote by $\mathcal{U}_0(\mathbf{K}_F)$ (resp. by $l\text{her}_0(\mathbf{R}_K)$) the full subcategory of $\mathcal{U}(\mathbf{K}_F)$ (resp. of $l\text{her}(\mathbf{R}_K)$) consisting of objects having no direct summands of the form $(F, 0, 0)$ (resp. no simple projective summands).

We have the following result proved in [31].

THEOREM 1.1. *Let \mathbf{K}_F be a special Schurian vector space category such that the ring \mathbf{R}_K is artinian. Then \mathbf{R}_K is an l -hereditary QF -2 ring and there exists a full and dense additive functor*

$$\Phi: \mathcal{U}(\mathbf{K}_F) \rightarrow l\text{her}_0(\mathbf{R}_K)$$

with the following properties:

(a) *If A is an indecomposable object in $\mathcal{U}(\mathbf{K}_F)$ then $\Phi(A) = 0$ if and only if A has one of the following forms $(F, 0, 0)$, $\hat{X}_i = (|X_i|, X_i, i\hat{d})$, $i = 1, \dots, n$.*

(b) *If A and B are objects in $\mathcal{U}(\mathbf{K}_F)$ having no summands of the form $(F, 0, 0)$, $\hat{X}_1, \dots, \hat{X}_n$ then every isomorphism from $\Phi(A)$ into $\Phi(B)$ has the form $\Phi(h)$ where $h: A \rightarrow B$ is an isomorphism.*

Theorem 1.1 will be used in the next Section.

2. Indecomposable modules over l -hereditary artinian PI -rings.

Throughout this section R will denote a basic l -hereditary artinian PI -ring of the triangular matrix form as in the introduction. We keep the terminology and notation in the introduction and Section 1.

A module X over R will be identified with a system $(X_i, {}_j\varphi_i)$ where X_i is a vector space over F_i and ${}_j\varphi_i: X_i \otimes {}_iM_j \rightarrow X_j$, $j = 1, \dots, n$, are F_j -linear maps satisfying the usual commutativity and associativity conditions (see [26, Sec. 3]). If no confusion will arise we will write simply (X_i) instead of $(X_i, {}_j\varphi_i)$.

For every j we consider the simple R -module $\tilde{F}_j = (X_i)$ with

$X_j = F_j$ and $X_i = 0$ for $i \neq j$. It is clear that $\tilde{F}_1, \dots, \tilde{F}_{n+1}$ is a complete list of simple R -modules.

A full additive subcategory \mathcal{A} of $\text{mod}(R)$ is said to be *cofinite* if all but a finite number of indecomposable objects in $\text{mod}(R)$ belong to \mathcal{A} . We denote by $\#\mathcal{A}$ the number of isomorphism classes of indecomposable objects in \mathcal{A} .

Finally, we recall that an additive functor is a *representation equivalence* if it is full, dense and reflects isomorphisms.

Throughout this section $\mathbf{R}(\mathbf{K})$ denotes the ring $\mathbf{R}_{\mathbf{K}}$ associated to the vector space category \mathbf{K}_F .

The aim of this section is to prove Theorem 2.5. Before we formulate it we prove some preliminary results.

The following simple lemma will be frequently used in this paper.

LEMMA 2.1. *Let R be an l -hereditary right QF-2 artinian ring and let m be the unique maximal element in the valued poset (I_R, \mathbf{d}) .*

(i) *If (I_R, \mathbf{d}) has a unique minimal element a such that $d_{am} = d_{ma} = 1$ and $R = P(\tilde{F}_a) \oplus P'$ then $R' = \text{End}(P')$ is an l -hereditary right QF-2 artinian and $\#l \text{ her}(R) = 1 + \#l \text{ her}(R')$.*

(ii) *Suppose that (I_R, \mathbf{d}) has an element m' such that there is an arrow $m' \rightarrow m$ in (I_R, \mathbf{d}) and $s \leq m'$ for all s in I_R , $s \neq m$. Let $R = \tilde{F}_m \oplus P''$. Then $R'' = \text{End}(P'')$ is an l -hereditary right QF-2 artinian ring and $\#l \text{ her}(R) = 1 + \#l \text{ her}(R'')$.*

PROOF. (i) If $X = (X_i)$ is an indecomposable module in $l \text{ her}(R)$ with $X_a \neq 0$ then there is an epimorphism $X \rightarrow \tilde{F}_a$. Hence $X = P(\tilde{F}_a)$ because the module $P(\tilde{F}_a)$ is projective-injective. Consequently (i) follows.

(ii) Consider the functor $T: l \text{ her}(R'') \rightarrow l \text{ her}(R)$ defined by $(X_i, {}_i\varphi_i)_{i, j \leq m'} \mapsto (Y_i, {}_i\psi_i)_{i, j \leq m}$ with $Y_i = X_i$ for $i \leq m'$ and $Y_m = X_m, {}_m\psi_m = id$. It is easy to see that every indecomposable module in $l \text{ her}(R)$ except the simple projective module \tilde{F}_m belongs to the image of T and the proof is complete.

Following Ringel [22] we define vector space categories arising from simple injective and simple projective modules. Let R be an l -hereditary artinian ring and (I_R, \mathbf{d}) the valued poset of R . Given a minimal point a in I_R (corresponding to the simple injective module \tilde{F}_a) we take the projective cover $P(\tilde{F}_a)$ and consider a decomposition

$R = P(\tilde{F}_a) \oplus P'$. We put

$$R_a = \text{End}(P'), \quad M^a = \text{rad } P(\tilde{F}_a) \quad \text{and} \quad F = F_a.$$

Then M^a is an F - R_a -bimodule and there is a ring isomorphism

$$R = \begin{bmatrix} F & {}_F M_{R_a}^a \\ 0 & R_a \end{bmatrix}.$$

Since R is l -hereditary the R_a -module M^a is l -hereditary and we have two vector space categories

$$\mathbf{K}_F^a = \text{Hom}_{R_a}(M^a, \text{mod}(R_a)) \quad \text{and} \quad \tilde{\mathbf{K}}_F^a = \text{Hom}_{R_a}(M^a, l \text{ her}(R_a))$$

which are the images of the categories $\text{mod}(R_a)$ and $l \text{ her}(R_a)$ under the functor $\text{Hom}_{R_a}(M^a, -)$ (see [22, p. 200]). Let us denote by $\text{mod}(R)_a$ the full subcategory of $\text{mod}(R)$ consisting of modules having no summands isomorphic to \tilde{F}_a . We have the following useful result.

PROPOSITION 2.2. *If the vector space category \mathbf{K}_F^a is special Schurian and $\mathbf{R}(\mathbf{K}^a)$ is the l -hereditary right QF-2 artinian ring associated to \mathbf{K}_F^a then*

(i) *There exists a full additive functor $H: \text{mod}(R) \rightarrow l \text{ her}(\mathbf{R}(\mathbf{K}^a))$ which establishes a representation equivalence between a subcategory of $\text{mod}(R)$ and a cofinite subcategory of $l \text{ her}(\mathbf{R}(\mathbf{K}^a))$.*

(ii) $\# \text{mod}(R) = \# \text{mod}(R_a) + \# l \text{ her}(\mathbf{R}(\mathbf{K}^a))$.

If $\tilde{\mathbf{K}}_F^a$ is special Schurian then

$$\# l \text{ her}(R) = \# l \text{ her}(R_a) + \# l \text{ her}(\mathbf{R}(\tilde{\mathbf{K}}^a)) - 1.$$

PROOF. It is easy to see that $\text{mod}(R)$ (resp. $l \text{ her}(R)$) is equivalent to the category of triples (V_F, X, t) where X is a finitely generated right R_a -module (resp. l -hereditary R_a -module) and $t: V \otimes_F M^a \rightarrow X$ is an R_a -homomorphism (resp. such that the map $t': V_F \rightarrow \text{Hom}_{R_a}({}_F M^a, X)$ adjoint to t is injective). Let H be the composed functor

$$\text{mod}(R)_a \xrightarrow{H'} \mathcal{U}(\mathbf{K}_F^a) \xrightarrow{\phi} l \text{ her}(\mathbf{R}(\mathbf{K}^a))$$

where H' is given by $(V_F, X, t) \mapsto (V_F, \text{Hom}_{R_a}(M^a, X), t')$. It follows

from [22, § 2.5, Lemma 2] and from the properties of the functor $\tilde{\Phi}$ that H has the required properties. Furthermore, if s is the number of all indecomposable modules X such that $\text{Hom}_{R_a}(M^a, X) \neq 0$ then obviously $\# \text{mod}(R_a) = s + \# \mathbf{K}_F^a$. Hence the equality (ii) is a consequence of Theorem 1.1 and [22, § 2.5, Lemma 2]. The second equality follows in a similar way.

Now we assume that c is a maximal point in (I_R, \mathbf{d}) . Then the simple module \tilde{F}_c is projective and we have a right module decomposition $R = \tilde{F}_c \oplus P''$. Hence there is a ring isomorphism

$$R = \begin{bmatrix} {}_cR & N^c \\ 0 & F \end{bmatrix}$$

where ${}_cR = \text{End}(P'')$, $N^c = \text{Hom}_R(\tilde{F}_c, P'')$ and $F = F_c$. We define a vector space category

$$\mathbf{L}_F^c = \text{mod}({}_cR) \otimes_{{}_cR} N_F^c$$

as the image of $\text{mod}({}_cR)$ under the functor $-\otimes_{{}_cR} N_F^c$.

PROPOSITION 2.3. *If \mathbf{L}_F^c is special Schurian then*

(i) *There exists a full additive functor $H: \text{mod}(R) \rightarrow \text{her}(\mathbf{R}(\mathbf{L}^c))$ which establishes a representation equivalence between an additive subcategory of $\text{mod}(R)$ and a cofinite additive subcategory of $\text{her}(\mathbf{R}(\mathbf{L}^c))$.*

(ii) $\# \text{mod}(R) = \# \text{mod}({}_cR) + \# \text{her}(\mathbf{R}(\mathbf{L}^c))$.

PROOF. It is clear that $\text{mod}(R)$ is equivalent to the category of triples (X, V_F, t) where X is in $\text{mod}({}_cR)$ and $t: X \otimes_{{}_cR} N_F^c \rightarrow V_F$ is a linear map. Since the category $\mathcal{U}(\mathbf{L}_F^c)$ is equivalent to the category of triples (X, V_F, t) where $t: |X|_F \rightarrow V_F$ is a linear map and $|X|_F = X \otimes_{{}_cR} N_F^c$ then there is a full and dense additive functor $H': \text{mod}(R) \rightarrow \mathcal{U}(\mathbf{L}_F^c)$ such that the composition of H' and the functor $\mathcal{G}: \mathcal{U}(\mathbf{L}_F^c) \rightarrow \text{her}(\mathbf{R}(\mathbf{L}^c))$ has the properties required for H . The equality (ii) can be proved similarly as (ii) in Proposition 2.2 (see [31; 1.5]).

In our discussion of pure semisimple rings we will need the following result.

PROPOSITION 2.4. *Suppose R and S are right artinian rings, \mathfrak{B} is a full additive subcategory of $\text{mod}(R)$ and let $T: \mathfrak{B} \rightarrow \text{mod}(S)$ be an*

additive functor which is full and reflects isomorphisms. If $\text{Im } T$ is cofinite in $\text{mod}(S)$ and the ring R is right pure semisimple then S is right pure semisimple too.

PROOF. Let $X_1 \xrightarrow{f_1} X_2 \rightarrow \dots \rightarrow X_n \xrightarrow{f_n} X_{n+1} \rightarrow \dots$ be a sequence of nonzero monomorphisms between indecomposable modules X_j in $\text{mod}(S)$. We shall show that there is an integer m such that f_j is an isomorphism for $j > m$. This is obviously the case whenever there are infinitely many indices j such that X_j does not belong to $\text{Im } T$. Then we can suppose that X_j belongs to $\text{Im } T$ for all j . By our assumption there is a sequence in $\text{mod}(R)$

$$Y_1 \xrightarrow{g_1} Y_2 \rightarrow \dots \rightarrow Y_n \xrightarrow{g_n} Y_{n+1} \rightarrow \dots$$

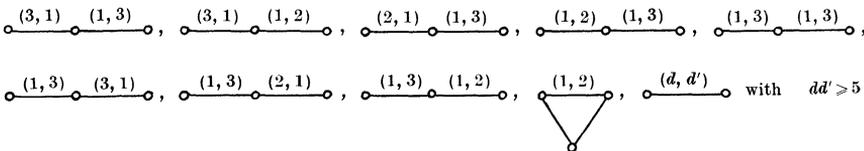
where Y_j is indecomposable, $T(Y_j) = X_j$ and $T(g_j) = f_j$ for all j . Since R is right pure semisimple then by [29, Theorem 1.3] and [25, Theorem 6.3] there is an integer m such that g_j are isomorphisms for $j > m$. Hence our claim follows and by [29, Theorem 1.3] S is right pure semisimple as we required.

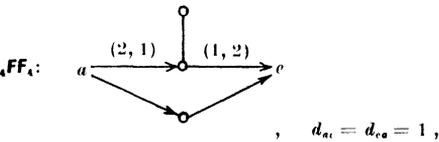
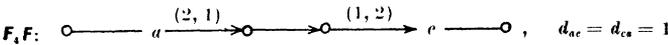
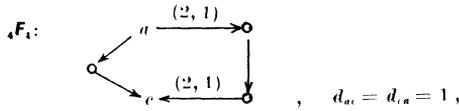
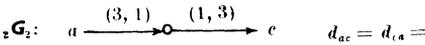
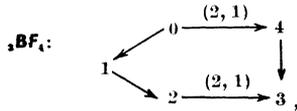
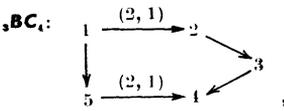
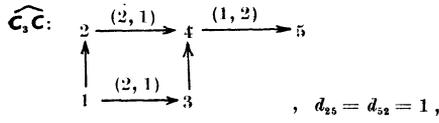
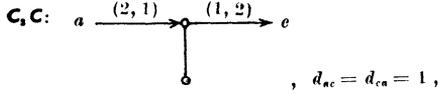
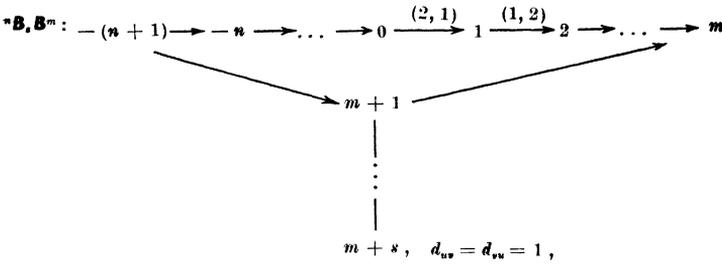
Following [15] we call a surjective map $f: (I_R, \mathbf{d}) \rightarrow (I_{R'}, \mathbf{d}')$ of valued posets a *contraction* if $f^{-1}(j)$ is connected and homogeneous for all j in $I_{R'}$.

Now we are able to prove the main result of this section.

THEOREM 2.5. *Let R be a basic l -hereditary artinian PI -ring and suppose that the valued poset (I_R, \mathbf{d}) is connected. Then the following statements are equivalent:*

- (1) R is of finite representation type.
- (2) R is right pure semisimple.
- (3) Neither (I_R, \mathbf{d}) nor its dual has no contractions and no full subposets of one of the following forms:
 - a) the extended Dynkin diagrams [9];
 - b) the minimal wild valued graphs:





where $\circ - \circ$ means either $\circ \rightarrow \circ$ or $\circ \leftarrow \circ$.

REMARK 1. Let us recall that Bautista [3, 4] has proved the equivalence of the statements (1), (3) and (4) under the assumption that R is a finite dimensional l -hereditary algebra over a field. He succesfully applies the almost split sequences. We can not follow the Bautista's proof in our more general situation because usually we have no almost split sequences. We will apply Propositions 2.2 and 2.3.

The proof of the implication (4) \rightarrow (1) will be divided into several lemmas. We start with the following useful remark.

REMARK 2. If F and G are division rings, and ${}_F M_G$ is an F - G -bimodule such that $\dim_F M = 1$, then for any nonzero element m in M there are a ring isomorphism $s: G \rightarrow F$ defined by $mx = s(x)m$ and an F - G -bimodule isomorphism ${}_F M_G \cong {}_F F_G$. If, in addition, $\dim M_G = 1$ then s is an isomorphism [3, 4].

LEMMA 2.6. *Let R be an l -hereditary artinian PI-ring. If (I_R, \mathbf{d}) is of one of the forms ${}_n \mathbf{C}_n, {}_2 \mathbf{G}_2, {}_4 \mathbf{F}_4$ then $\# \text{ mod } (R)$ is equal $n^2 + \frac{1}{2}n(n + 1) + 1, 12$ and 45 , respectively. Moreover, if $X = (X_i)$ is an indecomposable non projective-injective module then $X_e = 0$ provided that e is either a maximal or a minimal element in I_R .*

PROOF. Suppose (I_R, \mathbf{d}) has the form ${}_n \mathbf{C}_n$. It follows from Remark 2 that without loss of generality we can suppose that R has the form

$$R = \begin{bmatrix} F & F & F & \dots & F & F \\ & G & G & \dots & G & F \\ & & G & \dots & G & F \\ & & & \ddots & \vdots & \vdots \\ & 0 & & & G & F \\ & & & & & F \end{bmatrix}$$

where $G \subset F$ are division rings such that $\dim_G F = \dim F_G = 2$. We prove the lemma by applying Proposition 2.3 to R and $c = n + 1$. First we note that (I_R, \mathbf{d}) is the poset ${}_n \mathbf{C}_n$ with the natural linear ordering. The ring ${}_e R$ is obtained from R by omitting the last column, whereas N^e is the last column of R without the lower term F . The indecomposable objects in the vector space category L_F^e are determined by indecomposable ${}_e R$ -modules X for which $X \otimes N_F^e \neq 0$ or

equivalently $0 \neq (X \otimes N_F^c)^* = \text{Hom}_F(X \otimes N_F^c, F) = \text{Hom}_{eR}(X, (N^c)^*)$. It is clear that the eR -module $(N^c)^*$ is isomorphic to the unique indecomposable projective-injective module P_1 which is represented by the upper row in eR . Since eR is hereditary then $X \otimes N_F^c \neq 0$ if and only if X is one of the indecomposable projectives $(N_F^c)^* = P_1, P_2, \dots, P_n$ in $\text{mod}(eR)$ which correspond to the rows of eR . It follows that $P_1 \otimes N_F^c \cong {}_F F_F$ and $P_i \otimes N_F^c \cong {}_G F_F$ for $i = 2, \dots, n$. Now it is easy to see that the vector space category L_F^c is special Schurian and the ring $\mathbf{R}(L^c)$ associated to L_F^c is isomorphic to R .

We know from [10] that $\# \text{mod}(eR) = n^2$. By Lemma 2.1

$$\# l \text{ her}(R) = 1 + \# l \text{ her}(R')$$

where R' is obtained from R by omitting the upper row. From the definition of l -hereditary modules follows that the category $l \text{ her}(R')$ is equivalent to the category of $S_{n-1}(G)$ -spaces in the sense of Dlab and Ringel [8] and therefore [8, Proposition 2.5] yields $\# l \text{ her}(R') = \frac{1}{2}n(n+1)$. Then by Proposition 2.3 we have $\# \text{mod}(R) = n^2 + \frac{1}{2}n(n+1) + 1$, as required.

In order to prove the second part of the lemma we note that by [10] there are n^2 of those indecomposable R -modules $X = (X_i)$ for which $X_1 = 0$ and n^2 of those for which $X_{n+1} = 0$. Since we have calculated twice the modules X with $X_1 = 0 = X_{n+1}$ then together with the unique indecomposable projective-injective module (F, F, \dots, \dots, F) we have a list of $2n^2 - \frac{1}{2}n(n-1) + 1 = \# \text{mod}(R)$ indecomposable R -modules. Hence the lemma follows in the case ${}_n C_n$. In the remaining cases the proof is similar and we leave it to the reader (use [8, Propositions 3.1 and 4.2]).

We note that the second part of the lemma together with [10] gives a complete classification of indecomposable R -modules.

LEMMA 2.7. *Let $G \subset F$ be division rings such that $\dim_G F = \dim F_G = 2$ and let*

$$R = \begin{bmatrix} F & 0 & F & F \\ & G & G & F \\ & & 0 & G & F \\ & & & & F \end{bmatrix}, \quad T = \begin{bmatrix} F & 0 & 0 & F & F \\ & G & G & G & F \\ & & G & G & F \\ & & & 0 & G & F \\ & & & & & F \end{bmatrix}, \quad T' = \begin{bmatrix} F & F & 0 & F & F \\ & F & 0 & F & F \\ & & G & G & F \\ & & & 0 & G & F \\ & & & & & F \end{bmatrix},$$

$$E = \begin{bmatrix} F & F & F & F & F & F \\ & G & G & F & F & F \\ & & G & 0 & G & F \\ & & & F & F & F \\ 0 & & & & G & F \\ & & & & & F \end{bmatrix}, \quad E' = \begin{bmatrix} F & 0 & 0 & F & F & F \\ & G & G & F & F & F \\ & & G & 0 & G & F \\ & & & F & F & F \\ 0 & & & & G & F \\ & & & & & F \end{bmatrix}.$$

(1) R, T, T', E and E' are l -hereditary right QF-2 artinian rings and

$$(I_x, \mathbf{d}): \quad 2 \xrightarrow{(2,1)} 4 \xrightarrow{(1,2)} 5$$

$$\quad \quad \quad \uparrow$$

$$\quad \quad \quad 3, \quad d_{25} = d_{52} = 1,$$

$$(I_r, \mathbf{d}): \quad 2 \longrightarrow 3 \longrightarrow 4 \xrightarrow{(1,2)} 5, \quad d_{15} = d_{51} = 1,$$

$$\quad \quad \quad \quad \quad \downarrow \begin{matrix} 1 \\ (2,1) \end{matrix}$$

$$(I_r, \mathbf{d}): \quad 1 \longrightarrow 2 \xrightarrow{(2,1)} 4 \xrightarrow{(1,2)} 5$$

$$\quad \quad \quad \quad \quad \uparrow$$

$$\quad \quad \quad \quad \quad 3, \quad d_{25} = d_{52} = 1,$$

$$(I_x, \mathbf{d}): \quad 1 \xrightarrow{(2,1)} 2 \xrightarrow{(1,2)} 4 \xrightarrow{(2,1)} 5 \xrightarrow{(1,2)} 6$$

$$\quad \quad \quad \quad \quad \searrow \quad \nearrow$$

$$\quad \quad \quad \quad \quad 3, \quad d_{16} = d_{61} = d_{14} = d_{41} = d_{44} = d_{46} = 1$$

$$\quad \quad \quad \quad \quad d_{25} = d_{52} = 2,$$

$$(I_r, \mathbf{d}): \quad 2 \xrightarrow{(1,2)} 4 \xrightarrow{(2,1)} 5 \xrightarrow{(1,2)} 6$$

$$\quad \quad \quad \quad \quad \downarrow \quad \nearrow$$

$$\quad \quad \quad \quad \quad 1 \quad 3, \quad d_{25} = d_{52} = 2, \quad d_{46} = d_{64} = 1,$$

(2) $\# \text{ her}(R) = 10, \# l \text{ her}(T) = 24, \# l \text{ her}(T') = 15,$
 $\# l \text{ her}(E) = 14, \# l \text{ her}(E') = 28$ and $\# \text{ mod}(R) = 23$

(3) Any indecomposable R -module $X = \begin{pmatrix} X_2 & X_4 & X_5 \\ & X_3 & \end{pmatrix}$ with $X_2 \neq 0,$
 $X_3 \neq 0$ and $X_5 \neq 0$ is of one of the following forms:

$$D_1 = \begin{pmatrix} F & F & F \\ & G & \end{pmatrix}, \quad D_2 = \begin{pmatrix} F & F & F \\ & F & \end{pmatrix}, \quad D_3: (1, f)F \rightarrow F \oplus F \xrightarrow{\text{id}} F \oplus F,$$

$$\uparrow$$

$$G \oplus G$$

$$D_4: F \xrightarrow{u} F \oplus G \xrightarrow{v} F \oplus 0, \quad D_5: F \xrightarrow{u} F \oplus G \xrightarrow{v} F \oplus 0,$$

$$\uparrow \quad \uparrow$$

$$G \oplus 0 \quad F \oplus 0$$

$$D_6: (1, 1)F \rightarrow F \oplus F \xrightarrow{v} 0 \oplus F, \quad D_7: F \oplus F \xrightarrow{w} F \oplus F \xrightarrow{v} 0 \oplus F,$$

$$\uparrow \quad \uparrow$$

$$0 \oplus F \quad 0 \oplus F$$

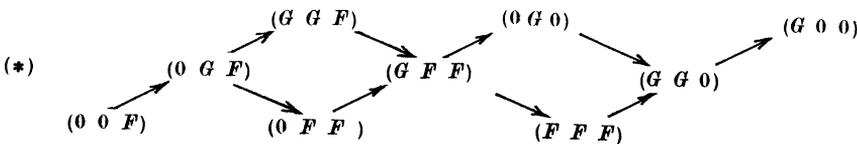
where $\{1, f\}$ is a fixed basis of ${}_aF$ as well as of F_a over G , p denote the natural projections and $u(x + fy) = (x + fy, x)$, $w(x + fy) = (2x + fy, x + 2fy)$, $x, y \in G$. Moreover $\text{End}(D_1) \cong G$ and $\text{End}(D_j) \cong F$ for $j = 2, \dots, 7$.

(4) Every indecomposable module $X = \begin{pmatrix} X_1 & X_2 & X_4 & X_5 & X_6 \\ & & X_3 & & \end{pmatrix}$ in $l \text{ her}(E)$

with $X_2 \neq 0$ is of one of the forms $\begin{pmatrix} F & F & F & F & F \\ & F & & & \end{pmatrix}$, $\begin{pmatrix} 0 & F & F & F & F \\ & F & & & \end{pmatrix}$, $\begin{pmatrix} 0 & G & F & F & F \\ & F & & & \end{pmatrix}$, $\begin{pmatrix} 0 & G & F & F & F \\ & G & & & \end{pmatrix}$.

PROOF. The proof of the statement (1) is obvious.

(2) First we will apply Proposition 2.2 to the ring R and $a = 2$. Note that the ring R_2 is hereditary, the valued graph of R_2 is $3 \rightarrow 4 \xrightarrow{(1,2)} 5$ and it follows from [8, Proposition 2.5] that the indecomposable R_2 -modules form a diagram



Since $M^2 = (0 F F)$ then the vector space category $\tilde{\mathbf{K}}_2^2$ has 3 indecomposable objects corresponding to the R_2 -modules M^2 , $(G F F)$ and

$(F F F)$. It follows that $\tilde{\mathbf{K}}_F^2$ is special Schurian,

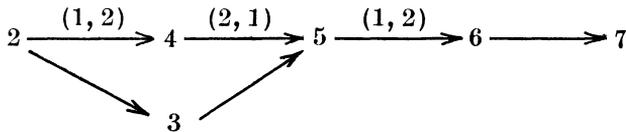
$$\mathbf{R}(\tilde{\mathbf{K}}^2) \cong \begin{bmatrix} F & F & F & F \\ & G & F & F \\ \mathbf{0} & & F & F \\ & & & F \end{bmatrix}$$

and the valued poset of $\mathbf{R}(\tilde{\mathbf{K}}^2)$ has the form $a \xrightarrow{(2,1)} c \xrightarrow{(1,2)} e \rightarrow \circ$, $d_{ac} = d_{ae} = 1$. By Lemma 2.1 and [8, Proposition 2.5] $\# \text{her}(\mathbf{R}(\tilde{\mathbf{K}}^2)) = 1 + 1 + 3 = 5$. Then Proposition 2.2 yields $\# \text{her}(R) = 6 + 5 - 1 = 10$, as we required.

Now we note that the category \mathbf{K}_F^2 has 5 indecomposable objects corresponding to the modules M^2 , $(G F F)$, $(F F F)$, $(0 G 0)$ and $(G G 0)$. It is easy to check that \mathbf{K}_F^2 is special Schurian, $\mathbf{R}(\mathbf{K}^2)$ has the form

$$R' = \begin{bmatrix} G & G & F & F & F & F \\ & G & 0 & G & F & F \\ & & F & F & F & F \\ & & & G & F & F \\ \mathbf{0} & & & & F & F \\ & & & & & F \end{bmatrix}$$

and its valued poset has the form



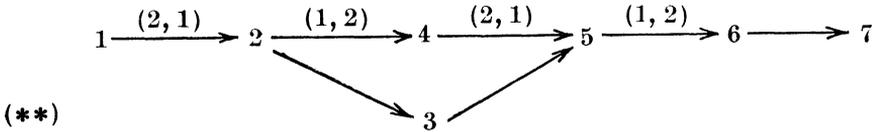
with $d_{46} = d_{64} = 1$ and $d_{25} = d_{52} = 2$. By Proposition 2.2 $\# \text{mod}(R) = 9 + \# \text{her}(R')$. Since we know from Lemma 2.1 that $\# \text{her}(R') = \# \text{her}(E)$ it remains to prove that this number equals 14. To prove it we consider the hereditary ring

$$S = \begin{bmatrix} F & 0 & 0 & F \\ & G & G & F \\ \mathbf{0} & & G & F \\ & & & F \end{bmatrix}.$$

The valued graph of S is $2 \rightarrow 5 \xrightarrow{(2,1)} 4 \leftarrow 3$. Now we will apply Proposition 2.2 to $R = S$ and $a = 2$. Note that $S_2 = R_2$, $M^2 = (0 \ 0 \ F)$ and hence the category $\tilde{\mathbf{K}}_r^2$ has 6 indecomposable objects corresponding to the modules of the form $(\cdot \cdot F)$ in the diagram (*). It follows that $\tilde{\mathbf{K}}_r^2$ is special Schurian, the ring $\mathbf{R}(\tilde{\mathbf{K}}^2)$ has the form

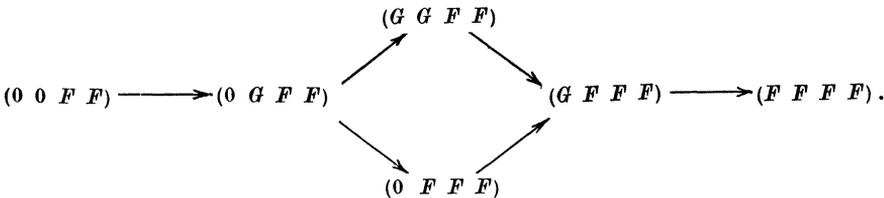
$$S' = \begin{bmatrix} F & F & F & F & F & F & F \\ & G & G & F & F & F & F \\ & & G & 0 & G & F & F \\ & & & F & F & F & F \\ & & & & F & F & F \\ & 0 & & & & F & F \\ & & & & & & F \end{bmatrix}$$

and the valued poset of S' is of the form



with $d_{16} = d_{61} = d_{14} = d_{41} = d_{46} = d_{64} = 1$ and $d_{25} = d_{52} = 2$. Since Lemma 2.1 yields $\#l\text{ her}(S') = 1 + \#l\text{ her}(E)$ then by Proposition 2.2 and [8, Proposition 3.1] we have $20 = \#l\text{ her}(S) = 6 + \#l\text{ her}(E)$ as we required. Consequently, $\#\text{ mod}(R) = 23$ and $\#l\text{ her}(E) = 14$.

In order to prove (2) for T we will apply Proposition 2.2 to $R = T$ and $a = 1$. The valued poset of the ring T_1 has the form $2 \rightarrow 3 \rightarrow 4 \xrightarrow{(1,2)} 5$ and $M^1 = (0 \ 0 \ F \ F)$. By [8, Proposition 2.5] those indecomposable modules X in $l\text{ her}(T_1)$ for which $\text{Hom}_{T_1}(M^1, X) \neq 0$ form the following diagram



It follows that the vector space category $\tilde{\mathbf{K}}_p^1$ is special Schurian, $\mathbf{R}(\tilde{\mathbf{K}}^1) \cong S'$ and the valued poset of $\mathbf{R}(\tilde{\mathbf{K}}^1)$ has the form (**). Then by Lemma 2.1, Proposition 2.2 and [8, Proposition 2.5] we get

$$\# \text{ l her } (T) = 10 + \# \text{ l her } (S') - 1 = 24 .$$

Now we will prove (2) for the ring T' by applying Proposition 2.2 to $R = T'$ and $a = 1$. Since $T'_1 = R$ and $M^1 = \begin{pmatrix} F & F & F \\ & 0 & \end{pmatrix}$ then by the statement (3) proved below the indecomposable modules X in $\text{l her } (R)$ for which $\text{Hom}_R(M^a, X) \neq 0$ form a chain $M^a \rightarrow D_3 \rightarrow D_1 \rightarrow D_2$. It follows that the vector space category $\tilde{\mathbf{K}}_p^1$ is special Schurian and the ring $\mathbf{R}(\tilde{\mathbf{K}}^1)''$ (see Lemma 2.1) is isomorphic to the ring $\mathbf{R}(\tilde{\mathbf{K}}^2)$ below the diagram (*). Hence $\# \text{ l her } (\mathbf{R}(\tilde{\mathbf{K}}^1)'') = 6$ and by Proposition 2.2 we have $\# \text{ l her } (T') = 10 + 6 - 1 = 15$.

Finally we prove (2) for the ring E' by applying Proposition 2.2 to $R = E'$ and $a = 1$. First we note that E'_1 is obtained from E by omitting the upper row and E'_1 -modules are those E -modules X for which $X_1 = 0$. Since $M^1 = \begin{pmatrix} 0 & 0 & F & F \\ & F & & \end{pmatrix}$ then by (3) and (4) proved below the indecomposable modules X in $\text{l her } (E'_1)$ for which $\text{Hom}(M^1, X) \neq 0$ form the following diagram

$$\begin{array}{ccccccc} M^1 \rightarrow & \begin{pmatrix} 0(1, f) & F & F & \oplus & F & F & \oplus & F \\ & G & \oplus & G & & & & \end{pmatrix} & \rightarrow & \begin{pmatrix} 0 & F & F & F \\ & G & & \end{pmatrix} & \rightarrow & \begin{pmatrix} 0 & F & F & F \\ & F & & \end{pmatrix} \\ & & & \downarrow & & \downarrow & & \\ & & & \begin{pmatrix} G & F & F & F \\ & G & & \end{pmatrix} & \rightarrow & \begin{pmatrix} G & F & F & F \\ & F & & \end{pmatrix} & \rightarrow & \begin{pmatrix} F & F & F & F \\ & F & & \end{pmatrix} . \end{array}$$

It follows that $\tilde{\mathbf{K}}_p^1$ is special Schurian and the ring $\mathbf{R}(\tilde{\mathbf{K}}^1)$ satisfies the conditions in Lemma 2.1 (ii) with $\mathbf{R}(\tilde{\mathbf{K}}^1)'' = S'$. Consequently, by Lemma 2.1 and Proposition 2.2 we get

$$\# \text{ l her } (E') = \# \text{ l her } (E) + \# \text{ l her } (S') - 1 = 28$$

and the statement (2) is proved.

(3) Let $X = \begin{pmatrix} X_2 & X_4 & X_6 \\ & X_3 & \end{pmatrix}$ be an indecomposable R -module. It

with $d_{ac} = d_{ca} = 1$ and if we apply Lemma 2.1 (ii) to $\mathbf{R}(\mathbf{K}^1)$ twice we get the ring R in Lemma 1.7. Hence $\# \text{her}(\mathbf{R}(\mathbf{K}^1)) = 12$ and the required equality follows from Proposition 2.2 and Lemma 2.7.

LEMMA 2.9. *Let R be an l -hereditary artinian PI-ring. If the valued poset of R has the form*

$$1 \rightarrow 2 \xrightarrow{(2,1)} 3 \rightarrow 4 \xrightarrow{(1,2)} 5 \rightarrow 6, \quad d_{25} = d_{52} = 1,$$

then $\# \text{mod}(R) = 49$. If (X_i) is an indecomposable R -module then either $X_2 = 0$ or $X_5 = 0$, or else (X_i) is of one of the forms: $(F F F F F F)$, $(F F F F F 0)$, $(0 F F F F F)$, $(0 F F F F 0)$.

PROOF. By Remark 2 we can suppose that there is a pair of division rings $G \subset F$ such that $\dim_G F = \dim F_G = 2$, $F_3 = F_4 = G$, $F_1 = F_2 = F_5 = F_6 = F$, ${}_1M_2 = {}_2M_5 = {}_5M_6 = {}_F F_F$, ${}_2M_3 = {}_F F_G$, ${}_3M_4 = {}_G G_G$ and ${}_4M_5 = {}_4M_6 = {}_G F_F$.

The ring $S = R_1$ is l -hereditary and its valued poset is of the form

$$2 \xrightarrow{(2,1)} 3 \rightarrow 4 \xrightarrow{(1,2)} 5 \rightarrow 6, \quad d_{25} = d_{52} = 1,$$

and $M^1 = (F F F F F)$. We will prove that $\# \text{mod}(S) = 32$ by applying Proposition 2.2 to $R = S$ and $a = 2$. The ring S_2 is hereditary, its valued graph has the form $3 \rightarrow 4 \xrightarrow{(1,2)} 5 \rightarrow 6$ and $M^2 = (F F F F) = E(\tilde{F}_6)$. Hence the vector space category \mathbf{K}_r^2 is special Schurian and has 4 indecomposable objects corresponding to the indecomposable injective S_2 -modules. Then the valued poset of the ring $\mathbf{R}(\mathbf{K}^2)$ has the form

$$3 \rightarrow 4 \xrightarrow{(1,2)} 5 \rightarrow 6 \rightarrow 7$$

and by Lemma 2.1 and [8, Proposition 2.5] $\# \text{her}(\mathbf{R}(\mathbf{K}^2)) = 8$. Then Proposition 2.2 and [10, Theorem 1.1] yield $\# \text{mod}(S) = 32$, as we claimed.

Since the following S -modules are indecomposable

$$\begin{aligned} W_1 &= (F 0 0 0 0), & W_4 &= (F F 0 0 0), & W_7 &= (F F F F 0), \\ W_2 &= (F G 0 0 0), & W_5 &= (F F G 0 0), & W_8 &= (F F F F F), \\ W_3 &= (F G G 0 0), & W_6 &= (F F F 0 0), \end{aligned}$$

and $\# \text{mod}(S_2) = 24$ then any indecomposable S -module X for which $X_2 \neq 0$ is of the form $W_i, i \leq 8$. Since $\dim \text{Hom}_S({}_F M^1_S, W_j)_F = 1$ for all j and $\text{Hom}_S(M^1, X) \neq 0$ if and only if $X_1 \neq 0$, then \mathbf{K}^1_F is special Schurian and has 8 indecomposable objects corresponding to the modules $W_j, j \leq 8$. It is easy to check that

$$\mathbf{R}(\mathbf{K}^1) \cong \begin{bmatrix} F & F & F & F & F & F & F & F & F \\ & G & G & F & F & F & F & F & F \\ & & G & 0 & G & F & F & F & F \\ & & & F & F & F & F & F & F \\ & & & & F & F & F & F & F \\ & & & & & F & F & F & F \\ & & & & & & F & F & F \\ & & & & & & & F & F \\ & & & & & & & & F \\ & & & & & & & & & 0 \\ & & & & & & & & & & F \end{bmatrix}.$$

Then, applying Lemma 2.1 (ii) and Lemma 2.7, we have

$$\# l \text{ her}(\mathbf{R}(\mathbf{K}^1)) = 17.$$

Now, it follows from Proposition 2.2 that $\# \text{mod}(R) = 32 + 17 = 49$.

In order to prove the second part of the lemma one can apply the same type of arguments as in the final part of the proof of Lemma 2.6.

LEMMA 2.10. *Let $R_{nm}, n \geq 0, m \geq 1$, be an l -hereditary artinian PI -ring such that the valued poset of R_{nm} has the form*

$$-n \rightarrow -(n-1) \rightarrow \dots \rightarrow 0 \xrightarrow{(2,1)} 1 \xrightarrow{(1,2)} 2 \rightarrow \dots \rightarrow m, \quad d_{0_2} = d_{2_0} = 1,$$

and suppose $F_j = F$ for $j \neq 1, F_1 = G \subset F$. Then

(i) $\# \text{mod}(R_{nm}) = m^2 + (n+1)(m+n+2)$.

(ii) *Every indecomposable R_{nm} -module $X = (X_i)_{-n \leq i \leq m}$ with $X_{-n} \neq 0$ is of one of the forms: $H = E(\tilde{F}_1) = (F \dots F G 0 \dots 0)$,*

$$H_j^n = (\underbrace{F \dots F}_j 0 \dots 0), \quad -n \leq j \leq m, \quad Q_i^n = (\underbrace{F \dots F}_{i \text{ terms}} F^2 \dots F^2 G^2 0 \dots 0),$$

$i = 1, \dots, n,$

where the maps $F \rightarrow F^2 = F \oplus F$, $F^2 \rightarrow G^2$ are defined by $1 \mapsto (1, f)$ and $(x + fy, x' + fy') \mapsto (x, y')$, respectively, and $\{1, f\}$ is a basis of F_a .

PROOF. In view of Remark 2 we can suppose that ${}_iM_j = F$ for all i and j . We prove the lemma by induction on n . The case $n = -1$ follows from [10]. Suppose $n \geq 0$ and apply Proposition 2.2 to $R = R_{nm}$ and $a = -n$. Note that $R_a = R_{n-1m}$ and $M^a = H_{m+n}^{n-1}$. By the inductive assumption the indecomposable R_a -modules Y for which $\text{Hom}(M^a, Y) \neq 0$ form a chain

$$H_1^{n-1} \leftarrow \dots \leftarrow H_n^{n-1} \leftarrow H \leftarrow Q_1^{n-1} \leftarrow \dots \leftarrow Q_{n-1}^{n-1} \leftarrow H_{n+1}^{n-1} \leftarrow \dots \leftarrow H_{n+m}^{n-1}.$$

It is obvious that $\text{End}(H_j^{n-1}) \cong F \cong \text{End}(Q_i^{n-1})$ for all i, j and $\text{End}(H) \cong \cong G$. Now it is easy to check that the vector space category \mathbf{K}_R^a is special Schurian and the valued poset of the ring $\mathbf{R}(\mathbf{K}^a)$ has the form

$$1 \rightarrow \dots \rightarrow n \xrightarrow{(2,1)} 0 \xrightarrow{(1,2)} 1' \rightarrow \dots \rightarrow n' - 1 \rightarrow n + 1 \rightarrow \dots \rightarrow n + m + 1,$$

with $d_{n_0} = d_{0n} = 1$. Thus, by Lemma 2.1, $\# \text{her}(\mathbf{R}(\mathbf{K}^a)) = m + 2n + 2$ and the equality (i) is a consequence of Proposition 2.2 and the inductive assumption. Since (ii) is an immediate consequence of (i) and the inductive assumption the proof of the lemma is complete.

LEMMA 2.11. *Let R be an l -hereditary artinian PI-ring. Then*

(i) *If the valued poset of R has the form ${}^n\mathbf{B}\mathbf{B}^m$ then*

$$\begin{aligned} \# \text{mod}(R) &= (m + s)^2 + (n + 1)(m + n + s + 2) + \\ &\quad + (s + 1)(m + 2n + \frac{1}{2}(s + 6)) + 1. \end{aligned}$$

If X is an indecomposable R -module nonisomorphic to $P(\tilde{F}_{m+s+1})$ then either $X_{m+s+1} = 0$ or $X_m = 0$.

(ii) *If the valued poset of R has the form ${}^m\mathbf{B}_s\mathbf{B}^m$ then R is of finite representation type.*

If X is an indecomposable R -module then either $X_{-n-1} = 0$ or $X_m = 0$,

or else X has the form

$$\begin{array}{l}
 (FF \dots F \dots FF) \\
 \left. \begin{array}{l} F^2 \\ \vdots \\ F^2 \end{array} \right\} i \text{ terms} \\
 \left. \begin{array}{l} F \\ \vdots \\ F \end{array} \right\} j \text{ terms} \quad i, j = 1, 2, \dots, s, i + j = s. \\
 0 \\
 \vdots \\
 0
 \end{array}$$

PROOF. (i) In view of Remark 2 we can suppose that $F_j = F$ for $j \neq 1$, $F_1 = G \subset F$ and ${}_i M_j = F$ for $i \neq m + s + 1$. We will prove (i) by applying Proposition 2.2 to the ring R and $a = m + s + 1$. The valued poset of the ring R_a has the form

$$\begin{array}{l}
 -n \rightarrow -(n-1) \rightarrow \dots \rightarrow 0 \xrightarrow{(2,1)} 1 \xrightarrow{(1,2)} 2 \rightarrow \dots \rightarrow m \\
 (*) \qquad \downarrow \\
 \qquad m + s \rightarrow \dots \rightarrow m + 2 \rightarrow m + 1, d_{02} = d_{20} = 1,
 \end{array}$$

and $M^a = \begin{pmatrix} F & F & \dots & F & F \\ F & \dots & F & F & \end{pmatrix}$. Let $X = \begin{pmatrix} X_{-n} & X_{-n+1} & \dots & X_m \\ X_{m+s} & \dots & X_{m+1} & \end{pmatrix}$ be an indecomposable R_a -module.

First we will prove that

$$\# \text{ mod } (R_a) = (m + s)^2 + (n + 1)(n + m + s + 2).$$

For this purpose we construct a sequence B_1, \dots, B_t of l -hereditary artinian PI -rings and a partial Coxeter functors

$$\text{mod } (R_a) \rightleftarrows \text{mod } (B_1) \rightleftarrows \dots \rightleftarrows \text{mod } (B_t)$$

in the sense of [2, § 4] (see also [28]) such that the following conditions are satisfied:

(a) $\# \text{ mod } (R_a) = \# \text{ mod } (B_1) = \dots = \# \text{ mod } (B_t)$;

(b) the valued poset of the ring B_t is obtained from the one of R_a by inverting the arrows between points $m, m + 1, \dots, m + s$;

(*c*) if X' denotes the image of X in $\text{mod}(B_i)$ under the composition of partial Coxeter functors then $X'_j = X_j$ for $j = -n, \dots, m$.

The partial Coxeter functors we are looking for can be defined as functors S_k^-, S_k^+ [28, § 1] where k is either a sink or a source in the graph $m + s - \dots - k - \dots - m + 1$ (contained in a corresponding poset (I_{B_i}, \mathbf{d})) with respect to an appropriate orientation. Then (*c*) immediately follows from the definition of the functors S_k^- and S_k^+ , whereas (*a*) follows from the properties $\mathbf{c}_1^+ - \mathbf{c}_4^+$ and $\mathbf{c}_1^- - \mathbf{c}_4^-$ in [28, § 1]. Now, the required equality follows from (*a*) and Lemma 2.10.

Suppose that X is not isomorphic to M^a and there is a nonzero map $f: M^a \rightarrow X$. Since $M^a = E(\tilde{F}_m)$ then $\text{Im } f$ contains an epimorphic image of $M^a/\tilde{F}_m = E(\tilde{F}_{m-1}) \oplus E(\tilde{F}_{m+1})$. It is easy to see that indecomposable epimorphic images of the module $E(\tilde{F}_{m+1})$ are of the forms

$$T_j = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ & & \underbrace{F \dots F}_{j \text{ terms}} & & 0 & \dots & 0 \end{pmatrix}, \quad j = 1, \dots, s,$$

whereas the indecomposable epimorphic images of $E(\tilde{F}_{m-1})$ are the following

$$L = E(\tilde{F}_1) = \begin{pmatrix} F & \dots & F & \dots & F & G & 0 & \dots & 0 \\ & & & & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$L_j = \begin{pmatrix} \overbrace{F \dots F}^{j \text{ terms}} & \dots & F & 0 & \dots & 0 \\ & & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}, \quad j = 1, \dots, n + m.$$

The modules L_j are injective for $j \neq n + 2$.

Now suppose $L_{n+2} \subset X$. Then, by (*c*), $L'_{n+2} \subset X'$ and in view of Lemma 2.10 either $X' = Q_j^n$ for some j or $X' = H_{n+2}^n$. Hence, applying again (*c*), we conclude that either $X = L_{n+2}$ or is of one of the forms

$$Q'_j = \begin{pmatrix} \overbrace{F \dots F}^{j \text{ terms}} & \dots & F & F^2 & \dots & F^2 & G^2 & 0 & \dots & 0 \\ & & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}, \quad j = 1, \dots, n.$$

It follows that indecomposable R_a -modules X for which $\text{Hom}(M^a,$

composable R -modules X for which $X_{m+s+1} = 0$ and exactly $(m-1)^2 + (n+s+2)(n+m+s+4)$ of those for which $X_m = 0$. Since we have calculated twice $(m-1)^2 + (n+1)(m+n+1) + \frac{1}{2}s(s+1)$ modules then the number of modules X for which either $X_{m+s+1} = 0$ or $X_m = 0$ is equal $\# \bmod (R) - 1$, as required.

(ii) Suppose the ordering of points $m+1, \dots, m+s$ in the poset (I_R, \mathbf{d}) is the following $m+s \rightarrow \dots \rightarrow m+2 \rightarrow m+1$. The general case can be reduced to this particular one by applying the partial Coxeter functors arguments used in the proof of part (i). Furthermore, in view of Remark 2 we can suppose that $F_j = F$ for $j \neq 1$, $F_1 = G \subset F$ and ${}_i M_j = F$ for $i \neq -(n+1)$.

We will apply Proposition 2.2 to the ring R and $a = -(n+1)$.

The valued poset of R has the form $(*)$ and $M^a = \begin{pmatrix} F & \dots & F & \dots & F & F \\ & & 0 & \dots & 0 & F \end{pmatrix}$. Using the same type of arguments as in the proof of the statement (i) we show that the indecomposable R_a -modules X for which $\text{Hom}(M^a, X) \neq 0$ form a diagram

$$L_1 \leftarrow \dots \leftarrow L_{n+1} \leftarrow L \leftarrow Q'_1 \leftarrow \dots \leftarrow Q'_n \leftarrow L_{n+2} \leftarrow \dots \leftarrow L_{n+m}$$

$$\begin{array}{c} \uparrow T'_s \leftarrow \dots \leftarrow T'_1 \\ \nearrow \nearrow \nearrow \nearrow \\ T'_s \leftarrow \dots \leftarrow T'_1 \end{array}$$

where

$$T'_j = \begin{pmatrix} F & \dots & F & \dots & F & F & \dots & F \\ & & 0 & \dots & 0 & \underbrace{F \dots F}_{j \text{ terms}} \end{pmatrix}, \quad T''_j = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ & & 0 & \dots & 0 & \underbrace{0F \dots F}_{j \text{ terms}} \end{pmatrix} \quad j = 1, \dots, s,$$

It is easy to check that the vector space category \mathbf{K}_R^a is special Schurian and the valued poset of $\mathbf{R}(\mathbf{K}^a)$ has the form

$$1 \rightarrow \dots \rightarrow n+1 \xrightarrow{(2,1)} 0 \xrightarrow{(1,2)} 1' \rightarrow \dots \rightarrow n' \rightarrow n+2 \rightarrow \dots \rightarrow n+m$$

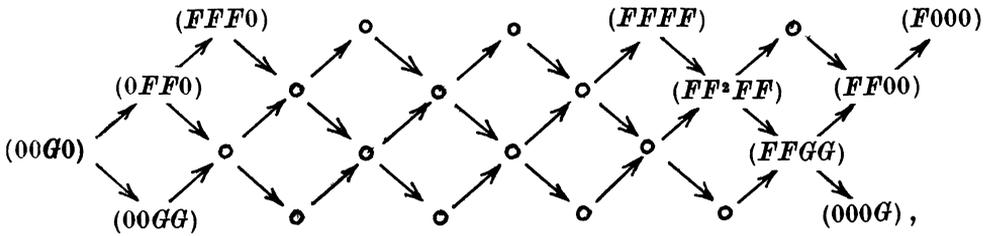
$$\begin{array}{c} \downarrow \swarrow \searrow \swarrow \searrow \\ \hat{s} \rightarrow \dots \rightarrow \hat{1} \rightarrow \hat{0} \end{array}$$

with $d_{n+1,1'} = d_{1',n+1} = 1$. Now, applying Lemma 2.1, Proposition 2.2 and the arguments used in the final part of the proof of (i) we get the first part of (ii). The proof of the second statement in (ii) is similar to that in (i) and we leave it to the reader. This completes the proof.

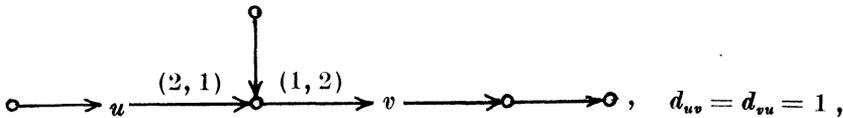
LEMMA 2.12. *If R is an l -hereditary artinian PI-ring and (I_R, \mathbf{d}) has the form ${}_3\mathbf{BF}_4$, then $\# \text{mod}(R) = 41$ and every indecomposable R -module X for which $X_0 \neq 0$ and $X_3 \neq 0$ is of one of the forms:*

$$\begin{pmatrix} F & F & F \\ F & F & F \end{pmatrix}, \quad \begin{pmatrix} F & F & F \\ F & F & G \end{pmatrix}, \quad \begin{pmatrix} F & F & G \\ F & F & G \end{pmatrix}, \quad \begin{pmatrix} F & F & F \\ F & F^2 & G^2 \end{pmatrix}, \quad \begin{pmatrix} F^2 & F & F \\ F^2 & F^2 & G^2 \end{pmatrix}.$$

PROOF. Without loss of generality we can suppose $F_0 = F_1 = {}_rF_2 = F, F_3 = F_4 = G \subset F$ and $\dim_G F = \dim F_G = 2$. We will apply Proposition 2.2 to R and $a = 0$. Then the ring R_a is hereditary, its valued graph has the form $1 \rightarrow 2 \xrightarrow{(2,1)} 3 \leftarrow 4$ and $M^a = (F F F F)$. We know from [10] that $\# \text{mod}(R) = 24$ and the indecomposable R -modules correspond to the positive roots of the Dynkin diagram \mathbf{F}_4 . It follows from the properties of the partial Coxeter functors [28] that the indecomposable R -modules form the following diagram



where arrows mean the existence of irreducible maps [22]. Then the category \mathbf{K}_r^a has exactly 5 indecomposable objects determined by the R_a -modules in the right side of the diagram. Hence we easily conclude that \mathbf{K}_r^a is special Schurian, the valued poset of the ring $\mathbf{R}(\mathbf{K}^a)$ has the form

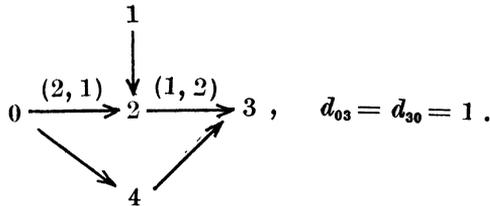


and we have $\# l \text{ her}(\mathbf{R}(\mathbf{K}^a)) = 2 + \# l \text{ her}(T') = 17$ according to Lemmas 2.1 and 2.7. Then the required equality follows from Proposition 2.2. The proof of the second statement is left to the reader.

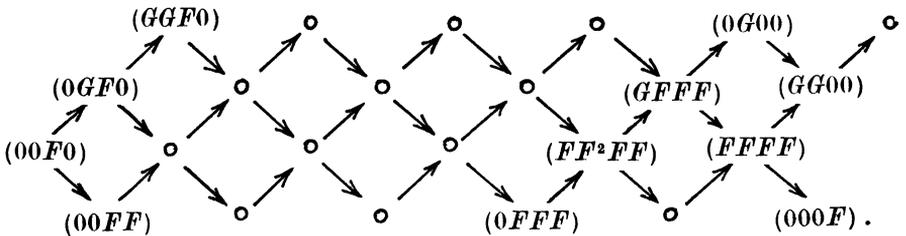
LEMMA 2.13. *If R is an l -hereditary artinian PI-ring and (I_R, \mathbf{d}) has the form $\mathbf{4FF}_4$ then $\# \text{mod}(R) = 54$ and every indecomposable R -module X for which $X_a \neq 0$ and $X_e \neq 0$ is of one of the forms:*

$$\begin{aligned} & \begin{pmatrix} G \oplus G \\ F & F \oplus F & F \oplus F \\ & & F \end{pmatrix}, \quad \begin{pmatrix} & & G \oplus G \\ F \oplus F & F & F \oplus F \\ & & F \end{pmatrix}, \quad \begin{pmatrix} & & G \oplus G \\ F \oplus F & F & F \oplus F \\ & & F \oplus F \end{pmatrix}, \\ & \begin{pmatrix} & & G \oplus G \\ F & F \oplus F & F \oplus F \\ & & F \oplus F \end{pmatrix}, \quad \begin{pmatrix} & & G \oplus G \\ F & F \oplus F & F \\ & & F \end{pmatrix}, \quad \begin{pmatrix} & & G \\ F & G \oplus G \oplus G & F \\ & & F \end{pmatrix}, \\ & \begin{pmatrix} & & G \oplus G \\ F & G \oplus G \oplus G & F \\ & & F \end{pmatrix}, \quad \begin{pmatrix} & & G \\ F & F & F \\ & & F \end{pmatrix}, \quad \begin{pmatrix} & & F \\ F & F & F \\ & & F \end{pmatrix}. \end{aligned}$$

PROOF. We suppose that (I_R, \mathbf{d}) has the form



The general case can be reduced to this particular one by applying the partial Coxeter functors arguments we use in the proof of Lemma 2.11. By Remark 2 we can suppose $F_1 = F_2 = G \subset F = F_0 = F_3 = F_4$. In the notation of Proposition 2.2 the ring R_0 is hereditary, the valued graph of R_0 has the form $1 \rightarrow 2 \xrightarrow{(1,2)} 3 \leftarrow 4$ and $M_0 = (0 \ F \ F \ F)$. The indecomposable R_0 -modules form the following diagram

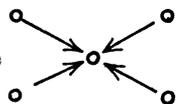


PROOF OF THEOREM 2.5. (4) \rightarrow (1) If the valued poset (I_R, \mathbf{d}) or its opposite is of one of the forms $a)$ or $b)$ then R is of finite representation type according to [10, 16]. If (I_R, \mathbf{d}) or its opposite is of one of the forms $c)$ then R is also of finite representation type because Remark 2 and the partial Coxeter functors arguments allow us to reduce to Lemmas 2.6-2.14.

The implication (1) \rightarrow (2) follows from [24, Theorem 3.1]. Since (3) \rightarrow (4) is proved in [4] it remains to prove (2) \rightarrow (3). We will do it by showing that any l -hereditary artinian PI -ring such that either (I_R, \mathbf{d}) or its opposite is of one of the forms $\mathbf{R}_1 - \mathbf{R}_9$ is not right pure semisimple. We recall from [28] that this is also the case if (I_R, \mathbf{d}) is of the form $a)$ or $b)$ in (3).

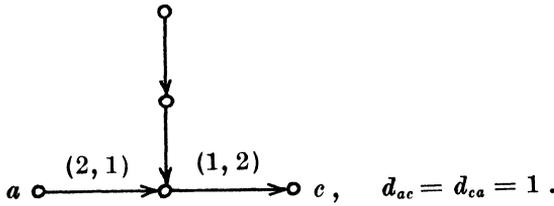
First we note that without loss of generality we can restrict our considerations to a fixed orientation of nondirected edges in each of the valued posets $\mathbf{R}_1 - \mathbf{R}_9$. This is an easy consequence of the partial Coxeter functors arguments in the proof of Lemma 2.11 and Proposition 2.4.

Now suppose that (I_R, \mathbf{d}) is of one of the forms $\mathbf{R}_1 - \mathbf{R}_7$. Then there exists a source \mathbf{a} in I_R such that the ring $R_{\mathbf{a}}$ is hereditary of finite representation type and the valued graph of $R_{\mathbf{a}}$ is homogeneous (we use the notation of Proposition 2.2). It follows that the vector space category $\mathbf{K}_F^{\mathbf{a}}$ is special Schurian and the valued poset of the ring $\mathbf{R}(\mathbf{K}^{\mathbf{a}})$ is homogeneous. Hence the category $l \text{ her } (\mathbf{R}(\mathbf{K}^{\mathbf{a}}))$ is equivalent to the category of representations of the poset $I_{\mathbf{R}(\mathbf{K}^{\mathbf{a}})}$ in the sense of Nazarova and Rojter [19] (see [5]). Since R is of infinite representation type [16], then by Proposition 2.2 the category $l \text{ her } (\mathbf{R}(\mathbf{K}^{\mathbf{a}}))$ is of infinite representation type and therefore the poset $I_{\mathbf{R}(\mathbf{K}^{\mathbf{a}})}$ is of infinite type. Then, in view of [12, 19] (see also [7]), by applying the Nazarova-Rojter differentiation procedure to the poset $I_{\mathbf{R}(\mathbf{K}^{\mathbf{a}})}$ in a finite number of steps we get a poset of weight ≥ 4 . This means that there exists a full additive subcategory \mathcal{A} of $l \text{ her } (\mathbf{R}(\mathbf{K}^{\mathbf{a}}))$ and a representation equivalence $\mathcal{A} \rightarrow l \text{ her } (S)$ where S is a hereditary ring of

the type . Since $l \text{ her } (S)$ is obviously cofinite in $\text{mod } (S)$

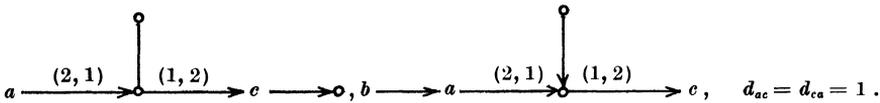
and by [28] S is not right pure semisimple then we conclude from Propositions 2.2 and 2.4 that R is not right pure semisimple, as required.

Next suppose (I_R, \mathbf{d}) has the form



In the notation of Proposition 2.2 the ring R_a is hereditary, the valued graph of R_a has the form $\circ \rightarrow \circ \rightarrow \circ \xrightarrow{(1,2)} \circ$ and $M^a = \begin{pmatrix} 0 & 0 & F & F \end{pmatrix}$. We note that there are nonzero maps from M^a into the R_a -modules $U = \begin{pmatrix} 0 & G & G & 0 \end{pmatrix}$, $W = \begin{pmatrix} G & G & F \end{pmatrix}$, whereas there is no nonzero map between U and W . Hence the full subcategory of $\mathfrak{U}(\mathbf{K}_F^a)$ consisting of objects (X, V, t) where $|X|_F$ is of the form $|U|^i \oplus |W|^j$ is cofinite in $\text{mod}(S)$, where S a hereditary artinian PI -ring with the valued graph $\circ \xrightarrow{(1,2)} \circ \xrightarrow{(2,1)} \circ$. We know from [28, Corollary 3.4] and [10] that S is not right pure semisimple. Then Propositions 2.2 and 2.4 imply that R is not right pure semisimple.

Finally, we suppose that (I_R, \mathbf{d}) is of one of the forms



Then $M^a = \begin{pmatrix} 0 & & & \\ F & F & 0 & \end{pmatrix}$, $M^b = \begin{pmatrix} 0 & & & \\ F & F & F & \end{pmatrix}$ and taking for U and W the R_a -modules $\begin{pmatrix} 0 & & & \\ G & 0 & 0 & \end{pmatrix}$, $\begin{pmatrix} G & & & \\ F & F & 0 & \end{pmatrix}$ and R_b -modules $\begin{pmatrix} 0 & & & \\ F & G & 0 & \end{pmatrix}$, $\begin{pmatrix} G & & & \\ F & F & 0 & \end{pmatrix}$ respectively we are in the same position as above. Hence we conclude that R is not right pure semisimple. The proof of the theorem is complete.

REMARK 3. The Lemmas 2.6-2.14 give an explicit description of indecomposable modules over non-homogeneous l -hereditary artinian PI -rings of finite representation type. In particular it is easy to see that for every such a module $X = (X_i)$ the dimension of the F_i -space X_i is at most 6 for all i . Moreover X is uniquely determined by its composition factors.

REMARK 4. The functor $\hat{\Phi}$ in Section 1 gives a simple method for reconstructing indecomposable R -modules from indecomposable modules in $l\text{ her }(\mathbf{R}(\mathbf{K}^a))$. In order to describe it suppose that (in the notation of Proposition 2.2) the category \mathbf{K}_F^a is special Schurian and we know the indecomposable R_a -modules as well as the indecomposable modules in $l\text{ her }(\mathbf{R}(\mathbf{K}^a))$. This was the case in all situations we consider in Lemmas 2.6-2.14.

Let us identify the R -modules and triples (V_F, Y, t) where Y is an R_a -module and $t: V_F \rightarrow \text{Hom}_F(M^a, Y)$ is an F -linear map.

If Y_1, \dots, Y_n are all indecomposable R_a -modules such that $\text{Hom}(M^a, Y) \neq 0$ we denote by \mathfrak{M}_R^a the full subcategory of $\text{mod}(R)$ consisting of modules having no direct summands of the forms $(F, 0, 0)$, (F, Y_j, t) where t is an isomorphism, $(0, Y, 0)$ where Y is indecomposable different from Y_1, \dots, Y_n . Then the functor $\hat{\Phi}$ induces a full additive functor

$$\hat{\Phi}: \mathfrak{M}_R^a \rightarrow l\text{ her }(\mathbf{R}(\mathbf{K}^a))$$

having the following properties:

(i) $\hat{\Phi}$ reflects isomorphisms.

(ii) Every indecomposable module in $l\text{ her }(\mathbf{R}(\mathbf{K}^a))$ except the unique simple projective one is of the form $\hat{\Phi}(Y)$ where Y is indecomposable.

(iii) If (V_F, Y, t) is an indecomposable module in \mathfrak{M}_R^a ,

$$Y = Y_1^{s_1} \oplus \dots \oplus Y_n^{s_n} \quad \text{and} \quad \hat{\Phi}(V_F, Y, t) = (M_i, {}_i\varphi_i)_{1 \leq i, j \leq n+1}$$

then

$$M_{n+1} = \text{Coker } t \quad \text{and} \quad s_i = \dim(M_i / \bar{M}_i)_{F_i}$$

where \bar{M}_i is the F_i -subspace of M_i defined in the proof of [31, Theorem 1.5]. Furthermore, if the modules Y_1, \dots, Y_n and $(M_i, {}_i\varphi_i)$ are uniquely determined by their composition factors then also (V_F, Y, t) is uniquely determined by its composition factors.

The property (iii) gives a method for constructing the indecomposable modules in \mathfrak{M}_R^a from their images under the functor $\hat{\Phi}$.

In a subsequent paper (joint with B. Klemp) we will give a diagrammatic characterization of l -hereditary right QF -2 artinian PI -

rings R for which the category $l\text{her}(R)$ is of finite representation type. We do it by applying the upper and lower differentiations in the sense of [7] and few simple reduction lemmas like the Lemma 2.1. The list of critical valued posets for this class coincides with that one for the finite type structures of division rings [8].

REFERENCES

- [1] M. AUSLANDER, *Representation theory of artin algebras - I*, Comm. Algebra, **1** (1974), pp. 177-268.
- [2] M. AUSLANDER - M. I. PLATZECK - I. REITEN, *Coxeter functors without diagrams*, Trans. Amer. Math. Soc., **250** (1979), pp. 1-46.
- [3] R. BAUTISTA, *On algebras close to hereditary artin algebras*, An. Inst. Mat. Univ. Nac. Autónoma México, **21**, No. 1 (1981), pp. 21-104.
- [4] R. BAUTISTA, *Classification of certain algebras of finite representation type*, Comunicacion Interna No. 51, 1978, Departamento de Matematicas, UNAM, México 20, D.F.
- [5] R. BAUTISTA - R. MARTINEZ, *Representations of partially ordered sets and 1-Gorenstein artin algebras*, Proc. Conf. on Ring Theory (Antwerp 1978), Marcel Dekker, Inc. New York and Basel, 1979.
- [6] R. BAUTISTA - L. COLAVITA - L. SALMERON, *On adjoint functors in representation theory*, Proceedings ICRA III, Puebla, 1980, in Lecture Notes in Math., No. 903, Springer-Verlag, 1981, pp. 9-25.
- [7] R. BAUTISTA - D. SIMSON, *Torsionless modules over l -hereditary 1-Gorenstein artinian rings*, to appear in Comm. Algebra, **12** (1984).
- [8] V. DLAB - C. M. RINGEL, *On algebras of finite representation type*, J. Algebra, **33** (1975), pp. 306-394.
- [9] V. DLAB - C. M. RINGEL, *Indecomposable representations of graphs and algebras*, Mem. Amer. Math. Soc., **173** (1976).
- [10] P. DOWBOR - D. SIMSON, *Quasi-Artin species and rings of finite representation type*, J. Algebra, **63** (1980), pp. 435-443.
- [11] JU. A. DROZD, *Coxeter transformations and representations of partially ordered sets*, Funkc. Anal. i Priložen., **8** (1974), pp. 34-42 (in Russian).
- [12] M. M. KLEINER, *Partially ordered sets of finite type*, Zap. Naučn. Sem. LOMI, **28** (1972), pp. 32-41 (in Russian).
- [13] M. M. KLEINER, *Induced modules and comodules and representations of BOCS' and DGS's*, Proceedings ICRA III, Puebla, 1980.
- [14] Z. LESZCZYŃSKI - D. SIMSON, *On triangular matrix rings of finite representation type*, J. London Math. Soc., **20** (1979), pp. 396-402.

- [15] M. LOUPIAS, *Indecomposable representations of finite partially ordered sets*, Lecture Notes in Mathematics 488 (Springer-Verlag, Berlin, Heidelberg, New York, 1975), pp. 201-209.
- [16] M. LOUPIAS, *Représentations indécomposables des ensembles ordonnés finis*, Thèse, Université François Rabelais de Tours, 1975.
- [17] S. MAC LANE, *Categories for the Working Mathematicians*, Springer-Verlag, New York, Heidelberg, Berlin, 1971.
- [18] L. A. NAZAROVA, *Representations of quivers of infinite type*, Izv. Akad. Nauk SSSR, **37** (1973), pp. 749-792.
- [19] L. A. NAZAROVA - A. V. ROJTER, *Representations of partially ordered sets*, Zap. Naučn. Sem. LOMI, **28** (1972), pp. 4-31 (in Russian).
- [20] L. A. NAZAROVA - A. V. ROJTER, *Kategorielle Matrizen-Probleme und die Brauer-Thrall-Vermutung*, Mittlungen Math. Sem. Giessen, **115** (1975), pp. 1-153.
- [21] C. M. RINGEL, *Report on the Brauer-Thrall conjectures: Rojter's theorem and the theorem of Nazarova and Rojter*, Lecture Notes in Mathematics, **831** (Springer-Verlag, Berlin, Heidelberg, New York, 1980), pp. 104-136.
- [22] C. M. RINGEL, *Tame algebras*, Lecture Notes in Mathematics, **831** (Springer-Verlag, Berlin Heidelberg, New York, 1980), pp. 137-287.
- [23] A. ROSENBERG - D. ZELINSKY, *Finiteness of the injective hull*, Math. Z., **70** (1959), pp. 372-380.
- [24] D. SIMSON, *Functor categories in which every flat object is projective*, Bull., Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys., **22** (1974), pp. 375-380.
- [25] D. SIMSON, *On pure global dimension of locally finitely presented Grothendieck categories*, Fund. Math., **96** (1977), pp. 91-116.
- [26] D. SIMSON, *Categories of representations of species*, J. Pure Appl. Algebra, **14** (1979), pp. 101-114.
- [27] D. SIMSON, *Pure semisimple categories and rings of finite representation type*, Corrigendum, J. Algebra, **67** (1980), pp. 25-256.
- [28] D. SIMSON, *Partial Coxeter functors and right pure semisimple hereditary rings*, J. Algebra, **71** (1981), pp. 195-218.
- [29] D. SIMSON, *On pure semisimple Grothendieck categories, II*, Fund. Math., **110** (1980), pp. 107-116.
- [30] D. SIMSON, *Right pure semisimple structures of division rings*, Proceedings of the Eleventh Annual Iranian Math. Conf., Mashhad, March 1980, pp. 186-201.
- [31] D. SIMSON, *Special Schurian vector space categories and l -hereditary right QF -2 rings*, to appear in Commentationes Math., **25** (1984).

Manoscritto pervenuto in redazione il 7 luglio 1982.

Note Added in Proof.

The main theorem of this paper can be extended to a class of triangular matrix rings R of the form written in the introduction with division PI -rings F_1, \dots, F_n and such that the bilinear maps $c_{ijk}: {}_iM_j \otimes {}_jM_k \rightarrow {}_iM_k$ are nonzero provided ${}_iM_j$ and ${}_jM_k$ are nonzero. To any such a ring R the valued poset (I_R, d) is associated and Theorem 2.5 is extended in a natural way. This is given in the author's notes *On methods for the computation of indecomposable modules over artinian rings*, published in Proc. Conf. Ring Theory and Algebraic Geometry, Chiba (Japan), 1982, pp. 143-170. A characterization of l -hereditary right QF -2 artinian PI -rings R with l -her (R) of finite type we have mentioned at the end of the paper is given by B. KLEMP and the author in the paper *A diagrammatic characterization of schurian vector space PI -categories of finite type* which will appear in Bull. Acad. Polon. Sci. in 1984.