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Bifurcation and total stability

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Bifurcation and Total Stability (*).

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1. Introduction.

In this paper we are concerned with the problem of bifurcation of invariant sets from an invariant set with respect to a family of flows. In particular, we will suppose that such flows are defined by a one-parameter family of ordinary differential equations:

$$\frac{dx}{dt} = f(\mu, x)\,, \tag{1}$$

where $x \in \mathbf{R}^n$, $\mu \in (-\bar{\mu}, \bar{\mu}) \subset \mathbf{R}$, $f \in C[(-\bar{\mu}, \bar{\mu}) \times \mathbf{R}^n, \mathbf{R}^n]$, f is locally Lipschitzian with respect to x, $f(\mu, 0) \equiv 0$. As is well known, bifurcation phenomenon is often associated with a drastic change of suitable stability properties. For example, let us suppose that the origin 0 of $\mathbf{R}^n$ be, with respect to (1), asymptotically stable for $\mu = 0$ and completely unstable (that is asymptotically stable in the past) for $\mu > 0$. Then, in a fixed neighborhood of 0, new compact invariant sets arise for $\mu > 0$ and μ small enough. These sets are disjoint from the origin, asymptotically stable and tend to the origin as μ tends to 0. Also these sets can be taken as the largest compact invariant sets, disjoint from the origin, contained in a fixed neighborhood of

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the origin. The above result is a corollary of a theorem given in [1, 2] where the general phenomenon of bifurcation of invariant sets from an invariant set is considered with respect to a one-parameter family of dynamical systems (not necessarily defined by differential equations).

In the proof of the previous result one uses, among other tools, Malkin's Theorem in order to get informations about the flow for $\mu > 0$. Malkin's Theorem assures us that, if 0 is asymptotically stable for $\mu = 0$, it is also totally stable (that is stable under persistent perturbations). Therefore, for $|\mu|$ small enough, the solutions of (1) which start from points « near » to the origin remain « near » to the origin. However, total stability does not imply, in general, asymptotic stability and interesting examples [3, 4] have been given in which an invariant set (in particular a critical point) is totally stable but not asymptotically stable. Therefore, it is important to know if bifurcation phenomenon still happens when one supposes that the origin is for $\mu = 0$ only totally stable. In Section 2 of this paper we are able to prove such a result by using a theorem given by P. Seibert in [5]. This theorem characterizes the total stability of a compact invariant set with respect to an autonomous system by means of the existence of a fundamental family of asymptotically stable neighborhoods of the set. However, in our hypotheses, the bifurcating sets cannot be taken in general as the largest invariant compact sets, disjoint from the origin, contained in a *fixed* neighborhood of the origin. In fact, the region of attraction of the neighborhoods of the origin, which exist because of Seibert's Theorem, could tend to the origin as these neighborhoods tend to the origin. This happens in the example which we consider in Section 2.

In Section 3 Hopf bifurcation in $\mathbf{R}^2$ is revisited. In [6] the problem of attractivity of bifurcating orbits was considered and it was proved that this property does not hold in general when the origin is asymptotically stable. Here we can add that, if the origin is stable for $\mu = 0$ without being asymptotically stable, the bifurcating orbits can never be all attracting. Therefore, if for $\mu = 0$ the origin is totally stable but not asymptotically stable (this is possible only if for $\mu = 0$ (1) is not analytical), then for every $\mu > 0$, μ small enough, an invariant set exists which is asymptotically stable and which is now an annulus bounded by closed orbits and tends to the origin as μ tends to 0. Nevertheless, for $\mu > 0$ and small enough there also exist bifurcating periodic orbits from the origin which are not attracting and which could be outside of the above annulus. This also happens in the example given in Section 2.

2. Preliminaries and results.

Let us consider system (1) for $\mu = 0$:

$$\dot{x} = f(0, x) =: f_0(x) \,. \tag{2}$$

We will denote by $C^0(x)$ the class of functions $g\colon (t, x) \in \mathbf{R}^+ \times \mathbf{R}^n \to \to g(t, x) \in \mathbf{R}^n$ which are continuous and locally Lipschitzian with respect to x. Let $M \subset \mathbf{R}^n$ be a compact subset and for any $\lambda > 0$ let $S(M, \lambda)$ be the subset $\{x \in \mathbf{R}^n, \varrho(x, M) < \lambda\}$, where ϱ is the usual distance.

2.1 Definition. A compact subset $M \subset \mathbf{R}^n$ is said to be *totally stable with respect to* (2) if for any $\varepsilon > 0$ there exist $\delta_1 = \delta_1(\varepsilon) > 0$, $\delta_2 = \delta_2(\varepsilon) > 0$ such that for any $t_0 \in \mathbf{R}^+$, for any $x_0 \in S(M, \delta_1)$ and for any $g \in C^0(x)$, $\|g(t, x) - f_0(x)\| < \delta_2$ on $\mathbf{R}^+ \times S(M, \varepsilon)$, we have $x_g(t, t_0, x_0) \in S(M, \varepsilon)$ for any $t \geqslant t_0$, where $x_g(t, t_0, x_0)$ denotes the solution of the equation $\dot{x} = g(t, x)$ passing through (t_0, x_0).

In [5] P. Seibert has given the following

2.2 Theorem [5, Th. A]. A compact subset $M \subset \mathbf{R}^n$ is totally stable w. r. to (2) if and only if M possesses a fundamental family of compact neighborhoods which are asymptotically stable w. r. to (2).

2.3 Remark. As observed in [5], it is easy to show that a compact subset $M \subset \mathbf{R}^n$ which is asymptotically stable w. r. to (2) possesses a fundamental family of neighborhoods which are asymptotically stable w. r. to (2). Therefore, by Th. 2.2, it is totally stable w. r. to (2) and Malkin's Theorem is a direct consequence of Th. 2.2.

Now we will state our main result. We will denote by $p_\mu(t, x)$ the dynamical system defined by (1). Further $\gamma_\mu^+(x)$ will denote the positive p_μ-semitrajectory through x and for a subset $A \subset \mathbf{R}^n$ we will set $\gamma_\mu^+(A) = \bigcup_{x \in A} \{\gamma_\mu^+(x)\}$.

2.4 Theorem. Let the origin $\{0\} \subset \mathbf{R}^n$ be totally stable with respect to (2) and completely unstable with respect to (1) for every $\mu \in (0, \bar{\mu})$. Then there exists $\mu^* \in (0, \bar{\mu})$ such that for every $\mu \in (0, \mu^*)$ there exists a compact subset $M'_\mu \subset \mathbf{R}^n$ with the following properties:

(*a*) M'_μ is p_μ-invariant;

(*b*) $M'_\mu \cap \{0\} = \emptyset$ and $\max\{\|x\| : x \in M'_\mu\} \to 0$ as $\mu \to 0$;

(*c*) M'_μ is p_μ-asymptotically stable.

2.5 Remark. Theorem 2.4 generalizes Th. 1.3 given in [1, Sec. III] when in this theorem one assumes $E = \mathbf{R}^n$, $M_\mu = \{0\}$ for every $\mu \in [0, \bar{\mu})$ and p_μ continuous dynamical systems defined by ordinary autonomous differential equations. The generalization consists in the assumption that the origin is totally stable w. r. to (2) instead of asymptotically stable. Our result could be established in the full setting of [1] but this leads to unnecessary complications in the proof.

Proof of Theorem 2.4. The line of the proof is the same as that of Th. 1.3 of [1, Sec. III]. Therefore, when possible, we will omit some details which can be found in [1].

By Th. 2.2, there exists a fundamental family of neighborhoods of $\{0\}$ which are asymptotically stable with respect to (2). Let $\lambda > 0$ and for $\varepsilon \in (0, \lambda]$ let A_ε be one of these neighborhoods such that $A_\varepsilon \subset S(0, \varepsilon)$. Because of asymptotic stability of A_ε there exists a compact neighborhood N_ε of A_ε and a function $V_\varepsilon\colon N_\varepsilon \to \mathbf{R}$, $V_\varepsilon \in C^1$, such that V_ε is positive definite in N_ε with respect to A_ε and

$$\dot{V}_\varepsilon^{(2)}(x) \leqslant -c(\varrho(x, A_\varepsilon)) , \qquad \forall x \in N_\varepsilon , \tag{3}$$

where $\dot{V}_\varepsilon^{(2)}$ is the derivative of V_ε along the solutions of (2) and $c\colon \mathbf{R}^+ \to \mathbf{R}^+$ is a function of class $\mathcal{K}$. Let $L_\varepsilon > 0$ such that $\|\operatorname{grad} V_\varepsilon(x)\| \leqslant L_\varepsilon$ for every $x \in N_\varepsilon$. Moreover, A_ε is totally stable with respect to (2), because of Malkin's Theorem, and therefore, by Def. 2.1, one can easily conclude that there exist three functions of class $\mathcal{K}$, say $h_1, h, \varkappa$, such that

$$\overline{\gamma_\mu^+\big(S(A_\varepsilon, h_1(\varepsilon))\big)} \subset N_\varepsilon , \qquad \forall \mu \in [0, \varkappa(\varepsilon)] \tag{4}$$

$$\overline{\gamma_\mu^+\big(S(A_\varepsilon, h(\varepsilon))\big)} \subset S(A_\varepsilon, h_1(\varepsilon)) , \qquad \forall \mu \in [0, \varkappa(\varepsilon)] \tag{5}$$

where $\varkappa$ is such that $\varkappa(\lambda) < \bar{\mu}$. Finally, because of the continuity of $f(\mu, x)$, there exists another function ψ of class $\mathcal{K}$ such that

$$\|f(\mu, x) - f(0, x)\| \leqslant \frac{c(h(\varepsilon))}{2L_\varepsilon} , \qquad \forall (\mu, x) \in [0, \psi(\varepsilon)] \times N_\varepsilon . \tag{6}$$

Let us set $\chi(\varepsilon) = \min\{\varkappa(\varepsilon), \psi(\varepsilon)\}$. χ is also a function of class $\mathcal{K}$ and (4), (5) and (6) hold true for $\mu \in [0, \chi(\varepsilon)]$. Let $\mu^* = \chi(\lambda)$. For $\mu \in (0, \mu^*)$ let $\varepsilon = \chi^{-1}(\mu)$ and let us consider the set $F_\mu = \overline{\gamma_\mu^+\big(S(A_\varepsilon, h(\varepsilon))\big)}$.

Because $A_\varepsilon \subset S(0, \varepsilon)$ and (5), we have $F_\mu \subset S(A_\varepsilon, h_1(\varepsilon)) \subset S(0, h_1(\varepsilon) + \varepsilon)$ with $\varepsilon = \chi^{-1}(\mu)$. Then as $\mu \to 0$ we have $\max\{\|x\| : x \in F_\mu\} \to 0$. Let $\tilde{F}_\mu$ denote the maximum invariant set contained in F_μ. $\tilde{F}_\mu$ is a non empty compact set and it is easy to show that the region of p_μ-negative attraction of the origin, say A_μ^-, is contained in $\tilde{F}_\mu$ (for details see [1]). Now we are able to define the family of sets $\{M'_\mu\}$ which satisfy conditions (a), (b), (c). Let us set $M'_\mu = \tilde{F}_\mu \setminus A_\mu^-$ for every $\mu \in (0, \mu^*)$. The sets M'_μ satisfy conditions (a), (b). To show that they satisfy also (c) it is enough to prove that they are p_μ-uniform attractors. As the points of $A_\mu^- \setminus \{0\}$ are uniformly p_μ-attracted from M'_μ, we have to prove only that $\tilde{F}_\mu$ is a uniform p_μ-attractor and for that it is enough that F_μ is a p_μ-uniform attractor. Indeed, if for the points of a neighborhood of F_μ we have $J_\mu^+(x) \subset F_\mu$, we have also $J_\mu^+(x) \subset \tilde{F}_\mu$ as $\tilde{F}$ is the maximum invariant set contained in F_μ. Let us prove that for $\varepsilon = \chi^{-1}(\mu)$ the points of $S(A_\varepsilon, h_1(\varepsilon)) \setminus F_\mu$ are p_μ-uniformly attracted from F_μ (by (5) $S(A_\varepsilon, h_1(\varepsilon))$ is a neighborhood of F_μ). Fix $x \in S(A_\varepsilon, h_1(\varepsilon)) \setminus F_\mu$, $\varepsilon = \chi^{-1}(\mu)$. By (4), $p_\mu(t, x) \in N_\varepsilon$ for every $t \geqslant 0$. It is easy to show that there exists $\tau_x > 0$ such that $p_\mu(\tau_x, x) \in S(A_\varepsilon, h(\varepsilon))$. In fact, if for every $t \geqslant 0$ $\varrho(p_\mu(t, x), A_\varepsilon) \geqslant h(\varepsilon)$ we have, by (3), (6), for $t \geqslant 0$

$$\dot{V}_\varepsilon^{(1)}(p_\mu(t, x)) = \dot{V}_\varepsilon^{(2)}(p_\mu(t, x)) + \operatorname{grad} V_\varepsilon(p_\mu(t, x)) \cdot [f(\mu, p_\mu(t, x)) -$$
$$- f_0(p_\mu(t, x))] \leqslant - c\big(\varrho(p_\mu(t, x), A_\varepsilon)\big) + L_\varepsilon \frac{c(h(\varepsilon))}{2L_\varepsilon} \leqslant - \frac{c(h(\varepsilon))}{2}$$

and

$$V_\varepsilon(p_\mu(t, x)) \leqslant V_\varepsilon(x) - \frac{c(h(\varepsilon))}{2}\, t, \tag{7}$$

which is absurd because V_ε is positive definite with respect to A_ε in N_ε. Also, by (7), τ_x can be chosen independent of x by taking

$$\tau_x = T = \frac{2}{c(h(\varepsilon))} \max_{x \in N_\varepsilon} V_\varepsilon(x) \, .$$

Therefore, F_μ is a uniform attractor and the proof is complete.

2.6 Remark. In the case $n = 2$, if one supposes that for every $\mu \in [0, \bar{\mu})$ the origin is an isolated equilibrium position of (1), then one can prove, as in [1], that the sets M'_μ of Th. 2.4 are annuli bounded by closed orbits of (1) containing the origin in their interior.

2.7 REMARK. If the origin is asymptotically stable with respect to (2) then the sets M'_μ satisfying (*a*), (*b*), (*c*) of Th. 2.4 can be taken as the largest compact invariant sets disjoint from the origin and contained in a fixed neighborhood of the origin [2]. Such a choice is no more possible when the origin is only totally stable. In fact, we can give the following example.

2.8 EXAMPLE. Let us consider the system

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x + \mu y - y f(x, y) \end{cases} \qquad \mu, x, y \in \mathbf{R} \tag{8}$$

with $f(x, y) = (x^2 + y^2)^s \sin^2 \pi/(x^2 + y^2)$, $s \geqslant 3$, for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. By setting $H = \frac{1}{2}(x^2 + y^2)$, along the solutions of (8) we have

$$\dot{H} = y^2 \left[\mu - (x^2 + y^2)^s \sin^2 \frac{\pi}{x^2 + y^2}\right]. \tag{9}$$

Therefore, for $\mu = 0$ (8) has a sequence of closed orbits which are the circles centered in the origin with radii $r_n = 1/\sqrt{n}$, $n = 1, 2, \ldots$, and the disks c_n bounded by these circles are asymptotically stable. Then, by Th. 2.2, the origin of $\mathbf{R}^2$ is totally stable with respect to (8) for $\mu = 0$. On the other hand, for $\mu > 0$ and small enough the origin is completely unstable with respect to (8). Thus, the hypotheses of Th. 2.4 are satisfied and for $\mu > 0$ and μ small enough there exist compact sets M'_μ which are invariant, disjoint from the origin, asymptotically stable with respect to (8) and $\max\{\|(x, y)\| : (x, y) \in M'_\mu\} \to 0$ as $\mu \to 0$. However, such sets are not the largest invariant sets contained in a fixed neighborhood of the origin. In fact, for $\mu > 0$ we have that the circles centered in the origin, with radii r satisfying the equation

$$\mu = r^{2s} \sin^2 \frac{\pi}{r^2}, \tag{10}$$

are closed orbits for system (8). If r'_μ is the minimum value of r satisfying (10) and c'_μ the corresponding closed orbit of (8), by (9) we have that c'_μ is attracting at least for inside orbits. The other circles will be alternately repelling and attracting. The largest compact invariant set disjoint from the origin contained in a fixed neighborhood of the origin is the annulus bounded by c'_μ and by another circle c''_μ which could be attracting or repulsing and does not satisfy (*b*) in Th. 2.4. Therefore, the set M'_μ cannot be identified with such largest invariant set.

3. Hopf bifurcation.

In this section we suppose in (1) $n = 2$, that is $f\colon (-\bar{\mu}, \bar{\mu}) \times \mathbf{R}^2 \to \mathbf{R}^2$, and $f \in C^2$. Further, we will denote by $\alpha(\mu) \pm i\beta(\mu)$ the eigenvalues of $D_x f(\mu, 0)$ and we suppose that

$$\alpha(0) = 0\,, \qquad \alpha'(0) > 0\,, \qquad \beta(0) > 0\,. \tag{11}$$

Also, we will denote by $V\colon (\mu, c) \in (-\mu^*, \mu^*) \times [0, c^*) \to V(\mu, c) \in \mathbf{R}$, $\mu^* > 0$, $c^* > 0$, the displacement function as defined in [6]. The non trivial closed orbits of (1) correspond to the non-null solutions of the equation $V(\mu, c) = 0$. Hopf's theorem assures that there exists a continuous function $\mu\colon [0, \tilde{c}) \to (-\tilde{\mu}, \tilde{\mu})$, $\tilde{c} \in (0, c^*)$, $\tilde{\mu} \in (0, \mu^*)$, $\mu(0) = 0$, such that for $\mu \in (-\tilde{\mu}, \tilde{\mu})$ the orbit of (1) passing through the point $(c, 0) \in \mathbf{R}^2$, $c \in (0, \tilde{c})$, is closed if and only if $\mu = \mu(c)$. As in [6] we will call the function $\mu(c)$ the bifurcation function and the closed orbits of (1) corresponding to the values of c for which $\mu = \mu(c)$ the bifurcating orbits from the origin.

In [6] the problem of attractivity of the bifurcating orbits has been considered and it has been shown (in contradiction to Th. 3B.4 of [7]) that the asymptotic stability of the origin with respect to (2) is not sufficient for such attractivity (see Remark 4.5 in [6]). Here we can prove the following theorems.

3.1 Theorem. Let the origin $0 \in \mathbf{R}^2$ be stable but not asymptotically stable w. r. to (2). Then there exists a sequence $\{\gamma_n\}_{n\in N}$ of closed orbits of (2) around the origin such that $\max\{\|x\|: x \in \gamma_n\} \to 0$ as $n \to \infty$.

Proof. Suppose that 0 is not p_0-asymptotically stable. As 0 is p_0-stable, there exists a fundamental family of compact p_0-positively invariant neighborhoods of the origin. Let $W_\varepsilon \subset S(0, \varepsilon)$, $\varepsilon > 0$, one of such neighborhoods. For $x \in W_\varepsilon$ we have $\Lambda^+(x) \subset W_\varepsilon$. Further, as 0 is not p_0-asymptotically stable, there exists $\bar{x} \in W_\varepsilon$ such that $0 \notin \Lambda^+(\bar{x})$. Because of (11) 0 is an isolated equilibrium position of (2) and we can suppose that in $S(0, \varepsilon)$ there are no other equilibria. Therefore, by Bendixon's theorem, $\Lambda^{+}(\bar{x})$ is a closed orbit of (2) around the origin.

3.2 THEOREM. Let $0 \in \mathbf{R}^2$ be totally stable but not asymptotically stable with respect to (2). Then, there exists $\hat{\mu} \in (0, \tilde{\mu})$ such that for $\mu \in (0, \hat{\mu})$ there are bifurcating orbits which are not attracting.

PROOF. The hypotheses of Th. 2.4 are satisfied and, by Remark 2.6, the sets M'_μ are annuli bounded by closed orbits of (1) containing the origin in their interior. As $\max\{\|x\|: x \in M'_\mu\}$ tends to 0 as μ tends to 0, the orbits which bound M'_μ are, for μ small enough, bifurcating orbits. Therefore, the bifurcation function $\mu(c)$ has to assume positive values for certain values of c arbitrarily small. Let $\hat{c} \in (0, \tilde{c})$ be such that $\mu(\hat{c}) = 0$. Such $\hat{c}$ exists because of Th. 3.1. Let us set $\hat{\mu} = \max\{\mu(c): c \in [0, \hat{c}]\}$. We have $\hat{\mu} > 0$ and for every $\mu \in (0, \hat{\mu})$ there exist at least two bifurcating orbits which cannot be all attracting.

3.3 REMARK. In Example 2.8 the bifurcation function is the function $\mu(c) = c^{2s} \sin^2 \pi/c^2$ and in a fixed neighborhood of the origin we have for $\mu > 0$ and small enough at least two bifurcating orbits which are not all attracting.

3.4 REMARK. Under the hypotheses of Th. 3.1 one could study the bifurcation problem of closed orbits from any of the closed orbits of the family $\{\gamma_n\}_{n\in N}$. For that, the results given in [8, 9] can be used. In example 2.8, the hypotheses of Th. 71 in [8] are satisfied and from any of the closed orbits which exist for $\mu = 0$ two closed orbits bifurcate for $\mu > 0$, whereas for $\mu < 0$ there are not closed bifurcating orbits.

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