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Some Special Properties of Solutions to Obstacle Problems.

CARLA MADERNA - SANDRO SALSA (*)

SUNTO - Si considera il classico problema dell'ostacolo e si provano alcune proprietà speciali della soluzione. In particolare si dà una maggiorazione ottimale della misura dell'insieme di coincidenza.

1. In this note we shall establish some special properties of solutions to the classical obstacle problem. Let G be a bounded domain in R^m , $\psi \in H^1(G)$ such that $\psi \leq 0$ on ∂G and $K = \{v \in H_0^1(G) : v \geq \psi \text{ in } G\}$ (the inequalities are to be intended in the sense of H^1 , see [4]). Consider the following variational inequality:

$$(1.1) \quad (\mathcal{L}u, v - u) = \int_G \sum_{i,j=1}^m a_{ij}(x) D_i u D_j (v - u) dx \geq \int_G g(v - u) dx, \\ v \in K.$$

If $g \in L^p(G)$, $p \geq 2m/(m + 2)$, $a_{ij} \in L^\infty(G)$ and

$$(1.2) \quad \sum_{i,j=1}^m a_{ij}(x) \xi_i \xi_j \geq |\xi|^2 \quad \text{for every } \xi \in R^m \text{ and for a.e. } x \in G,$$

then it is well known, [4], that (1.1) has a unique solution $u \in K$.

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Incidentally, notice that if $a_{ij} = a_{ji}$ ($i, j = 1, 2, \dots, m$), u minimizes in K the functional

$$J(v) = \int_G \left\{ \sum_{i,j=1}^m a_{ij}(x) D_i v D_j v - 2vg \right\} dx .$$

We shall denote by $I = I(u)$ the coincidence set, i.e. $I \equiv \{x \in G: u(x) = \psi(x)\}$. The regularity of the solution, the topology of I and the regularity of the boundary ∂I have been extensively studied by various authors (see, for a bibliography, [4], [1]).

Our aim, in this note, is to give some information on I from a different point of view, by using a symmetrization technique. As a typical result, for instance, we prove a sharp estimate for the measure of the coincidence set involving only the measure of G and the data g and ψ (see section 2).

Other information about various characteristic parameter involved in the obstacle problem are given in section 3.

2. In this section we need some assumptions on the coefficients a_{ij} , on g and the obstacle ψ , for ∂I to be a regular hypersurface and for u to be in $C^{1,\alpha}(\bar{G}) \cap C^2(\bar{G} \setminus \bar{I})$. For instance (see the quoted papers in the introduction) we may suppose

$$(2.1) \quad a_{ij} \in C^1(\bar{G}), \quad g \in C(\bar{G}), \quad \psi \in C^2(\bar{G}), \quad \psi = -1 \text{ on } \partial G .$$

The last assumption on ψ is actually a normalization condition; we could work equally well assuming $\psi < 0$ on ∂G .

Set now

$$(2.2) \quad f = g - \mathfrak{L}\psi = g + \sum_{i,j=1}^m D_j(a_{ij}(x) D_i \psi), \quad f_- = \max(-f, 0)$$

and for $0 < \lambda \leq |G|$ ⁽¹⁾,

$$(2.3) \quad \Phi(\lambda) = \int_{\lambda}^{|G|} m^{-2} C_m^{-2/m} r^{-2+2/m} dr \int_0^{r-\lambda} f_-^*(s) ds .$$

⁽¹⁾ If E is a measurable subset of R^m we write $|E|$ for the Lebesgue measure of E .

Here $C_m = \pi^{m/2}/(\Gamma(1 + m/2))$ and f_-^* is the decreasing rearrangement of f_- , in the sense of Hardy and Littlewood, that is

$$f_-^*(s) = \inf \{t > 0: |x \in G: f_-(x) > t| < s\}.$$

The following theorem establishes an optimal estimate for the measure of the coincidence set of the solution to (1.1).

THEOREM 1. *Assume that (2.1) and the ellipticity condition (1.2) hold.*

Then either $|I| = 0$ or $|I| \leq \Lambda$, where Λ is the (unique) solution of the equation $\Phi(\lambda) = 1$.

Moreover Λ is the measure of the coincidence set of the solution to the following obstacle problem:

$$(2.4) \quad \begin{cases} \int_{\sigma} (|Dv|^2 - 2Fv) \, dx = \min \\ v \in H = \{v \in H_0^1(G^*): v \leq 1 \text{ in } G^*\}, \end{cases}$$

where $F(x) = f_-^*(C_m|x|^m - \Lambda)$ if $C_m|x|^m - \Lambda > 0$, $F(x) = 0$ otherwise.

Here G^* is the ball centered at the origin such that $|G^*| = |G|$.

PROOF. Let u be the solution to (1.1) and suppose $|I| \neq 0$. The function $w = 1 - u + \psi$ satisfies the following set of conditions:

$$(2.5a) \quad \Delta w = -f \quad \text{in } G \setminus I$$

$$(2.5b) \quad w \leq 1 \text{ in } G, \quad w = 1 \text{ in } I, \quad w = 0 \text{ on } \partial G.$$

Furthermore

$$(2.6) \quad \sum_{i,j=1}^m a_{ij} \nu_i D_j w = 0 \quad \text{on } \partial I$$

where $\nu = (\nu_1, \nu_2, \dots, \nu_m)$ is the normal to ∂I .

Setting $E_t = \{x \in G: w(x) > t\}$, we have, for a.e. t , $0 < t < 1$,

$$\int_{\partial E_t} \sum_{i,j=1}^m a_{ij} D_i w D_j w / |Dw| H_{m-1}(dx) = \int_{E_t \setminus I} \Delta w \, dx = \int_{E_t \setminus I} (-f) \, dx \leq \int_{E_t \setminus I} f_- \, dx$$

where H_{m-1} denotes the $(m - 1)$ -dimensional measure. We have used Green's formula and the conditions (2.5a), (2.6), observing that $Dw/|Dw|$ is the inner unit normal to ∂E_t .

The ellipticity condition (1.2) and a well known theorem of Hardy-Littlewood [3], give now, for a.e. t , $0 < t < 1$,

$$\int_{\partial E_t} |Dw| H_{m-1}(dx) \leq \int_0^{\mu(t)-|I|} f_{-}^{*}(s) ds$$

where $\mu(t)$ is the distribution function of w , that is $\mu(t) = |E_t|$.

Acting as in [5], we get for a.e. t , $0 < t < 1$,

$$t \leq m^{-2} C_m^{-2/m} \int_{\mu(t)}^{|G|} r^{-2+2/m} dr \int_0^{r-|I|} f_{-}^{*}(s') ds' .$$

Letting $t \rightarrow 1$, we obtain

$$(2.7) \quad 1 \leq m^{-2} C_m^{-2/m} \int_{|I|}^{|G|} r^{-2+2/m} dr \int_0^{r-|I|} f_{-}^{*}(s') ds' ,$$

taking (2.5b) into account.

Observe now that $\Phi(\lambda)$, defined by (2.3), is a strictly decreasing function of λ from $\Phi(0^+) \geq 1$ (by (2.7)) to $\Phi(|G|) = 0$, therefore there exists a unique λ such that $\Phi(\lambda) = 1$.

Clearly (2.7) gives $\Phi(\lambda) \leq \Phi(|I|)$, from which the first part of the thesis. The final assertion of the theorem follows observing that the solution of (2.4) is given by the following function:

$$U(x) = \begin{cases} m^{-2} C_m^{-2/m} \int_{C_m|x|^m}^{|G|} r^{-2+2/m} dr \int_0^{r-\lambda} f_{-}^{*}(s) ds & x \in G^* - I^* \\ 1 & x \in I^* \end{cases}$$

where I^* is the ball centered at the origin with $|I| = \lambda$.

REMARK. Note that (2.7) could be used to give sufficient conditions in order to have $|I| = 0$. For instance, we quote the following

one, which follows easily using Holder inequality:

$$\text{if } |G|^{(2p-m)/mp} < C_m^{2/m} m p^{-1} (2p - m) \|f_-\|_{L^p(G)}^{-1} \quad (m/2 < p < \infty)$$

then $|I| = 0$.

The information contained in theorem 1 can be restated in a particularly meaningful way in a special case. More precisely:

COROLLARY. *Let γ be a fixed negative constant and consider the following class of obstacle problems:*

$$(2.8) \quad \left\{ \begin{array}{l} (\mathfrak{L}u, v - u) \geq \int_G \gamma(v - u) \, dx \\ v \in K = \{v \in H_0^1(G) : v \leq 1 \text{ in } G\}, \end{array} \right.$$

where:

G is a bounded domain with fixed measure,

$\mathfrak{L} = \sum_{i,j=1}^m D_i(a_{ij}D_j)$ is a second order linear elliptic operator with $C^1(\bar{G})$ coefficients and lower ellipticity constant equal to 1.

Then the measure of the coincidence set of the solution to (2.8) is maximum when $\mathfrak{L} = -\Delta$ and G is a ball.

In this case note that the equation $\Phi(\lambda) = 1$ writes

$$\frac{m}{2(m-2)} \lambda^{2/m} - \frac{|G|^{-1+2/m}}{m-2} \lambda - \frac{1}{2} |G|^{2/m} + \frac{m}{\gamma} C_m^{2/m} = 0 \quad \text{if } m > 2,$$

$$\lambda \log(\lambda/|G|) - \lambda + |G| - \frac{4\pi}{\gamma} = 0 \quad \text{if } m = 2.$$

3. In this section we prove some relations among various characteristic parameter appearing in the obstacle problem.

For the sake of simplicity we take $g = 0$.

THEOREM 2. *Consider the variational inequality (1.1) with $g = 0$ and $a_{ij} \in L^\infty(G)$. Assume $\psi \in L^\infty(G) \cap H^1(G)$ and, in the sense of H^1 ,*

$$\psi < 0 \text{ on } \partial G, \quad t_1 = \sup \{\psi(x), x \in \partial I\} > 0.$$

Then

$$(3.1) \quad (\|\psi\|_\infty - t_1)|I| \geq \frac{m}{M} C_m^{1/m} t_1 \left[\int_G |D\psi|^2 dx \right]^{-1} \left[\int_{\psi > t_1} (\psi - t_1)^{m/(m-1)} dx \right]^{(m-1)/m},$$

where M denotes the upper ellipticity constant.

PROOF. Clearly we take $t_1 < \|\psi\|_\infty$ otherwise there is nothing to prove.

Let α be the Lipschitz continuous function defined by

$$\alpha(t) = \begin{cases} \frac{t_1}{\|\psi\|_\infty - t_1} (t_1 - t) & \text{if } t_1 < t \leq \|\psi\|_\infty. \\ 0 & \text{if } t \leq 0, t > \|\psi\|_\infty \\ t & \text{if } 0 < t < t_1 \end{cases}$$

Set $v(x) = u(x) + \alpha \circ u(x)$ where u is the solution of (1.1). By Corollary A.5, pag. 54 of [4], v belongs to the convex set K . Inserting v in (1.1) we get

$$\frac{t_1}{\|\psi\|_\infty - t_1} \int_{u > t_1} \sum_{i,j=1}^m a_{ij}(x) D_i u D_j u dx \leq \int_{0 < u \leq t_1} \sum_{i,j=1}^m a_{ij}(x) D_i u D_j u dx.$$

Clearly

$$(3.2) \quad \int_{0 < u \leq t_1} \sum_{i,j=1}^m a_{ij}(x) D_i u D_j u dx \leq M \int_G |D\psi|^2 dx.$$

On the other hand, if $\mu(t)$ denotes the distribution function of u , we have

$$(3.3) \quad \begin{aligned} \int_{u > t_1} \sum_{i,j=1}^m a_{ij}(x) D_i u D_j u dx &\geq \int_{u > t_1} |Du|^2 dx \geq \\ &\geq (\mu(t_1))^{-1} \left(\int_{u > t_1} |Du| dx \right)^2 = (\mu(t_1))^{-1} \left[\int_{t_1}^{\|\psi\|_\infty} P(u(x) > r) dr \right]^2. \end{aligned}$$

Here we have used the ellipticity condition, Schwartz inequality

and the total variation formula of Fleming-Rishel [2]; $P(u(x) > r)$ denotes the perimeter in the sense of De Giorgi of the set $\{x \in G: u(x) > r\}$.

The isoperimetric inequality gives now

$$(3.4) \quad \int_{t_1}^{\|v\|_\infty} P(u(x) > r) \, dr \geq \int_{t_1}^{\|v\|_\infty} m C_m^{1/m} \mu(r)^{1-1/m} \, dr \geq m C_m^{1/m} \left(\int_{v > t_1} (\psi - t_1)^{m/(m-1)} \, dx \right)^{(m-1)/m}.$$

The last inequality follows from

$$\|h\|_{L^p(R^m)} \leq \int_0^{+\infty} (\text{meas } \{x \in R^m: |h(x)| > t\})^{1/p} \, dt$$

which holds for every h in $L^p(R^m)$, $1 < p < \infty$.

From (3.2), (3.3), (3.4) and noticing that $\mu(t_1) < |I|$, we get (3.1).

Other information can be obtained integrating the conormal derivative of the solution u on the boundary of the level set $\{x \in G: u(x) > t\}$ for t sufficiently small. More precisely we have (we restrict ourselves to $m > 2$ for brevity):

THEOREM 3. *Consider the variational inequality (1.1) with $g = 0$. Suppose (1.2), (2.1) hold and $\max \{\psi(x), x \in G\} > 0$. Then*

$$(3.5) \quad t_0 \equiv \inf \{\psi(x), x \in \partial I\} < \frac{M C_m^{-2/m}}{m(m-2)} \{ |I|^{-1+2/m} - |G|^{-1+2/m} \} \int_{\partial I} |D\psi| H_{m-1}(dx).$$

Moreover equality holds in (3.5) if and only if ψ is radial, G is a ball centered at the origin and $\Omega = -\Delta$.

PROOF. If $0 < t < t_0$, since $D\psi = Du$ on ∂I , Green's formula in the set $\{x \in G: u(x) > t\} \cap (G \setminus I)$ gives

$$\int_{u=t} |Du| H_{m-1}(dx) = \int_{\partial I} \sum_{i,j=1}^m a_{ij}(x) D_i \psi v_j H_{m-1}(dx) \leq M \int_{\partial I} |D\psi| H_{m-1}(dx)$$

where $v_j = D_j u / |Du|$.

As in the proof of theorem 1, we get, for the distribution function $\mu(t)$ of u the following inequality:

$$(3.6) \quad t \leq \frac{MC_m^{-2/m}}{m(m-2)} \int_{\partial I} |D\psi| H_{m-1}(dx) \{ \mu(t)^{-1+2/m} - |G|^{-1+2/m} \}$$

for almost every t , $0 < t < t_0$.

(3.5) follows letting t go to t_0 and observing that $\mu(t_0^-) \geq |I|$.

REMARK. Suppose E is a compact subset of G with smooth boundary and consider the capacity potential u of E with respect to G ; in other words u is the solution of the variational inequality

$$\int_G \sum_{i,j=1}^m a_{ij}(x) D_i u D_j (v - u) dx \geq 0$$

for any $v \in K = \{v \in H_0^1(G) : v \geq 1 \text{ in } E\}$.

The \mathcal{L} -capacity of E with respect to G is defined by

$$\text{cap}_{\mathcal{L}}(E) = \int_G \sum_{i,j=1}^m a_{ij}(x) D_i u D_j u dx .$$

The level sets method of theorem 3 gives in this case the well known result:

$$\text{cap}_{\mathcal{L}}(E) \geq m(m-2) C_m^{2/m} \{ |E|^{-1+2/m} - |G|^{-1+2/m} \} \quad (m > 2)$$

with equality if and only if $\mathcal{L} = -\Delta$ and E, G are concentric balls.

A slight modification in the proof of theorem 3 and Lemma 4.2 of [2] pag. 117 give:

COROLLARY. *Let the assumptions in theorem 3 be satisfied. If moreover G is convex then:*

$$(3.7) \quad t_0 \leq MC_m^{-1/m} \|D\psi\|_{\infty} \{ |G|^{1/m} - |I|^{1/m} \} .$$

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