

RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

RICHARD D. CARMICHAEL

STEPHEN P. RICHTERS

Distributional boundary values in \mathcal{D}'_{L^p} . V

Rendiconti del Seminario Matematico della Università di Padova,
tome 70 (1983), p. 55-76

http://www.numdam.org/item?id=RSMUP_1983__70__55_0

© Rendiconti del Seminario Matematico della Università di Padova, 1983, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (<http://rendiconti.math.unipd.it/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

Distributional Boundary Values in \mathcal{D}'_{L^p} . - V

RICHARD D. CARMICHAEL - STEPHEN P. RICHTERS (*)

1. Introduction and preliminary results.

The purpose of this paper is twofold. First we present some new results concerning the Hardy H^p functions in tubes and generalizations of H^p functions in tubes. Secondly we add information to the analysis of [5] where we have related H^p functions with the distributional boundary value process.

Let C be an open convex cone in \mathbb{R}^n , n dimensional Euclidean space, and let $T^C = \mathbb{R}^n + iC$ be the corresponding tube in \mathbb{C}^n , n dimensional complex space. In [10] we have proved a growth property for the Hardy $H^p(T^C)$ spaces, $1 < p < \infty$, corresponding to the tube T^C ; in section 2 of this paper we improve the growth property obtained in [10]. We add information to the analysis of [5, sections 2-4] in sections 3 and 4 of this paper and use the growth properties of H^p functions obtained in section 2 in our analysis in sections 3 and 4. In [9] spaces of holomorphic functions in tubes which generalize the $H^p(T^C)$ functions have been defined, and representation results for these new spaces in terms of Fourier-Laplace integrals have been obtained. Section 5 of the present paper is concerned with spaces of functions related to those of [9]; we show that the functions which we define in section 5, which are generalizations of H^p functions corresponding to the quadrants in \mathbb{R}^n , can be characterized directly in terms of the H^p functions corresponding to these quadrants.

(*) Indirizzo degli AA.: Department of Mathematics Wake Forest University Winston-Salem, North Carolina, 27109 U.S.A.

In the remainder of this introductory section we introduce notation, definitions, and results which are needed for the analysis of our basic results in this paper contained in sections 2-5.

The n -dimensional notation to be used in this paper will be as described in [3, section II] and [4, section II] with the following slight change in the definition of the differentiation operator D_t^α , $t \in \mathbb{R}^n$, where α is an n -tuple of nonnegative integers. Here we put

$$D_t^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n} \quad \text{where} \quad D_j = \frac{-1}{2\pi i} \frac{\partial}{\partial t_j}, \quad j = 1, \dots, n.$$

Also we let $\bar{0}$ denote the n -tuple $(0, 0, \dots, 0)$ in \mathbb{R}^n .

The definition of a cone in \mathbb{R}^n and related terminology such as projection of a cone C , compact subcone $C' \subset C$, and dual cone $C^* = \{t \in \mathbb{R}^n: \langle t, y \rangle \geq 0 \text{ for all } y \in C\}$ of a cone C are given in [4, section II]. For the cones in \mathbb{R}^n that are the quadrants, we adopt the same notation as in [6, section 1] which is as follows. Let $\sigma = (\sigma_1, \dots, \sigma_n)$, n being the dimension, be an n -tuple where $\sigma_j = \pm 1$, $j = 1, \dots, n$, throughout this paper. There are 2^n such n -tuples σ , and for each such σ we put $C_\sigma = \{y \in \mathbb{R}^n: \sigma_j y_j > 0, j = 1, \dots, n\}$. The 2^n C_σ are cones in \mathbb{R}^n which we call quadrants.

Let C be any open connected cone in \mathbb{R}^n . In this paper $d(y)$ will denote the distance from $y \in C$ to the boundary of C and is given by the formula ([25, p. 159], [24])

$$(1.1) \quad d(y) = \inf_{t \in \text{pr}(C^*)} \langle t, y \rangle, \quad y \in C,$$

where $\text{pr}(C^*)$ denotes the projection of C^* which is the intersection of C^* with the unit sphere in \mathbb{R}^n .

The Hardy $H^p(T^B)$ space, $0 < p < \infty$, corresponding to any open proper subset B of \mathbb{R}^n and the corresponding tube $T^B = \mathbb{R}^n + iB$, is the set of all holomorphic functions $f(z)$ in T^B , $z = x + iy \in T^B$, for which there is a finite constant $A > 0$, such that

$$\|f(x + iy)\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x + iy)|^p dx \right)^{1/p} \leq A, \quad \text{for all } y \in B.$$

$H^\infty(T^B)$ is the set of all bounded holomorphic functions in T^B . (See [20, pp. 90-91] for $H^p(T^B)$, $0 < p < \infty$.) Throughout this paper, by $f(z) \in H^p(T^B)$, $0 < p < \infty$, (or $h(x) \in L^p = L^p(\mathbb{R}^n)$, $1 \leq p < \infty$), we of course

mean that $f(z) \in H^p(T^B)$ for some p , $0 < p \leq \infty$ (or $h(x) \in L^p$ for some p , $1 \leq p < \infty$.)

The function spaces \mathcal{S} , \mathcal{D}_{L^p} , $\mathcal{B} \equiv \mathcal{D}_{L^\infty}$, and \mathcal{B} and generalized function spaces \mathcal{S}' and \mathcal{D}'_p are contained in [18, pp. 199-205 and pp. 233-248].

The Fourier transform of an L^1 function $\varphi(t)$ is given by

$$(1.2) \quad \mathcal{F}[\varphi(t); x] = \int_{\mathbb{R}^n} \varphi(t) \exp(2\pi i \langle x, t \rangle) dt;$$

and we have the usual corresponding definition for the inverse Fourier transform of $\varphi(t)$, which is denoted as $\mathcal{F}^{-1}[\varphi(t); x]$, by replacing $2\pi i \langle x, t \rangle$ by $-2\pi i \langle x, t \rangle$ in (1.2). The Fourier and inverse Fourier transforms for L^p functions $h(t)$, $1 < p \leq 2$, and L^q functions $g(x)$, $(1/p) + (1/q) = 1$, are defined by the usual limit in the mean process and will be denoted by $\mathcal{F}[h(t); x]$ and $\mathcal{F}^{-1}[g(x); t]$, respectively; we refer to [2] and [13]. The Fourier and inverse Fourier transforms for the \mathcal{S}' distributions are defined in [18, pp. 248-252]; if $V \in \mathcal{S}'$ we denote its Fourier transform by $\mathcal{F}[V]$ and its inverse Fourier transform by $\mathcal{F}^{-1}[V]$.

All distribution terminology, such as distributional differentiation and support of a distribution V , denoted $\text{supp}(V)$, will be that of [18].

Let C be an open connected cone in \mathbb{R}^n . Throughout this paper, by $y \rightarrow \bar{0}$, $y \in C$, we mean $y \rightarrow \bar{0}$, $y \in C' \subset C$, for every compact subcone C' of C . Let $f(z)$ be a function of $z = x + iy \in T^c$, and let U be a distribution or generalized function. By $f(x + iy) \rightarrow U$ in the weak topology of the distribution or generalized function space as $y \rightarrow \bar{0}$, $y \in C$, we mean $\langle f(x + iy), \psi(x) \rangle \rightarrow \langle U, \psi \rangle$ as $y \rightarrow \bar{0}$, $y \in C$, for each fixed element ψ in the corresponding test function space. By $f(x + iy) \rightarrow U$ in the strong topology of the distribution or generalized function space as $y \rightarrow \bar{0}$, $y \in C$, we mean $\langle f(x + iy), \psi(x) \rangle \rightarrow \langle U, \psi \rangle$ as $y \rightarrow \bar{0}$, $y \in C$, with the convergence being uniform for ψ on arbitrary bounded sets in the corresponding test function space. U is called the weak or strong, respectively, distributional boundary value of $f(z)$.

We now define the Cauchy and Poisson kernel functions corresponding to certain tubes. Let C be an open convex cone such that \bar{C} does not contain any entire straight line. The Cauchy kernel function corresponding to the tube $T^c = \mathbb{R}^n + iC$ is

$$(1.3) \quad K(z - t) = \int_{C^*} \exp(2\pi i \langle z - t, \eta \rangle) d\eta, \quad t \in \mathbb{R}^n, z \in T^c,$$

where $C^* = \{t: \langle t, y \rangle \geq 0 \text{ for all } y \in C\}$ is the dual cone of C . The Poisson kernel function corresponding to T^c is

$$(1.4) \quad Q(z; t) = \frac{K(z-t)\overline{K(z-t)}}{K(2iy)} = \frac{|K(z-t)|^2}{K(2iy)}, \quad t \in \mathbb{R}^n, z \in T^c.$$

Because of [22, p. 222, Lemma 1] we need the assumption on \bar{C} stated above in this paragraph in order for $Q(z; t)$ to be well defined. We have obtained properties of $K(z-t)$ and $Q(z; t)$ in [8] and [9]. Korányi [16, Prop. 2] and Stein and Weiss [20, p. 105] have noted that the Poisson kernel function $Q(z; t)$ is an approximate identity; see also [8, p. 213, Lemma 6]. In the following lemma we collect facts from these references which we need in this paper.

LEMMA 1.1. $K(z-t)$ is a holomorphic function of $z \in T^c$ for fixed $t \in \mathbb{R}^n$. For $1 \leq p \leq 2$ and fixed $z \in T^c$, $K(z-t) \in \dot{\mathfrak{B}} \cap \mathcal{D}_{L^q}$ for all q , $(1/p) + (1/q) = 1$; and $Q(z; t) \in \dot{\mathfrak{B}} \cap \mathcal{D}_{L^q}$ for all q , $1 \leq q \leq \infty$, with both $K(z-t)$ and $Q(z; t)$ being considered as functions of $t \in \mathbb{R}^n$. Further, $Q(z; t)$ satisfies the following approximate identity properties:

$$Q(z; t) \geq 0, \quad t \in \mathbb{R}^n, z \in T^c;$$

$$\int_{\mathbb{R}^n} Q(z; t) dt = 1, \quad z \in T^c;$$

if $\delta > 0$,

$$\lim_{\substack{z \rightarrow t_0 \\ z \in T^c}} \int_{|t-t_0| > \delta} Q(z; t) dt = 0$$

uniformly for all $t_0 \in \mathbb{R}^n$.

From Lemma 1.1 and the definitions of $Q(z; t)$ and $K(z-t)$ in (1.4) and (1.3), respectively, we have

$$(1.5) \quad 0 \leq Q(z; t) = \frac{|K(z-t)|^2}{K(2iy)} \leq \frac{(K(iy))^2}{K(2iy)}, \quad t \in \mathbb{R}^n, z = x + iy \in T^c.$$

We use (1.5) in the proof of Theorem 2.2 below.

From the definition (1.3), the fact that $\langle y, t \rangle > 0$ for all $y \in C$ and $t \in \text{pr}(C^*)$ [25, p. 158], and a calculation as in [25, p. 159, lines 2

and 3] we have

$$K(iy) = \int_{C^*} \exp(-2\pi\langle y, \eta \rangle) d\eta = (2\pi)^{-n}(n-1)! \int_{\text{pr}(C^*)} \langle t, y \rangle^{-n} dt, \quad y \in C,$$

where n is the dimension; and from this representation of $K(iy)$, $y \in C$, and the equality (1.1) for $d(y)$, the distance from $y \in C$ to the boundary of C , we immediately obtain the estimate

$$(1.6) \quad |K(iy)| \leq A(C)(d(y))^{-n}, \quad y \in C,$$

where n is the dimension and the constant $A(C)$, which depends only on the cone C and not on $y \in C$, is given by

$$A(C) = (2\pi)^{-n}(n-1)! \int_{\text{pr}(C^*)} 1 dt$$

with this integral over $\text{pr}(C^*)$ being the finite surface area of $\text{pr}(C^*)$. Inequality (1.6) will be used in the proof of Theorem 2.2 below.

At several places in this paper we shall use a certain C^∞ multiplying function which we now describe. Let C be an open connected cone in \mathbb{R}^n and let C^* be its dual cone. As indicated in [25, p. 144, lines 8-12] there exists a function $\lambda(t) \in C^\infty$, $t \in \mathbb{R}^n$, with the properties that for any n -tuple α of nonnegative integers

$$(1.7) \quad |D_i^\alpha(\lambda(t))| \leq M_\alpha, \quad t \in \mathbb{R}^n,$$

where M_α is a constant which depends only on α ; and for any $\varepsilon > 0$

$$(1.8) \quad \begin{aligned} \lambda(t) &= 1 \text{ for } t \text{ on an } \varepsilon \text{ neighborhood of } C^*, \\ \lambda(t) &= 0 \text{ for } t \in \mathbb{R}^n \text{ but not on a } 2\varepsilon \text{ neighborhood of } C^*. \end{aligned}$$

In this paper we use two results from our recent paper [11] which we state here. Let C be an open connected cone and let $A \geq 0$ be a real number. For any real number $m > 0$ and any compact subcone C' of C put $T(C'; m) = \mathbb{R}^n + i(C' \setminus (C' \cap N(\bar{0}, m)))$. We say that a function $f(z)$ belongs to the class $H(A; C)$ if $f(z)$ is holomorphic in the tube $T^c = \mathbb{R}^n + iC$ and if for every compact subcone C' of C and every $m > 0$ there exists a constant $M(C', m)$ depending on C'

and on $m > 0$ such that

$$(1.9) \quad |f(x + iy)| \leq M(C', m)(1 + |z|)^N \exp(2\pi(A + \beta)|y|),$$

$$z = x + iy \in T(C'; m),$$

for all $\beta > 0$, where N is a nonnegative real number which does not depend on C' or on $m > 0$.

THEOREM 1.1. [11] *Let $f(z) \in H(A; C)$ where $A \geq 0$ and C is an open connected cone. Let $f(x + iy) \rightarrow U$ in the weak topology of S' as $y \rightarrow \bar{0}$, $y \in C$, where U is unique. Then $U \in S'$; there exists a unique element $V \in S'$ such that $\text{supp}(V) \subseteq \{t \in \mathbb{R}^n : u_C(t) \leq A\}$ and $U = \mathcal{F}[V]$ in S' ; and $f(z) = \langle V, \exp(2\pi i \langle z, t \rangle) \rangle$, $z \in T^C$.*

In Theorem 1.1, $u_C(t)$ is the indicatrix function of the cone C defined in [4, p. 252]. The growth (1.9) is more general than [5, p. 138, (1)].

Let C be an open connected cone and let C' be an arbitrary compact subcone of C . We now consider the growth

$$(1.10) \quad |f(x + iy)| \leq M(C')(1 + |z|)^N \left(\sum_{j=1}^r a_j |y|^{-k_j} \right), \quad z = x + iy \in T^{C'},$$

where $M(C')$ is a constant which depends on $C' \subset C$, $N \geq 0$ is a real constant, $r \geq 0$ is an integer, the $a_j \geq 0$, $j = 1, \dots, r$, are real constants, and the $k_j \geq 0$, $j = 1, \dots, r$, are real constants. Here N , r , the a_j , $j = 1, \dots, r$, and the k_j , $j = 1, \dots, r$, depend only on f and on C and are independent of $C' \subset C$. Any function $f(z)$ which is holomorphic in $T^C = \mathbb{R}^n + iC$ and which satisfies (1.10) will have a unique strong distributional boundary value in S' according to [22, p. 235]. We have the following theorem which is proved in [11].

THEOREM 1.2. [11] *Let C be an open convex cone such that \bar{C} does not contain any entire straight line. Let $f(z)$ be holomorphic in T^C and satisfy (1.10). Let the unique strong S' boundary value of $f(z)$, which exists, be $h(x) \in L^p$, $1 \leq p \leq \infty$. Then $f(z) \in H^p(T^C)$, $1 \leq p \leq \infty$, and*

$$(1.11) \quad f(z) = \int_{\mathbb{R}^n} h(t) Q(z; t) dt, \quad z \in T^C.$$

2. Growth of H^p functions in tubes.

In [10] we have proved the following result.

THEOREM 2.1. *Let C be any open convex cone in \mathbb{R}^n such that \bar{C} does not contain any entire straight line. Let $f(z) \in H^p(T^C)$, $1 \leq p < \infty$. For any compact subcone C' of C there exists a constant $M(C')$ depending on C' (and of course on f) such that*

$$(2.1) \quad |f(x + iy)| \leq M(C') |y|^{-n/p}, \quad z = x + iy \in T^{C'} = \mathbb{R}^n + iC'.$$

$H^\infty(T^C)$ is by definition the space of all bounded holomorphic functions in T^C ; hence we did not include the case $p = \infty$ in Theorem 2.1 since the growth of $H^\infty(T^C)$ functions is known by definition for $z \in T^C$.

Consider the quadrants C_σ . From analysis of Madych [17], if $f(z) \in H^p(T^{C_\sigma})$, $0 < p < \infty$, there exists a constant M , depending only on f and the quadrant C_σ , such that

$$(2.2) \quad |f(x + iy)| \leq M \left(\sum_{j=1}^n |y_j| \right)^{-1/p}, \quad z = x + iy \in T^{C_\sigma}.$$

The growth (2.2) is a better growth than (2.1) for the case that $C = C_\sigma$ in that (2.2) holds for all points in the tube T^{C_σ} . For arbitrary open convex cones as in Theorem 2.1 we are now able to prove a growth result for $H^p(T^C)$ functions, $1 \leq p < \infty$, which holds for all $z \in T^C$ and not just in tubes defined by compact subcones of C as obtained in Theorem 2.1. After proving this new growth result in Theorem 2.2 below, we then show that the growth of Theorem 2.1 becomes a special case of our new growth. Again we do not include $p = \infty$ in the following theorem because of the boundedness of $H^\infty(T^C)$ functions by definition.

THEOREM 2.2. *Let C be any open convex cone in \mathbb{R}^n such that \bar{C} does not contain any entire straight line. Let $f(z) \in H^p(T^C)$, $1 \leq p < \infty$. There exists a constant $M(C)$ depending on the cone C (and of course on f) such that*

$$(2.3) \quad |f(x + iy)| \leq M(C) |y|^{n/p} (d(y))^{-2n/p}, \quad z = x + iy \in T^C,$$

where $d(y)$ is the distance of $y \in C$ to the boundary of C given by (1.1).

PROOF. We apply [16, Prop. 4] to $f(z) \in H^p(T^c)$, $1 \leq p < \infty$, and obtain a function $h \in L^p \equiv L^p(\mathbb{R}^n)$ such that

$$f(z) = \int_{\mathbb{R}^n} h(t) Q(z; t) dt, \quad z \in T^c,$$

where $Q(z; t)$ is the Poisson kernel function defined in (1.4). Using this equality, the fact that $Q(z; t)$, $z \in T^c$, $t \in \mathbb{R}^n$ is an approximate identity as noted in Lemma 1.1, Jensen's inequality [13, p. 91, 2.4.19], and (1.5), we obtain for $z = x + iy \in T^c$ that

$$(2.4) \quad |f(x + iy)|^p = \left| \int_{\mathbb{R}^n} h(t) Q(z; t) dt \right|^p \leq \\ \leq \int_{\mathbb{R}^n} |h(t)|^p Q(z; t) dt \leq \frac{(K(iy))^2}{K(2iy)} \int_{\mathbb{R}^n} |h(t)|^p dt.$$

By [10, Lemma 2] there exists a constant $B(C)$ depending only on C such that

$$(2.5) \quad K(2iy) \geq B(C) |y|^{-n}, \quad y \in C.$$

Combining (2.4), (2.5), and (1.6) we get

$$(2.6) \quad |f(x + iy)|^p \leq (A(C))^2 (B(C))^{-1} |y|^n (d(y))^{-2n/p} \int_{\mathbb{R}^n} |h(t)|^p dt, \\ z = x + iy \in T^c.$$

Taking the p th. root of both sides of (2.6) yields

$$|f(x + iy)| \leq \left((A(C))^2 / B(C) \right)^{1/p} \|h\|_{L^p} |y|^{n/p} (d(y))^{-2n/p}, \quad z = x + iy \in T^c,$$

which is the desired growth (2.3) with

$$M(C) = \left((A(C))^2 / B(C) \right)^{1/p} \|h\|_{L^p}.$$

The proof of Theorem 2.2 is complete.

From [23, p. 932, (3.6)], for each compact subcone $C' \subset C$ of the cone C in Theorems 2.1 or 2.2 there exists a number $\delta > 0$ depending

on C' such that

$$(2.7) \quad \delta|y| \leq d(y) \leq |y|, \quad y \in C' \subset C,$$

where again $d(y)$ is the distance from y to the boundary of C . Thus given $f(z) \in H^p(T^c)$, $1 \leq p < \infty$, we apply (2.3) to $f(z)$ and use the left inequality of (2.7) to obtain for any compact subcone C' of C that

$$|f(x + iy)| \leq M(C)|y|^{n/p}(\delta|y|)^{-2n/p} = M(C)\delta^{-2n/p}|y|^{-n/p},$$

$$z = x + iy \in T^{c'},$$

where the constant $(M(C)\delta^{-2n/p})$ now depends on C' since $\delta > 0$ does. Thus our previously known growth (2.1) for $H^p(T^c)$ functions, $1 \leq p < \infty$, is a special case of our newly proved growth (2.3).

3. Representation of H^p functions by Fourier-Laplace integrals.

In this section and the next we add information to [5, sections 2, 3, and 4]. In the present section we show that [5, Theorem 2 and Corollary 1] hold with no growth assumption in them, and we make the conclusions of these results more precise. We also extend results of these types to all values of p , $0 < p \leq \infty$. Our additional information to [5, Theorem 2 and Corollary 1] and related results are contained in the following theorem.

THEOREM 3.1. *Let $f(z) \in H^p(T^c)$, $0 < p \leq \infty$, where C is an open convex cone in \mathbb{R}^n such that \bar{C} does not contain any entire straight line.*

I. *If $1 \leq p < \infty$ there exists a function $h \in L^p$ such that*

$$(3.1) \quad f(z) = \int_{\mathbb{R}^n} h(t)Q(z; t) dt, \quad z \in T^c.$$

If $1 \leq p < \infty$, $f(x + iy) \rightarrow h(x)$ in L^p and in the strong topology of S' as $y \rightarrow \bar{0}$, $y \in C$. If $p = \infty$, $f(x + iy) \rightarrow h(x)$ in the weak-star topology of L^∞ and in the strong topology of S' as $y \rightarrow \bar{0}$, $y \in C$.

II. *If $0 < p < 1$ and $C = C_\sigma$ is any of the 2^n quadrants, there exist unique elements $U \in S'$ and $V \in S'$ with $\text{supp}(V) \subseteq C_\sigma^* = \bar{C}_\sigma$ and $U = = \mathcal{F}[V]$ in S' such that $f(x + iy) \rightarrow U$ in the strong topology of S' as*

$y \rightarrow \bar{0}$, $y \in C$, and

$$(3.2) \quad f(z) = \langle V, \exp(2\pi i \langle z, t \rangle) \rangle, \quad z \in T^c.$$

III. If $p = 1$ there exists a function $g \in L^\infty$ with $\text{supp}(g) \subseteq C^*$ as an element of \mathcal{S}' such that $h = \mathcal{F}[g]$ in \mathcal{S}' and

$$(3.3) \quad f(z) = \int_{\mathbb{R}^n} g(t) \exp(2\pi i \langle z, t \rangle) dt, \quad z \in T^c.$$

IV. If $1 < p \leq 2$ there exists a function $g \in L^q$, $(1/p) + (1/q) = 1$, with $\text{supp}(g) \subseteq C^*$ almost everywhere such that $h = \mathcal{F}[g]$ in \mathcal{S}' and (3.3) holds.

V. If $2 < p < \infty$ there exists $V \in \mathcal{S}'$ with $\text{supp}(V) \subseteq C^*$ such that $h = \mathcal{F}[V]$ in \mathcal{S}' and (3.2) holds for this $V \in \mathcal{S}'$.

PROOF. The existence of $h \in L^p$, $1 \leq p < \infty$, such that (3.1) holds is a result of Korányi [16, Prop. 4]. The convergence of $f(x + iy)$ to $h(x)$ in the L^p norm topology if $1 \leq p < \infty$ and in the weak-star topology of L^∞ if $p = \infty$ is obtained by combining (3.1) with [16, Prop. 3 (c) and 3 (d)]. If $1 \leq p < \infty$, Hölder's inequality as in [5, p. 141, (3)] and the L^p convergence of $f(x + iy)$ to $h(x)$ yield that $f(x + iy) \rightarrow h(x)$ in the weak topology of \mathcal{S}' as $y \rightarrow \bar{0}$, $y \in C$. Similarly, for $p = \infty$ the definition of the weak-star topology of L^∞ yields that $f(x + iy) \rightarrow h(x)$ in the weak topology of \mathcal{S}' as $y \rightarrow \bar{0}$, $y \in C$, for the case $p = \infty$. But \mathcal{S} is a Montel space ([22, p. 21], [12, p. 510]). Thus for $1 \leq p < \infty$, $f(x + iy) \rightarrow h(x)$ in the strong topology of \mathcal{S}' as $y \rightarrow \bar{0}$, $y \in C$, by [12, p. 510, Corollary 8.4.9]. This proves part I.

To prove part II recall the growth (2.2). Since $f(z) \in H^p(T^{c_\sigma})$, $0 < p < 1$, satisfies (2.2) then by the analysis of [21] or [22, section 26.3, p. 235] there is a unique element $U \in \mathcal{S}'$ such that $f(x + iy) \rightarrow U$ in the strong topology of \mathcal{S}' as $y \rightarrow \bar{0}$, $y \in C_\sigma$. The growth (2.2) is a special case of the growth (1.9) with $A = 0$ in (1.9). Thus by Theorem 1.1 there is a unique element $V \in \mathcal{S}'$ with $\text{supp}(V) \subseteq \{t \in \mathbb{R}^n: u_c(t) \leq 0\} = C_\sigma^* = \bar{C}_\sigma$, $U = \mathcal{F}[V]$ in \mathcal{S}' , and (3.2) holds. This proves part II. (The exact same argument together with the analysis of [7] in place of Theorem 1.1 also proves part II.)

To prove part III we first recall from part I of this theorem that $f(x + iy) \rightarrow h(x) \in L^1 \subset \mathcal{S}'$ in the strong topology of \mathcal{S}' as $y \rightarrow \bar{0}$, $y \in C$.

By Theorem 2.1, $f(z)$ satisfies the growth (2.1) for $p = 1$ here; hence $f(z)$ satisfies the growth (1.9) with $A = 0$ in (1.9). Thus we can apply Theorem 1.1 and obtain a unique $V \in \mathcal{S}'$ such that $\text{supp}(V) \subseteq C^*$, $h = \mathcal{F}[V]$ in \mathcal{S}' , and (3.2) holds for this V . But $h \in L^1$ implies $g(t) = \mathcal{F}^{-1}[h(x); t]$ exists in the function sense and hence in \mathcal{S}' also, and $g \in L^\infty \subset \mathcal{S}'$. Since the inverse Fourier transform \mathcal{F}^{-1} is an isomorphism of \mathcal{S}' onto \mathcal{S}' then $V = \mathcal{F}^{-1}[h] = g$ in \mathcal{S}' , and hence $\text{supp}(g) \subseteq C^*$ as an element of \mathcal{S}' and $h = \mathcal{F}[g]$ in \mathcal{S}' . Let $\lambda(t) \in C^\infty$ be a function corresponding to our present cone C and its dual cone C^* which satisfies (1.7) and (1.8). Then $(\lambda(t) \exp(2\pi i \langle z, t \rangle)) \in \mathcal{S}$ as a function of $t \in \mathbb{R}^n$ for $z \in T^c$ [25, p. 119], and $(\lambda(t) \exp(2\pi i \langle z, t \rangle)) = \exp(2\pi i \langle z, t \rangle)$ on $\text{supp}(g) = \text{supp}(V)$. Recalling that (3.2) holds for our present $f(z) \in H^1(T^c)$ and the above constructed $V = g \in \mathcal{S}'$, we thus have for $z \in T^c$ that

$$\begin{aligned} f(z) &= \langle V, \exp(2\pi i \langle z, t \rangle) \rangle = \langle V, \lambda(t) \exp(2\pi i \langle z, t \rangle) \rangle \\ &= \langle g(t), \lambda(t) \exp(2\pi i \langle z, t \rangle) \rangle = \int_{\mathbb{R}^n} g(t) \exp(2\pi i \langle z, t \rangle) dt \end{aligned}$$

and as in [25, p. 144] these equalities are independent of the $\lambda(t)$ chosen. Thus (3.3) is obtained and the proof of part III is complete.

In part IV the desired representation (3.3) where $g \in L^q$, $(1/p) + (1/q) = 1$, and $\text{supp}(g) \subseteq C^*$ almost everywhere follows as a special case of [9, Corollaries 4.1 and 4.2]. We can prove directly from the representation (3.3) that $f(x + iy) \rightarrow \mathcal{F}[g] \in \mathcal{S}'$ in the strong topology of \mathcal{S}' as $y \rightarrow \bar{0}$, $y \in C$, by the techniques of [8, Theorem 10, (42)] or [25, pp. 144-145] and using [12, p. 510, Corollary 8.4.9]. Recall from part I that for the present case of $1 < p \leq 2$, $f(x + iy) \rightarrow h(x) \in L^p \subset \mathcal{S}'$ in the strong topology of \mathcal{S}' as $y \rightarrow \bar{0}$, $y \in C$. Thus $h = \mathcal{F}[g]$ since the \mathcal{S}' boundary value of $f(x + iy)$ is unique because of the growth (2.1) and [22, section 26.3, p. 235]. The proof of part IV is complete.

Part V remains to be proved. From Theorem 2.1 for $2 < p < \infty$, $f(z)$ satisfies (2.1). For $p = \infty$ $f(z)$ is bounded on T^c by definition. Thus for $2 < p < \infty$ $f(z)$ satisfies (1.9) with $A = 0$ there. From this and the fact, from part I, that $h(x) \in L^p \subset \mathcal{S}'$ is the strong \mathcal{S}' boundary value of $f(x + iy)$, we apply Theorem 1.1 to obtain an element $V \in \mathcal{S}'$ with $\text{supp}(V) \subseteq C^*$ such that $h = \mathcal{F}[V]$ in \mathcal{S}' and (3.2) holds for $z \in T^c$ for this $V \in \mathcal{S}'$. This completes the proof of Theorem 3.1.

As noted in the proof of Theorem 3.1, (3.3) follows in part IV for $1 < p \leq 2$ as a special case of [9, Corollaries 4.1 and 4.2]. For $p = 2$

this representation of $H^2(T^c)$ functions has been known for many years; see [1] and [20, p. 101, Theorem 3.1]. Our analysis of [9, Corollaries 4.1 and 4.2] extended this Fourier-Laplace integral representation to the $H^p(T^c)$ spaces, $1 < p < 2$. Now our proof of part III of Theorem 3.1 for the case $p = 1$ yields the representation (3.3) for functions in the space $H^1(T^c)$ also.

Parts III, IV, and V of Theorem 3.1 show that no growth assumption is needed in the statements of [5, Theorem 2 and Corollary 1] to obtain these results. Of course we also have the Cauchy integral representation of [5, Corollary 1] in Theorem 3.1 above by the proof of [5, Corollary 1].

Using analysis as in [11] we can also prove that if the cone C in case V of Theorem 3.1 is contained in or is any of the 2^n quadrants C_σ then the element V in case V , the case for $2 < p \leq \infty$, can be concluded to be in \mathcal{D}'_L , which is a proper subspace of S' . Thus in this situation the conclusion of the existence of $V \in \mathcal{D}'_L \subset S'$ in case V is somewhat more precise than can be concluded for arbitrary cones C as stated in case V of Theorem 3.1 above.

4. Converse results.

In the case $p = \infty$ the following Theorem 4.1 generalizes [5, Theorem 6] to arbitrary tubes like those of Theorem 3.1; Theorem 4.1 includes [5, Corollary 2] for $p = 2$ and also obtains these types of results for $1 < p < 2$ and $2 < p < \infty$. [5, Theorem 4] remains as a related result. The results of Theorem 4.1 are converse results to those of Theorem 3.1, and the proof of Theorem 4.1 is interesting as an application of Theorem 1.2.

THEOREM 4.1. *Let C be an open convex cone such that \bar{C} does not contain any entire straight line. Let $h(x) \in L^p$, $1 < p < \infty$, such that $h(x) = \mathcal{F}[V]$ in S' for some $V \in S'$ with $\text{supp}(V) \subseteq C^*$. There exists $f(z) \in H^p(T^c)$ such that*

$$\lim_{\substack{v \rightarrow \bar{0} \\ v \in C}} f(x + iy) = h(x)$$

in L^p if $1 < p < \infty$, in the weak-star topology of L^∞ if $p = \infty$, and in the strong topology of S' for $1 < p < \infty$.

PROOF. The case $p = 2$ has already been proved in [5, Corollary 2] and this case also follows by the analysis below. Corresponding to the cone C and its dual cone C^* , let $\lambda(t) \in C^\infty$ be a function which satisfies (1.7) and (1.8). As noted in the proof of Theorem 3.1, $(\lambda(t) \exp(2\pi i \langle z, t \rangle)) \in \mathcal{S}$ as a function of $t \in \mathbb{R}^n$ for $z \in T^c$. For the assumed $V \in \mathcal{S}'$ with $\text{supp}(V) \subseteq C^*$ we put

$$(4.1) \quad f(z) = \langle V, \lambda(t) \exp(2\pi i \langle z, t \rangle) \rangle = \langle V, \exp(2\pi i \langle z, t \rangle) \rangle, \quad z \in T^c.$$

By [8, p. 208, Theorem 8] $f(z)$ is holomorphic in T^c and satisfies (1.10) for $z \in T^{c'}$ where C' is any compact subcone of C . By the proof of [8, Theorem 10, (42)] we have

$$f(x + iy) = \langle V, \lambda(t) \exp(2\pi i \langle x + iy, t \rangle) \rangle \rightarrow \mathcal{F}[V]$$

in the weak topology of \mathcal{S}' as $y \rightarrow \bar{0}$, $y \in C$; and hence this convergence also holds in the strong \mathcal{S}' topology by [12, p. 510, Corollary 8.4.9] as we have noted before in this paper. But by hypothesis, $\mathcal{F}[V] = h(x) \in L^p$, $1 \leq p < \infty$. Applying Theorem 1.2 we conclude that $f(z) \in H^p(T^c)$, $1 \leq p < \infty$, and we have proved the convergence $f(x + iy) \rightarrow \mathcal{F}[V] = h(x)$ as $y \rightarrow \bar{0}$, $y \in C$, in the strong topology of \mathcal{S}' in any case $1 \leq p < \infty$. By Theorem 1.2 we also have

$$(4.2) \quad f(z) = \int_{\mathbb{R}^n} h(t) Q(z; t) dt, \quad z \in T^c.$$

The L^p convergence for $1 \leq p < \infty$ and the weak-star convergence for $p = \infty$ of $f(x + iy) \rightarrow h(x)$ as $y \rightarrow \bar{0}$, $y \in C$, now follow from (4.2) and [16, Prop. 3 (e) and 3 (d)]. The proof of Theorem 4.1 is complete.

The results of [5, section 4] extend some of the results of [5, sections 2 and 3] to functions holomorphic in disconnected tubular cones. The results of sections 3 and 4 of this paper can similarly be extended to this more general situation. We leave this to the interested reader to consider; for the ideas to do this are now obvious from our analysis above and from [5, section 4].

5. Generalization of H^p functions for tubes over quadrants.

In [9] we have defined holomorphic functions in tubes which generalize and properly contain the H^p functions. These new functions are defined as follows. Let B be an open proper subset of \mathbb{R}^n . Let $p > 0$ and $A \geq 0$. $V_A^p(T^B)$ is the space of all holomorphic functions $f(z)$ in T^B which satisfy

$$(5.1) \quad \|f(x + iy)\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x + iy)|^p dx \right)^{1/p} \leq M_{A,f} \exp(2\pi A |y|),$$

$y \in B,$

if $0 < p < \infty$ or

$$(5.2) \quad |f(x + iy)| \leq M_{A,f} \exp(2\pi A |y|), \quad z = x + iy \in T^B,$$

if $p = \infty$, where the constant $M_{A,f}$ depends on $A \geq 0$ and on f and does not depend on $y \in B$ in (5.1) or on $z \in T^B$ in (5.2). Notice that $H^p(T^B) = V_0^p(T^B)$, $0 < p < \infty$. We always have $H^p(T^B) \subseteq V_A^p(T^B)$, $0 < p < \infty$, $A \geq 0$. There are tubes T^B and values of p such that $H^p(T^B)$ is a proper subset of $V_A^p(T^B)$. In [9] we represented the $V_A^p(T^B)$ functions, $1 < p \leq 2$, $A \geq 0$, in terms of a Fourier-Laplace integral of a function $g(t)$ having certain properties. If $B = C$, an open connected cone in \mathbb{R}^n , the representing function $g(t)$ is in L^q , $(1/p) + (1/q) = 1$, and has support in $\{t \in \mathbb{R}^n: u_c(t) \leq A\}$ almost everywhere.

We are now going to define special cases of the $V_A^p(T^B)$ functions for the cases that $B = C_\sigma$, any of the 2^n quadrants. These new spaces are also generalizations of the $H^p(T^{C_\sigma})$ functions and have the interesting property that they can be directly characterized in terms of the $H^p(T^{C_\sigma})$ functions as we prove in Theorem 5.1 below.

Let $b = (b_1, \dots, b_n)$ be an n -tuple of nonnegative real numbers. Let $0 < p < \infty$. $W_{b,\sigma}^p \equiv W_0^p(T^{C_\sigma})$ is the space of all holomorphic functions $f(z)$ in T^{C_σ} which satisfy

$$(5.3) \quad \|f(x + iy)\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x + iy)|^p dx \right)^{1/p} \leq M_{b,f} \exp(2\pi \langle b, (|y_1|, \dots, |y_n|) \rangle), \quad y \in C_\sigma,$$

where the constant $M_{b,f}$ depends on the n -tuple b and on f but not on $y = (y_1, y_2, \dots, y_n) \in C_\sigma$.

Similarly we define $W_{b,\sigma}^\infty \equiv W_b^\infty(T^{c_\sigma})$ to be the space of all holomorphic functions $f(z)$ in T^{c_σ} which satisfy

$$(5.4) \quad |f(x + iy)| \leq M_{b,f} \exp(2\pi \langle b, (|y_1|, \dots, |y_n|) \rangle), \\ z = x + iy \in T^{c_\sigma},$$

where again the constant $M_{b,f}$ depends only on the n -tuple b and on f but not on $z \in T^{c_\sigma}$.

If $b = \bar{0}$, the n -tuple of zeros, we have $H^p(T^{c_\sigma}) = W_{\bar{0},\sigma}^p$, $0 < p \leq \infty$. Otherwise we have the proper containment $H^p(T^{c_\sigma}) \subset W_{b,\sigma}^p$ by Theorem 5.1 below. For example for any T^{c_σ} and any $b = (b_1, \dots, b_n)$ such that $b_j > 0$ for at least one j , $j = 1, \dots, n$, we have

$$f(z) = \prod_{j=1}^n ((i + \sigma_j z_j)^{-1} \exp(-2\pi i b_j \sigma_j z_j)), \quad (z_1, \dots, z_n) \in T^{c_\sigma},$$

is in $W_{b,\sigma}^p$ but not in $H^p(T^{c_\sigma})$. For the case $B = C_\sigma$, notice also that the growths (5.1) and (5.2) are more general than (5.3) and (5.4), respectively.

The $W_{b,\sigma}^p$ functions are interesting in that they can be directly characterized in terms of the $H^p(T^{c_\sigma})$ functions as we now show.

THEOREM 5.1. *Let $0 < p \leq \infty$. $f(z) \in W_{b,\sigma}^p$ if and only if*

$$(5.5) \quad f(z) = \exp(-2\pi i \langle b, (\sigma_1 z_1, \dots, \sigma_n z_n) \rangle) F(z), \quad z \in T^{c_\sigma},$$

where $F(z) \in H^p(T^{c_\sigma})$.

PROOF. Let $f(z) \in W_{b,\sigma}^p$. Put

$$(5.6) \quad F(z) = \exp(2\pi i \langle b, (\sigma_1 z_1, \dots, \sigma_n z_n) \rangle) f(z), \quad z \in T^{c_\sigma}.$$

$F(z)$ is holomorphic in T^{c_σ} since $f(z)$ is. For $y \in C_\sigma$ we have $\sigma_j y_j = |y_j|$, $j = 1, \dots, n$, and

$$(5.7) \quad |\exp(2\pi i \langle b, (\sigma_1 z_1, \dots, \sigma_n z_n) \rangle)| \leq \\ \leq \prod_{j=1}^n \exp(-2\pi b_j \sigma_j y_j) = \exp(-2\pi \langle b, (|y_1|, \dots, |y_n|) \rangle), \quad z \in T^{c_\sigma}.$$

This combined with the growth (5.3) of $f(z) \in W_{b,\sigma}^p$, $0 < p < \infty$, yields

$$(5.8) \quad \|F(x + iy)\|_{L^p} \leq \exp(-2\pi\langle b, (|y_1|, \dots, |y_n|)\rangle) \|f(x + iy)\|_{L^p} \\ \leq M_{b,f} \exp(-2\pi\langle b, (|y_1|, \dots, |y_n|)\rangle) \exp(2\pi\langle b, (|y_1|, \dots, |y_n|)\rangle) = M_{b,f}$$

for $y \in C_\sigma$; hence $F(z) \in H^p(T^{C_\sigma})$, $0 < p < \infty$. Similarly, by (5.7) and (5.4) in the case $p = \infty$ we have

$$|F(x + iy)| \leq \exp(-2\pi\langle b, (|y_1|, \dots, |y_n|)\rangle) |f(x + iy)| \leq M_{b,f}, \\ z = x + iy \in T^{C_\sigma},$$

where $M_{b,f}$ is the constant from (5.4) corresponding to $f(z) \in W_{b,\sigma}^\infty$; hence, $F(x + iy) \in H^\infty(T^{C_\sigma})$. Thus for $f(z) \in W_{b,\sigma}^p$, $0 < p \leq \infty$, we have that $F(z)$ defined in (5.6) is in $H^p(T^{C_\sigma})$, and the representation (5.5) is immediate from (5.6).

Conversely, let $f(z)$ be defined by (5.5) with $F(z) \in H^p(T^{C_\sigma})$, $0 < p \leq \infty$. Then $f(z)$ is holomorphic in T^{C_σ} since $F(z)$ is. Arguing as in (5.7) and (5.8) we have for $y \in C_\sigma$ and $0 < p < \infty$ that

$$(5.9) \quad \|f(x + iy)\|_{L^p} \leq \exp(2\pi\langle b, (|y_1|, \dots, |y_n|)\rangle) \|F(x + iy)\|_{L^p} \\ \leq A \exp(2\pi\langle b, (|y_1|, \dots, |y_n|)\rangle)$$

where A is the bound on $\|F(x + iy)\|_{L^p}$ since $F(z) \in H^p(T^{C_\sigma})$. If $p = \infty$ we have for $z \in T^{C_\sigma}$

$$(5.10) \quad |f(x + iy)| \leq \exp(2\pi\langle b, (|y_1|, \dots, |y_n|)\rangle) |F(x + iy)| \\ \leq A \exp(2\pi\langle b, (|y_1|, \dots, |y_n|)\rangle)$$

where A is the bound on $|F(x + iy)|$ since $F(z) \in H^\infty(T^{C_\sigma})$ in this case. (5.9) and (5.10) yield that $f(z)$ defined by (5.5) is in $W_{b,\sigma}^p$, $0 < p \leq \infty$, as desired. The proof of Theorem 5.1 is complete.

Considerable information is known about the $H^p(T^{C_\sigma})$ functions including boundary value properties and integral representations. Information concerning $H^p(T^{C_\sigma})$ functions combined with Theorem 5.1 yield information about the $W_{b,\sigma}^p$ functions as we now show.

COROLLARY 5.1. *Let $f(z) \in W^p_{b,\sigma}$, $1 \leq p < \infty$. There is a function $h(x) \in L^p$ such that*

$$(5.11) \quad \lim_{\substack{y \rightarrow \bar{0} \\ y \in C_\sigma}} f(x + iy) = h(x)$$

for almost every $x \in \mathbb{R}^n$ and

$$(5.12) \quad \lim_{\substack{y \rightarrow \bar{0} \\ y \in C_\sigma}} \|f(x + iy) - h(x)\|_{L^p} = 0.$$

There is a function $g(t) \in L^p$ such that

$$(5.13) \quad f(z) = \exp(-2\pi i \langle b, (\sigma_1 z_1, \dots, \sigma_n z_n) \rangle) \int_{\mathbb{R}^n} g(t) Q(z; t) dt, \quad z \in T^{c_\sigma},$$

where

$$Q(z; t) = (\pi)^{-n} \prod_{j=1}^n \frac{\sigma_j y_j}{(t_j - x_j)^2 + y_j^2}, \quad z = x + iy \in T^{c_\sigma}, \quad t \in \mathbb{R}^n,$$

is the Poisson kernel function corresponding to the generalized half plane T^{c_σ} .

PROOF. By Theorem 5.1 there is an $F(z) \in H^p(T^{c_\sigma})$, $1 \leq p < \infty$, such that (5.5) holds. By [20, pp. 115-118] there is a function $g(x) \in L^p$ such that (5.11) and (5.12) hold corresponding to $F(z) \in H^p(T^{c_\sigma})$ and $g(x) \in L^p$. Put

$$(5.14) \quad h(x) = \exp(-2\pi i \langle b, (\sigma_1 x_1, \dots, \sigma_n x_n) \rangle) g(x), \quad x \in \mathbb{R}^n.$$

Then $h(x) \in L^p$, $1 \leq p < \infty$, and the convergence (5.11) of $f(x + iy)$ to $h(x)$ follows immediately from that of $F(x + iy)$ to $g(x)$. To show (5.12), we have for any $y \in C_\sigma$

$$(5.15) \quad \int_{\mathbb{R}^n} |f(x + iy) - h(x)|^p dx = \\ = \int_{\mathbb{R}^n} |f(x + iy) - \exp(-2\pi i \langle b, (\sigma_1 z_1, \dots, \sigma_n z_n) \rangle) g(x) \\ + \exp(-2\pi i \langle b, (\sigma_1 z_1, \dots, \sigma_n z_n) \rangle) g(x) - h(x)|^p dx \\ \leq 2^p (I_1 + I_2)$$

where

$$I_1 = \int_{\mathbf{R}^n} |f(x + iy) - \exp(-2\pi i \langle b, (\sigma_1 z_1, \dots, \sigma_n z_n) \rangle) g(x)|^p dx$$

and

$$I_2 = \int_{\mathbf{R}^n} |\exp(-2\pi i \langle b, (\sigma_1 z_1, \dots, \sigma_n z_n) \rangle) g(x) - h(x)|^p dx.$$

The representation (5.5) and the fact that (5.12) holds for $\|F(x + iy) - g(x)\|_{L^p}$ prove that $I_1 \rightarrow 0$ as $y \rightarrow \bar{0}$, $y \in C_\sigma$. Further, by (5.14)

$$I_2 \leq \left| \prod_{j=1}^n (\exp(2\pi b_j \sigma_j y_j) - 1) \right|^p \int_{\mathbf{R}^n} |g(x)|^p dx$$

which yields $I_2 \rightarrow 0$ as $y \rightarrow \bar{0}$, $y \in C_\sigma$, since $g \in L^p$. Thus (5.12) follows from (5.15) since the term $2^p(I_1 + I_2) \rightarrow 0$ as $y \rightarrow \bar{0}$, $y \in C_\sigma$.

Because of [16, Prop. 4 and Prop. 3 (c)] and the fact that the $g(t) \in L^p$ obtained in the second sentence of this proof is the L^p limit of $F(z) \in H^p(T^{C_\sigma})$, it follows that

$$(5.16) \quad F(z) = \int_{\mathbf{R}^n} g(t) Q(z; t) dt, \quad z \in T^{C_\sigma}.$$

Then (5.13) follows from (5.16) and (5.5). This completes the proof of Corollary 5.1.

Because of (5.5) and [16, Prop. 4] any function $f(z) \in W_{b,\sigma}^\infty$ also has the representation (5.13) for some $g \in L^\infty$; and by [16, Prop. 3 (d)] the Poisson integral, and hence the element $F(z)$ in (5.5), in this representation converges to g in the weak-star topology of L^∞ as $y \rightarrow \bar{0}$, $y \in C_\sigma$.

We also have the following information for $W_{b,\sigma}^p$ functions $f(z)$, $1 < p \leq \infty$. Let $g(t)$ be the L^p function in the representation (5.13) yielded by the proof of Corollary 5.1 if $1 < p < \infty$ and by the remark in the preceding paragraph if $p = \infty$. We have

$$(5.17) \quad \|f(x + iy)\|_{L^p} \leq \|g\|_{L^p} \exp(2\pi \langle b, (|y_1|, \dots, |y_n|) \rangle),$$

$$y \in C_\sigma, 1 < p < \infty;$$

$$(5.18) \quad |f(x + iy)| \leq \|g\|_{L^\infty} \exp(2\pi \langle b, (|y_1|, \dots, |y_n|) \rangle),$$

$$z = x + iy \in T^{C_\sigma}, \quad p = \infty;$$

$$(5.19) \quad \limsup_{\substack{y \rightarrow 0 \\ y \in C_\sigma}} \|f(x + iy)\|_{L^p} \leq \|g\|_{L^p}, \quad 1 < p < \infty;$$

and

$$(5.20) \quad \limsup_{\substack{y \rightarrow 0 \\ y \in C_\sigma}} |f(x + iy)| \leq \|g\|_{L^\infty}, \quad p = \infty.$$

Obviously (5.19) and (5.20) follow immediately from (5.17) and (5.18), respectively. (5.18) follows immediately from the representation (5.13) for $p = \infty$ as noted in the preceding paragraph and Lemma 1.1. To obtain (5.17) we again use (5.13) and have

$$(5.21) \quad \|f(x + iy)\|_{L^p} \leq \exp(2\pi \langle b, (|y_1|, \dots, |y_n|) \rangle) \left\| \int_{\mathbf{R}^n} g(t) Q(z; t) dt \right\|_{L^p}.$$

But

$$\int_{\mathbf{R}^n} g(t) Q(z; t) dt = (g * Q_y)(x), \quad y \in C_\sigma, \quad x \in \mathbf{R}^n,$$

where

$$Q_y(t) = (\pi)^{-n} \prod_{j=1}^n \frac{\sigma_j y_j}{t_j^2 + y_j^2};$$

and by [14, p. 397]

$$(5.22) \quad \|g * Q_y\|_{L^p} \leq \|Q_y\|_{L^1} \|g\|_{L^p} = \|g\|_{L^p}, \quad 1 < p < \infty,$$

since $\|Q_y\|_{L^1} = 1$ by Lemma 1.1. Combining (5.22) with (5.21) we obtain (5.17). (5.17) and (5.18) are interesting in that they show that the constant in the defining growths (5.3) and (5.4) for $W_{b,\sigma}^p$, $1 < p < \infty$, and $W_{b,\sigma}^\infty$, respectively, can be taken to be the L^p norm of the boundary value of the $H^p(T^{C_\sigma})$ function $F(z)$ in the representation (5.5) of any element $f(z) \in W_{b,\sigma}^p$, $1 < p < \infty$.

Another result of the representation (5.5) in Theorem 5.1 is that we can explicitly write the pointwise growth of a function $f(z) \in W_{b,\sigma}^p$. Recall the growth (2.2) for $H^p(T^{C_\sigma})$ functions, $0 < p < \infty$. Then for

$f(z) \in W_{b,\sigma}^p$, $0 < p < \infty$, we use (5.5) and (2.2) to yield

$$(5.23) \quad |f(x + iy)| \leq M \left(\prod_{j=1}^n |y_j| \right)^{-1/p} \exp \left(2\pi \langle b, (|y_1|, \dots, |y_n|) \rangle \right), \\ z = x + iy \in T^{c_\sigma}, \quad 0 < p < \infty,$$

where M is a constant which depends only on f and not on $z \in T^{c_\sigma}$. Of course if $p = \infty$ $f(z)$ satisfies (5.4) by definition.

We summarize the analysis contained in the preceding two paragraphs in the following result.

COROLLARY 5.2. *Let $f(z) \in W_{b,\sigma}^p$, $0 < p < \infty$. If $0 < p < \infty$, (5.23) holds; if $p = \infty$, (5.4) holds. If $1 < p < \infty$ we have (5.17) and (5.19); if $p = \infty$ we have (5.18) and (5.20).*

Because of the pointwise growth property (5.23) for $0 < p < \infty$ and the definition (5.4) for $p = \infty$, we may use analysis exactly as in the proof of Theorem 3.1 above and results which we have obtained in [7] and [9] to also obtain distribution information concerning the $W_{b,\sigma}^p$ functions. The techniques to do so are exactly those of the proof of Theorem 3.1 using facts from [7] and [9]; thus we state our results, which are corollaries to Theorem 5.1, and omit the proofs.

COROLLARY 5.3. *Let $f(z) \in W_{b,\sigma}^p$, $0 < p \leq \infty$. There exists a unique element $V \in \mathcal{S}'$ with $\text{supp}(V) \subseteq \{t: -b, \leq \sigma_j t_j < \infty, j = 1, \dots, n\}$ such that*

$$(5.24) \quad f(z) = \langle V, \exp(2\pi i \langle z, t \rangle) \rangle, \quad z \in T^{c_\sigma};$$

$$(5.25) \quad \lim_{\substack{y \rightarrow 0 \\ y \in C_\sigma}} f(x + iy) = \mathcal{F}[V] \in \mathcal{S}'$$

in the strong topology of \mathcal{S}' ;

$$(5.26) \quad f(x + iy) = \mathcal{F}[\exp(-2\pi \langle y, t \rangle) V], \quad z = x + iy \in T^{c_\sigma},$$

as an equality in \mathcal{S}' ; and

$$(5.27) \quad \{f(x + iy): y \in C_\sigma, |y| \leq R\} \text{ is a strongly bounded set in } \mathcal{S}' \text{ where } \\ R > 0 \text{ is arbitrary but fixed.}$$

Further, if $1 < p \leq 2$, $V = g(t) \in L^q$, $(1/p) + (1/q) = 1$.

COROLLARY 5.4. *Let $f(z) \in W_{b,\sigma}^p$, $1 \leq p < \infty$. The boundary values $h(x) \in L^p$ and $\mathcal{F}[V] \in \mathcal{S}'$ of $f(z)$ from Corollaries 5.1 and 5.3, respectively, satisfy $h = \mathcal{F}[V]$ in \mathcal{S}' .*

REFERENCES

- [1] S. BOCHNER, *Group invariance of Cauchy's formula in several variables*, Ann. of Math., **45** (1944), pp. 686-707.
- [2] S. BOCHNER - K. CHANDRASEKHARAN, *Fourier Transforms*, Princeton University Press, Princeton, N. J., 1949.
- [3] R. D. CARMICHAEL, *Distributional boundary values in \mathcal{D}'_{L^p}* , Rend. Sem. Mat. Univ. Padova, **43** (1970), pp. 35-53.
- [4] R. D. CARMICHAEL, *Distributional boundary values in \mathcal{D}'_{L^p} - II*, Rend. Sem. Mat. Univ. Padova, **45** (1971), pp. 249-277.
- [5] R. D. CARMICHAEL, *Distributional boundary values in \mathcal{D}'_{L^p} - III*, Rend. Sem. Mat. Univ. Padova, **48** (1973), pp. 137-158.
- [6] R. D. CARMICHAEL, *Distributional boundary values in \mathcal{D}'_{L^p} - IV*, Rend. Sem. Mat. Univ. Padova, **63** (1980), pp. 203-214.
- [7] R. D. CARMICHAEL, *Functions analytic in an octant and boundary values of distributions*, J. Math. Anal. Appl., **33** (1971), pp. 616-626.
- [8] R. D. CARMICHAEL, *Generalized Cauchy and Poisson integrals and distributional boundary values*, SIAM J. Math. Anal., **4** (1973), pp. 198-219.
- [9] R. D. CARMICHAEL - E. K. HAYASHI, *Analytic functions in tubes which are representable by Fourier-Laplace integrals*, Pacific J. Math., **90** (1980), pp. 51-61.
- [10] R. D. CARMICHAEL - S. P. RICHTERS, *Growth of H^p functions in tubes*, Internat. J. Math. Math. Sci., **4** (1981), pp. 435-443.
- [11] R. D. CARMICHAEL - S. P. RICHTERS, *Holomorphic functions in tubes which have distributional boundary values and which are H^p functions*, SIAM J. Math. Anal., **14** (1983), to appear.
- [12] R. E. EDWARDS, *Functional Analysis*, Holt, Rinehart and Winston, New York, 1965.
- [13] H. FEDERER, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.
- [14] E. HEWITT - K. STROMBERG, *Real and Abstract Analysis*, Springer-Verlag, New York, 1969.
- [15] Y. KATZNELSON, *An Introduction to Harmonic Analysis*, John Wiley and Sons, New York, 1968.
- [16] A. KORÁNYI, *A Poisson integral for homogeneous wedge domains*, J. Analyse Math., **14** (1965), pp. 275-284.

- [17] W. R. MADYCH, *Distributions with strong maximal functions in $L^p(\mathbb{R}^n)$* , preprint.
- [18] L. SCHWARTZ, *Théorie des Distributions*, Hermann, Paris, 1966.
- [19] L. SCHWARTZ, *Mathematics for the Physical Sciences*, Addison-Wesley, Reading, Mass, 1966.
- [20] E. M. STEIN - G. WEISS, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, N. J., 1971.
- [21] H. G. TILLMANN, *Darstellung der Schwartzschen Distributionen durch analytische Funktionen*, Math. Z., **77** (1961), pp. 106-124.
- [22] V. S. VLADIMIROV, *Methods of the Theory of Functions of Many Complex Variables*, M.I.T. Press, Cambridge, Mass., 1966.
- [23] V. S. VLADIMIROV, *Generalized functions with supports bounded by the side of an acute convex cone*, Siberian Math. J., **9** (1968), pp. 930-937.
- [24] V. S. VLADIMIROV, *The Laplace transform of tempered distributions*, Global Anal. Appl., International Sem. Course Trieste 1972, Vol. III (1974), pp. 243-270.
- [25] V. S. VLADIMIROV, *Generalized Functions in Mathematical Physics*, Mir Publishers, Moscow, 1979.

Manoscritto pervenuto in redazione il 3 marzo 1982.