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Contributions to foundations of probability calculus on the basis of the modal logical calculus $MC^v$ or $MC^v_*$

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Contributions to Foundations of Probability Calculus on the Basis of the Modal Logical Calculus $MC^\vee$ or $MC^\vee_*$.

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PART III

An Analysis of the Notions of Random Variables and Probability Spaces, Based on Modal Logic.

12. Introduction.

The present paper (Part 3) is concerned with foundations of the probability calculus. We want, first, to analyze the notion of casual or random variables, which is usually dealt with as a primitive, on the basis of $MC^\vee_*$ [MC$^\vee$] (through $TP_*$ [TP]) and in particular by use of absolute concepts and their extensionalizations [N. 3]—cf. [3]. More precisely two versions of this intuitive notion are widely used in the literature; so to say, one is physical (or natural) and the other is purely mathematical. We emphasize and analyze this distinction by defining physical (or casual) random variable [N. 13] and absolute random variable [N. 14] rigorously, within $TP_*$ [TP].

Second, within the object language itself we define the (standard) physical notion of a probability space relative to an assertion $\alpha$, in a natural way connected with $MC^\vee_*$ (or $MC^\vee$) [N. 15]. Among these

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spaces there are the maximal ones. These may be infinite, finite, or the trivial one, which is substantially formed by the ranges of \( \alpha, \sim \alpha, \alpha \wedge \sim \alpha, \) and \( \alpha \vee \sim \alpha, \) and exists for every choice of \( \alpha. \) In N. 17 any set formed with propositions having a probability relative to \( \alpha \) and closed under conjunction is proved to belong to a probability space relative to \( \alpha. \) Incidentally this theorem constitutes a bridge between, so to say, the geometrical theories of probability and the existential ones, we mean between those that start with (a rather mathematical notion of) probability spaces (and define probability as a measure on them) and the probability theories such as von Mises’s, or Reichenbach’s (or our theories \( TP_\ast \) and \( TP \)) that start with existence axiom.

In the afore-mentioned treatment of probability based on the theory \( TP_\ast \) \( [TP], \) probability is a primitive notion that has the form of a function of propositions [proposition ranges]: hence probability cannot be reintroduced as a (primitive) measure on a (probability) space. Instead one has to define when a measure on a probability space relative to the trial \( \alpha \) is a probability measure.

Let us note that our theory \( TP_\ast \) or \( TP \) happens to agree with Freudenthal and De Finetti’s views about the approach to probability calculus—cf. [10], [8]—, in that e.g. they just criticize the usual systems of probability axioms which substantially are definitions of particular purely mathematical measure spaces, and among other things they consider these systems to say «little if anything about probability »—cf. [10] p. 261.

It must be added that unlike [10] and [8] the present paper has been written with no didactical purposes. However it can also be regarded as a theoretical support to Freudenthal and De Finetti’s views.

13. A first notion of random variable.

In the literature two notions of random variables are substantially used: so to say, a physical one and a mathematical notion. The first one, with which this section is concerned, is defined e.g. by Castelnuovo in [6] p. 30 as follows: we shall call (physical or casual) «random variable a variable quantity \( x \) that can take various real values \( x_1, x_2, ..., x_n \) according as one of the incompatible events \( E_1, E_2, ..., E_n \) having the known probabilities \( p_1, p_2, ..., p_n \) of sum \( p_1 + p_2 + ... + p_n = 1 \) occurs». He obviously means that \( x \) takes value \( x_i \) whenever \( E_i \) occurs (\( i = 1, ..., n \)).
It is obvious that « variable quantity » is used here and, for instance, in the assertion « $x = \sin \omega t$ is a variable quantity » (which refers to a harmonic motion of period $2\pi/\omega$) in very different senses. The first is essentially modal in that $x$ can assume (physically or casually) various real values $x_1, x_2, \ldots, x_n$, that is, $x$ can happen to equal the privileged absolute real numbers $x_1$ to $x_n$ ($x_i \in \mathbb{R}$, i.e. $x_1$ is a modally fixed real number for $i = 1, \ldots, n$); thus the variable real number $x$ is what we simply call a real number: $x \in \mathbb{R}$—cf. (3.4). Instead in the last assertion both $t$ and $x$ are used as (modally) fixed real numbers ($t, x \in \mathbb{R}$); $x$ varies only in that various values (in $\mathbb{R}$) are naturally assigned to the variable $t$.

Incidentally, remembering the semantics of $ML'$ or $ML^*$, from [3] or [5] respectively, we see that the casual variable $x$ varies in connection with one given value-assignment $\mathcal{U}$ to variables, in that $\mathcal{U}$ assigns $x$ an intension or quasi-intension, and the extension $\xi(\gamma)$ of $x$ in the elementary (possible) case $\gamma$ varies when $\gamma$ describes the class $\Gamma$ of these cases. In the second case, which can be dealt with e.g. the extensional semantics presented in [11], the values of $x = (\sin \omega t)$ varies in that it takes different values in correspondence with different choices of $\mathcal{U}$ that assign $t$ values in $\mathbb{R}$ (that exhaust $\mathbb{R}$) and a same value to $\omega$. Let us add that in the first case $\gamma$ varies in the same way as $t$ and $x$ do in the second one; but $\gamma$ is a metalinguistic variable, whereas $t$ and $x$ belong to the object language.

In order to define (physical or) random variables in $MC^*$ we first consider a particular case: let the proposition (trial) $\alpha$ necessarily imply exactly one among the propositions (events) $\alpha_1$ to $\alpha_n$; and let $\alpha_i$ have the probability $p_i$ with respect to $\alpha$ ($i = 1, \ldots, n$). In this case the events $\alpha_1$ to $\alpha_n$ will be said to determine a probability space relative to the trial $\alpha$; and this will be expressed in $MC^*$ by $DPrS(\alpha, \alpha_1, \ldots, \alpha_n)$:

$$DPrS(\alpha, \alpha_1, \ldots, \alpha_n) \equiv_D$$

$$\equiv_D \left( \alpha \supset \bigvee_{i=1}^n \alpha_i \right) \land \left( \bigwedge_{i=1}^{n-1} \left( \alpha \cdot \supset \bigwedge_{j=i+1}^n \left( \alpha_i \supset \sim \alpha_j \right) \land \land_{i=1}^n \alpha \cdot \supset \alpha_i \cdot (\alpha \equiv \alpha_i) \right) \right).$$

Now we can define (real) physical (or casual) random variables associated with the events $\alpha_1$ to $\alpha_n$ and trial $\alpha$ ($\text{ThRV}$):

$$\Delta \in \text{ThRV}_{\alpha, \alpha_1, \ldots, \alpha_n} \equiv_D$$

$$\equiv_D DPrS(\alpha, \alpha_1, \ldots, \alpha_n)(\alpha \supset \Delta \in \mathbb{R}(\alpha)) \land \left( \exists x_i \in \mathbb{R}(\alpha_1 \supset \Delta = x_i) \right)$$

where $x_1$ to $x_n$ are distinct variables that do not occur in $\Delta$. 
Obviously (13.2) is a definition scheme; and on its basis we cannot define physical random variables but only those of them that are capable of at most \( n \) values (\( \mathcal{HRV}_n, \; n = 1, 2, \ldots \)).

\[
(13.3) \quad \Delta \in \mathcal{HRV}_n \equiv_D (\exists \alpha, \alpha_1, \ldots, \alpha_n) \Delta \in \mathcal{HRV}_{\alpha, \alpha_1, \ldots, \alpha_n}.
\]

The considerations above are tightly connected with ordinary (particular) definitions of random variables such as Castelnuovo's. Now, to generalize \( \mathcal{HRV} \) (into \( \mathcal{HRV} \) below), we define a general notion of **physical random variables relative to the trial \( \alpha \) (\( \mathcal{HRV}_\alpha \))**:

\[
(13.4) \quad \Delta \in \mathcal{HRV}_\alpha \equiv_D \\
\equiv_D \diamond \alpha \land (\alpha \supset \Delta \in \mathbb{R}^{(\omega)})(\forall a \in \mathbb{R})(\alpha \in \Delta \succ a) \quad [\succ \equiv ^\omega \succ] 
\]

that is, \( \Delta \in \mathcal{HRV}_\alpha \) iff \( \alpha \) happen, \( \Delta \) is a variable real number and for every (modally) fixed real number \( a \) the probability of the event \( \Delta > a \) relative to \( \alpha \) exists. **Physical random variables (\( \mathcal{HRV} \))** are defined by

\[
(13.5) \quad \Delta \in \mathcal{HRV} \equiv_D (\exists \alpha) \Delta \in \mathcal{HRV}_\alpha.
\]

14. **A second notion of random variable.**

For instance Daboni says—cf. [7], p. 54—that a random number « numero aleatorio »—\( x \) is a \( P \)-measurable real-valued function defined on a partition \( \Omega \) on which a probability distribution \( P \) has been assigned.

In connection with the propositions \( \alpha \) and \( \alpha_1 \) to \( \alpha_n \) considered in N. 13 we can intuitively regard the set \( \mathcal{N}_n = \{1, \ldots, n\} \) formed with the indices in \( \alpha_1 \) to \( \alpha_n \), as representing a probability space to be associated with (\( \alpha \) and) \( \alpha_1 \) to \( \alpha_n \). For \( A \subseteq \mathcal{N}_n \), i.e. for \( A \in \mathcal{A} \) where \( \mathcal{A} \) is the class \( \mathcal{SN}_n \) of the subsets of \( \mathcal{N}_n \), let us identify \( P(A) \) with the sum of the \( p_i \)'s with \( i \in A \). Thus an instance of the afore-mentioned probability distribution is obtained.

Now let \( f \) be any real valued function defined on \( \mathcal{N}_n \), hence it is \( P \)-measurable. It is a random variable according to e.g. Daboni's definition above. We can say that \( f \) is a (non standard) absolute random variable relative to the trial \( \alpha \) and the representation \( r \rightarrow \alpha_r \) of the events \( \alpha_1 \).
Setting $x_i = f(i)$, we have a variable number called a random variable by Dore (1).

Remark that the mathematical function $f$ corresponds naturally to—or represents—the physical random variable $x$ relative to the trial $\alpha$, that equals $f(i)$ in case $\alpha_i$ occurs, for $i = 1, \ldots, n$; formally

\begin{equation}
(14.1) \quad x = \{ f \in \mathbb{R}^N \mid \forall i \leq n \land \alpha_i \supseteq x = f(i) \}.
\end{equation}

Obviously, for instance, any permutation $s \rightarrow \rho_s$ of $N^n$ changes $r \rightarrow \alpha_r$ into another representation $s \rightarrow \alpha_{\rho_s}$ of $\alpha_1$ to $\alpha_n$ on $N_n$; and it changes $f$ in the absolute random variable $f'$: $f'(s) = f(r_s)$ ($f'$ is relative to $\alpha$ and the representation $s \rightarrow \alpha_{\rho_s}$ of $\alpha_1$ to $\alpha_n$ on $N_n$). Both $f$ and $f'$ naturally correspond to the same physical random variable $x$. Thus $f$ and $f'$ may be regarded as equivalent. Even in the simple use above we have many equivalent absolute random variables—more than $n!$, cf. the last part of footnote (1). This multiplicity depends on the analogous multiplicity for what can be taken to be the probability space in the same case.

The multiplicity above increases when the probability space can be identified with $S_\alpha$, the set of n-tuples of real numbers. Therefore it is useful to fix once for all a unique standard probability space in every situation of the preceding kind. When one accepts to base the theory on nature being dealt with on the (modal) logical calculus $MC^\alpha$ or $MC^\upsilon$, a certain choice of the probability space is natural. For instance, in the aforementioned use connected with $\alpha$ and $\alpha_1$ to $\alpha_n$, we can replace $N_n$ with the indicator $\mu_\alpha$ of $\alpha$—cf. (4.9)—and $\Omega$ with the family of subsets of $\mu_\alpha$ generated by $\mu_{\alpha_1}$ to $\mu_{\alpha_n}$—cf. [7], p. 52-4.

(1) In [9] pp. 88-89 Dore says: "Il numero variabile $x_i$ che contrassegna l'evento $A_i$ ed è suscettibile di assumere uno degli $r$ valori costituenti l'insieme inerente ai fatti aleatori $A_i$, si suol designare come variabile aleatoria (o stocastica o casuale); la qualità di aleatorietà essendo appunto caratterizzata dal fatto che ad ognuno dei valori di cui essa variabile è suscettibile corrisponde un valore $P_i$ della probabilità tale che $\sum P_i = 1$.

L'insieme delle $P_i$ può considerarsi come una "distribuzione" di probabilità tra i valori $x_i$ della variabile $\nu$.

Obviously $x_i$ is a variable number in that $i$ is variable. Furthermore his probability distribution to the value $x_1$ to $x_n$ (which are fixed real numbers) shows that Dore gives $\{x_1, \ldots, x_n\}$ the role of our $N_n$. 
Then the above function $f$ has to be replaced by the mapping $\tilde{f}$ of $\mathcal{I} \mathcal{X}$ onto $\mathcal{N}_n$ such that $\tilde{f}(u) = f(r)$ for $u \in \mathcal{I} \mathcal{X}_r \ (r = 1, \ldots, n)$.

On the basis of the intuitive considerations above we can define in $MC^*$ or $MC^\nu$ (standard) absolute random variables relative to the trial $\mathcal{I}$ ($\text{ARV}_\mathcal{I}$), before defining (standard) probability space relative to $\mathcal{I}$, as follows:

(14.2) \[ f \in \text{ARV}_\mathcal{I} \equiv_D \Diamond \land f \in \mathbb{R}^\mathcal{I} \land (\forall a \in \mathbb{R}). \mathcal{I} \exists (\exists u)(|u \land f(u) > a), \]

where

(14.3) \[ f \in \mathbb{R}^\mathcal{I} \equiv_D (u)(u \in \mathcal{I} \supset f(u) \in \mathbb{R})(u \notin \mathcal{I} \supset f(u) = \Diamond \mathcal{I}). \]

Now we can define absolute random variables (ARV) by

(14.4) \[ f \in \text{ARV} \equiv_D (\exists \mathcal{I}) f \in \text{ARV}_\mathcal{I}. \]

15. A first notion of probability spaces.

If $\text{DPrS}_{\mathcal{I}_1, \ldots, \mathcal{I}_n}$ holds—cf. (13.1)—then according to the intuitive considerations on $\Omega$ made in N. 14, in any $\mathcal{L}$ of the calculi $MC^*$ and $MC^\nu$ we can define (formally) the standard (probability) space $\text{TrS}_{\mathcal{I}_1, \ldots, \mathcal{I}_n}$ associated with the trial $\mathcal{I}$ and the events $\mathcal{I}_1$ to $\mathcal{I}_n$ after [7], by

(15.1) \[ \text{TrS}_{\mathcal{I}_1, \ldots, \mathcal{I}_n} \equiv_D \{\mathcal{I}_1, \ldots, \mathcal{I}_n\}^{(x)} \]

\[ = (\lambda x) \bigvee_{i=1}^n x = \Diamond \mathcal{I}_i—\text{cf. [3], p. 68}. \]

Incidently it is useful to mean trial simply as a proposition that can be true. However we think it convenient to define notions involving a trial $\mathcal{I}$ in such a way that they are meaningful (but uninteresting) also in case $\mathcal{I}$ cannot occur. The analogue holds with other definitions such as (15.3) below.

In order to generalize this notion into a general notion of standard probability space associated with the trial or proposition $\mathcal{I}$ ($\text{TrSp}_\mathcal{I}$) we first remark that

(15.2) \[ \Diamond \mathcal{I} \mathcal{U} = \Diamond \mathcal{I} \supset \mathcal{I} \equiv \Diamond \text{prop}_{\mathcal{I} \mathcal{U}}. \]
where

\[ (15.3) \quad \text{prop}_{\mathcal{U}} \equiv_{D} \text{prop}(\mathcal{U}) \equiv_{D} (\exists u).u \in \mathcal{U} \land |_{u}, \]

which, among other things, shows how variables restricted to El can be used also in \( \text{MC}^{c} \), as propositional variables. Then we introduce (in \( \mathcal{L} \)) the class \( \text{PrS}_{\alpha} \) of probability \( \alpha \)-subranges, which are the ranges for propositions that have a probability relative to the proposition (trial) \( \alpha \):

\[ (15.4) \quad \text{PrS}_{\alpha} \equiv_{D} (\lambda \mathcal{U})(\mathcal{U} \subseteq \text{El})(\alpha \in \text{prop}_{\mathcal{U}}). \]

Now we can define in \( \mathcal{L} \), in the usual extensional way, when a family \( \mathcal{A} \) of subsets of a set \( S \) is an (extensional) Boolean Algebra [or \( \sigma \)-Algebra] on \( S = \bigcup \mathcal{A} \) (BoolAlg [Bool \( \sigma \)-Alg]). In case \( S \), \( \mathcal{A} \), and the elements of \( \mathcal{A} \) are absolute, we call \( \mathcal{A} \) absolute (\( \mathcal{A} \) BoolAlg [\( \mathcal{A} \) Bool \( \sigma \)-Alg]). Incidentally, by Theor. 41.1 (V) in [3], \( \neg F \subseteq G \land F \in \text{MConst} \land \land G \in \text{Abs} \supset F \in \text{Abs} \), we can write

\[ (15.5) \quad \mathcal{A} \subseteq \mathcal{A} \text{ BoolAlg} \equiv_{D} \emptyset \in \mathcal{A} \land \bigcup \mathcal{A} \in \mathcal{A} \land (\mathcal{A}, \bigcup \mathcal{A} \in \text{Abs}) \land (\forall F, G). \]

\[ \ldots (F, G \in \mathcal{A}) \supset F \cap G \in \mathcal{A} \land \bigcup (\mathcal{A}) \rightarrow F \in \mathcal{A} \land F \in \text{MConst} \cdot \]

\[ (15.6) \quad \mathcal{A} \subseteq \mathcal{A} \text{ Bool } \sigma \text{-Alg} \equiv_{D} \mathcal{A} \subseteq \mathcal{A} \text{ BoolAlg} \land (\forall f \in \mathcal{A}^{N}) \supset \bigcup_{n \in \mathbb{N}} f(n) \in \mathcal{A}. \]

In connection with \( \text{MC}^{c} \) or \( \text{MC}^{r} \) it is natural to define the standard probability spaces associated with a proposition \( \alpha \) to be the absolute Boolean \( \sigma \)-Algebras on \( \text{PrS}_{\alpha} \)

\[ (15.7) \quad \Sigma \in \text{PrSp}_{\alpha} \equiv_{D} \Sigma \subseteq \mathcal{A} \text{ BoolAlg} \land \Sigma \subseteq S^{(\text{mo})} \text{PrS}_{\alpha} \]

where \( S^{(\text{mo})} \), modally constant subset, is defined—cf. [4] p. 62—by

\[ (15.8) \quad S^{(\text{mo})} F \equiv_{D} (\lambda G).G \subseteq F \land G \in \text{MConst}. \]

Remark that for every proposition \( \alpha \)—e.g. \( \text{prop}_{\mathcal{U}} \) see (15.3), (where \( \mathcal{U} \subseteq \text{El} \))—we have

\[ (15.9) \quad \vdash \mathcal{A}_{\alpha} = \hat{\{\alpha, \emptyset\}} \supset \mathcal{A}_{\alpha} \in \text{PrSp}_{\alpha} \]

(and \( \hat{\alpha} \) is \( \emptyset \) in case \( \alpha \) cannot occur).
Indeed $\models \mathfrak{S}_{a,a} = 1$, and $\mathfrak{S}_{a,a \land \neg a} = 0$ by (6.1) and (10.8), and $\models \mathcal{A}_a \in \text{Bool } \sigma$-Alg. Thus, for every $\alpha$, $\text{PrSp}_\alpha$ is non-empty. However $\text{PrSp}_\alpha$ might contain only the trivial probability space $\mathcal{A}_a$ above. It is logically possible that this holds for every $\alpha$. However in the next sections we prove that some non-trivial probability spaces exist in connection with proposition $\alpha$ for which $\text{PrSp}_\alpha$ is non trivial in a suitable sense—see below (16.1).

16. Towards an existence theorem for probability spaces.

In the next section we prove a non-trivial existence theorem for probability spaces, which, so to say, is a bridge between geometrical theories of probability and existential ones—cf. N. 12. In this section we state some preliminaries for the goal above. First we introduce the class $\mathcal{FPrS}_\alpha$ of $\cap$-closed families of probability $\alpha$-subranges of $\text{PrSp}_\alpha$:

\[
\mathcal{FPrS}_\alpha \equiv (\lambda \mathcal{G}). \mathcal{G} \in S(\text{inc}) \text{PrSp}_\alpha(\forall \mathcal{U}, \mathcal{U} \in \mathcal{G}) \mathcal{U} \cap \mathcal{U} \in \mathcal{G}
\]

Hence if $\mathcal{G} \in \mathcal{FPrS}_\alpha$ and $\alpha, \beta \in \mathcal{G}$, then $\nu(\alpha \land \beta) \in \mathcal{G}$, which justifies the notation $\mathcal{FPrS}_\alpha$. We can say that $\text{PrSp}_\alpha$ is trivial when $F \in \mathcal{FPrS}_\alpha$ yields $F = \alpha$ or $F = \emptyset$.

The Boolean Algebra $\overline{\mathcal{G}}$ generated by $\mathcal{G}$ can be easily defined in $\mathcal{L}$. For the sake of brevity we don’t make this definition explicit. However we write the theorem in $\mathcal{MC}'$:

\[
\models \mathcal{G} \in \mathcal{FPrS}_\alpha \land (\nu \beta, \nu \gamma \in \overline{\mathcal{G}}) \supset [\nu \mathcal{F}(\beta, \gamma) \in \overline{\mathcal{G}}]
\]

where $\mathcal{F}(\beta, \gamma)$ is any wff constructed by means of $\beta, \gamma, \sim, \land$, and parentheses. Its easy proof is based on (4.11)—cf. [1]. On the other hand we have

\[
\models \mathcal{G} \in \mathcal{FPrS}_\alpha \supset \overline{\mathcal{G}} \in \mathcal{FPrS}_\alpha.
\]

Indeed let $\mathcal{G} \in \mathcal{FPrS}_\alpha$ and $F \in \overline{\mathcal{G}}$. We prove that $F \in \text{PrSp}_\alpha$, so that $\overline{\mathcal{G}} \in \mathcal{FPrS}_\alpha$ (being $\mathcal{G}$ $\cap$-closed). If $F \in \overline{\mathcal{G}}$, then for some sequence $H_1, \ldots, H_n$ we have $H_n = F$ and, for $i = 1$ to $n$, either (a) $H_i \in \mathcal{G}$ or (b) $H_i = \nu x - H_i$ for some $j < i$, or (c) $H_i = H_j \cap H_s$ for some $r < i$ and $s < i$. 
Let $\mathcal{G}'$ be the set of the $H_i$'s that fulfill (a). Hence $F \in \mathcal{G}'$ and $\mathcal{G}'$ is finite, say $\mathcal{G}' = \{B_1, \ldots, B_m\}$. Define $\beta_r \equiv \text{prop}_{\alpha} (r = 1, \ldots, m)$—see (15.3). Since $B_r \in \mathcal{G}(\in F \Pr S_\alpha)$, $\mathcal{F}_{\alpha, \beta_t \wedge \ldots \wedge \beta_t}$ exists for any $l \in e \mathcal{N}_m = \{1, \ldots, m\}$ and any injection $s \rightarrow i_s$ of $N_i$ into $\mathcal{N}_m$; let it equal $\mathcal{P}_{i_1, \ldots, i_l} \cdot$

(16.4) \[ \overline{P}_{i_1, \ldots, i_l} = \mathcal{F}_{\alpha, \beta_t \wedge \ldots \wedge \beta_t} \cdot \]

Furthermore let $\beta_s'$ be either $\beta_s$ or $\sim \beta_s$ ($s = 1, \ldots, r$). If the probability exists, its value $p'$ is a well known function of the $\mathcal{P}_{i_1, \ldots, i_l}$ that can be calculated on the basis of $A5.4$ and $A5.6$.

In particular $\overline{P}'$ is independent of the propositions $\alpha, \beta_1, \ldots, \beta_n$ that fulfill (16.4). Hence by the existence rule $10.1$—cf. [2]—, (d) $\mathcal{F}_{x, \beta_t \wedge \ldots \wedge \beta_t}$ exists ($r = 1, \ldots, m$).

As is well known, $F(\in \mathcal{G}')$ has the form $C_1 \cup \ldots \cup C_h$ where $C_1$ to $C_h$ are (different and hence) disjoint sets of the form $B_t' \wedge \ldots \wedge B_t'$, where $B_t' = \beta_t$ or $\alpha \sim B_r$, ($r = 1, \ldots, m$). Therefore $F$ is the range of $\gamma_1 \lor \ldots \lor \gamma_h$, where $\gamma_s = \text{prop}(C_1) (s = 1, \ldots, h)$. Hence $\gamma_1 \wedge \ldots \wedge C = \gamma_s$ ($s = 2, \ldots, h$) so that, by $A5.6$ $\mathcal{F}_{x, \gamma_t \lor \ldots \lor \gamma_s}$ exists ($s = 1, \ldots, h$). For $s = h$ this implies that $F \in \Pr S_\alpha$. We conclude that (16.3) holds. q.e.d.

17. A non-trivial existence theorem for probability spaces.

We now briefly prove the following non-trivial existence theorem in $TP^*$ (or $TP$)

(17.1) \[ \vdash \mathcal{A} \in F \Pr S_\alpha^c \cap (\exists S) S \in \Pr S_\alpha \land \mathcal{A} \subseteq S \text{—cf. (15.7).} \]

It is an immediate consequence of the assertions

(a) any $\mathcal{A} \in F \Pr S_\alpha^c$ belongs to a maximal element of the family $F \Pr S_\alpha^c$ (ordered by set inclusion).

(b) any maximal element of $F \Pr S_\alpha^c$ is a Boolean $\sigma$-Algebra.

In order to prove them briefly, let us remember that the relation $\subseteq$ induces a partial order $\leq$ on the family $F_\alpha = F \Pr S_\alpha^c$. Let $\mathcal{F}$ be any subchain, i.e. a subset of $F_\alpha$ simply ordered and non-empty; and
set $\mathcal{K} = \bigcup \Gamma$. Since $\mathcal{F} \in \Gamma$ implies $\mathcal{F} \subseteq \text{PrS}_a$, we have $\mathcal{K} \in \text{PrS}_a$ ($\mathcal{K} \subseteq \text{PrS}_a$).

Furthermore, let $F, G \in \mathcal{K}$. Then for some elements $\mathcal{F}$ and $\mathcal{G}$ of $\Gamma$, $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Since $\Gamma$ is a chain, it is not restrictive to assume $\mathcal{F} \subseteq \mathcal{G}$ ($\in \mathcal{F} \text{PrS}_a$). Hence $F \cap G \in \mathcal{G}$ ($\subseteq \mathcal{K}$). We conclude that $\mathcal{K} \in \mathcal{F} \text{PrS}_a$.

By this result the lattice $\langle \mathcal{F} \text{PrS}_a, \subseteq \rangle$ can be said to be inductive. Then by Zorn’s lemma any $\mathcal{A} \in \mathcal{F} \text{PrS}_a$ belongs to some maximal element $\mathcal{G}$ of $\mathcal{F} \text{PrS}_a$. Hence (a) holds.

To prove (β) let $\mathcal{G}$ be any maximal element of $\mathcal{F} \text{PrS}_a$, so that $\mathcal{G} \in \mathcal{F} \text{PrS}_a$ by (16.3) and, since $\mathcal{G} \subseteq \mathcal{G}$, $\mathcal{G}(= \mathcal{G})$ is a Boolean algebra.

Now we can assume $B_i \in \mathcal{G}$ ($i = 1, 2, \ldots$). We also set briefly

$$17.2 \quad C_1 =_D B_1, \quad C_r =_D B_r \sim \bigcup_{s=1}^{r-1} B_s \quad (r = 2, 3, \ldots), \quad C_\infty =_D \bigcup_r C_r,$$

whence $\vdash C_\infty = \bigcup_r B_r$, and (2)

$$17.3 \quad \gamma_r =_D \text{prop}(C_r), \quad \gamma_\infty =_D (\exists r \in \mathbb{N}) \gamma_r.$$

Then

$$17.4 \quad \vdash \nu \gamma_r = C_r, \quad \vdash \nu \gamma_\infty = C_\infty, \quad \vdash (\forall r, s).r \leq_N s \wedge \alpha \gamma_r \wedge \gamma_s.$$

Since $\mathcal{G}$ is a Boolean algebra, $C_r \in \mathcal{G}$ ($\subseteq \mathcal{F} \text{PrS}_a$) for every $r \in \mathbb{N}$, so that by (17.4) and (16.1) $\alpha \exists \nu \gamma_r$ for some $p_r$. Then by (17.3), (17.4), (15.6), and A5.9 $\alpha \exists \nu \gamma_\infty$ for $q = \sum_{r=1}^{n} p_r$. Hence by (17.4), $C_\infty \in \mathcal{F} \text{PrS}_a$.

Now let $D \in \mathcal{G}$ and set $\delta =_D \text{prop}(D)$. Then—cf. (17.3)3

$$17.5 \quad D \cap C_\infty = \iota(\delta \wedge \gamma_\infty) = \iota(\exists r \in \mathbb{N})(\delta \wedge \gamma_r).$$

(2) To translate (17.2-3) into $\mathcal{L}$ rigorously, one can replace $B_i$ with $B(r)$, $C_r$ with $C(r)$, and (17.2) with $C = (\eta)g(1) = \wedge B(1) \wedge (\forall r \geq_N 2) g(r) = \wedge B^{1,2}(r) \sim \bigcup_{s=1}^{r-1} B(s) \wedge (x \notin N g(x) = \wedge a^*$. Now it is clear that $\gamma_r$ and $\gamma_\infty$ are two wffs of $\mathcal{L}$.

(3) Of course $\langle B_i \rangle$ stands for $B(i)$.

Likewise $\langle C \rangle$ in (17.2) has to be regarded as a functor, while $\langle \gamma \rangle$ in (17.3-4) acts as an attribute.
Since $D, C_r \in \mathcal{F} (e F PrS'_\alpha), D \cap C_r \in \mathcal{F} (r = 1, 2, \ldots)$, so that $\alpha \ni_m  \exists \delta \wedge \gamma_r$ for some $p_r$. Furthermore by (17.4)_3, we have $(\forall r, s), r < \aleph_0 s \wedge \exists \delta \gamma_r, C^\alpha_n \sim (\delta \gamma_r)$. Then by A5.9 $\exists q (\exists r (r \in \aleph_0) (\delta \wedge \gamma_r))$ for some $q$. Then by (17.5) $D \cap C_\infty \in PrS_\alpha$. We conclude that $\mathcal{G}^* \in \mathcal{F} PrS'_\alpha$, for $\mathcal{G}^* = \cap \mathcal{G} \cup \{C_\infty\}^{(\omega)}$. Since $\mathcal{G}$ is a maximal element of $\mathcal{F} PrS'_\alpha$, we have $C_\infty \in \mathcal{G}$. Thus ($\beta$) holds. q.e.d.

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