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Rotating Drops in a Vessel.

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1. Introduction.

Many problems related to surface tension phenomena have been considered in the last years. A successful approach, based on the principle of virtual work, leads to a variational formulation of the physical problem in which a certain functional (representing the global energy of the system under consideration) has to be minimized subject to some «natural» constraints. In particular, results on existence and regularity have been obtained recently for the capillary tube and for the sessile and pendent drop (see for example [1], [2], [3], [4], [8], [9], [10], [12], [13], [15], [16], [17] and [24]).

Moreover, a problem of rotating drops related with astrophysics and nuclear physics was considered by the third author in [1] (for a numerical approach to this last problem see [2], [3], [4] and [21]).

In this paper we consider a problem which arises, for instance, in the construction of spincasting contact lenses (see [5] and [20]). We study the existence and the regularity of equilibrium configurations for a rotating incompressible fluid in an infinite vessel $\mathcal{V}$.

One would like to find the domain occupied by a body of fluid contained in $\mathcal{V}$ when it rotates with constant angular velocity.

The hypotheses are essentially on $\mathcal{V}$; we suppose that the walls of $\mathcal{V}$—that is, its boundary $\partial\mathcal{V}$—are given as the graph of a func-

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tion $\psi\colon \mathbb{R}^n \to \mathbb{R}$, where ψ goes to infinity at least as $c|y|^2$ when its argument y goes to infinity. Throughout this paper we suppose for semplicity that $\psi(y) = |y|^\alpha$; this hypothesis is not essential, the behaviour at infinity and a « trace inequality » (see (1.3)) being sufficient for the existence and interior regularity of the equilibrium configurations.

The energy of any allowable configuration is given by volume terms (gravity and kinetic forces) and surface contributions. The problem is to minimize the energy functional among all the subsets of prescribed volume.

In order to write the surface energy in a convenient way one needs a suitable expression for the surface area. That is, an expression which makes the energy functional lower semicontinuous in some topology in which one has good compactness theorems.

Our expression for the surface area of the boundary of a set is the « perimeter of a set », as introduced in [6] and [14]. It is well known that a family of sets which have perimeters bounded by a constant is compact in the L^1_{loc}-topology. Unfortunately, this is not sufficient for our purpose, since we have imposed a volume constraint on the admissible configurations.

Thus, for the existence programme we must improve the L^1_{loc}-convergence on the minimizing sequences. This is obtained by taking into account the gravity contribution as in [16].

A method introduced in [18] was already used in [1] to study the problem of a rotating drop in the space. Here we modify the method in an appropriated way to obtain the boundedness and regularity of the equilibrium configurations. It should be pointed out that the same method can actually be used to prove the existence of local minima for small angular velocity, even in the case $0 < \alpha < 2$ (where the existence of absolute minima fails to hold). In this case one should consider a definition of a local minimum analogous to that introduced in [1]: a local minimum is a configuration which minimizes with respect to « small » perturbations.

In the first section we introduce some preliminary results and notation from perimeter theory and formulate the problem in a precise way. Section two is devoted to the existence programme, while in section four we prove the boundedness and regularity of the equilibrium configurations. In section three we prove the existence of « interior » points for the equilibrium figures: this is a crucial step for the regularity programme.

1. Notations and definitions.

Let $x = (y, z) \in \mathbb{R}^n$, $y \in \mathbb{R}^{n-1}$, $z \in \mathbb{R}$

$$\psi(y) = |y|^\alpha, \quad \alpha > 0$$

$$\mathcal{V} = \{(y, z) \in \mathbb{R}^n, z > \psi(y)\}.$$

The global energy of a liquid drop E in the vessel $\mathcal{V}$ rotating around the z-axis with constant angular velocity $\sqrt{2\Omega}$ under the combined action of surface and volume forces is given by

$$\mathcal{F}_\Omega(E) = \int_{\mathcal{V}} |D\varphi_E| + \int_{\partial\mathcal{V}} \cos\theta\varphi_E \, dH_{n-1} + \int_E H(y, z)\, dy\, dz \tag{1.1}$$

where $H(y, z) = gz - \Omega|y|^2$, $g > 0$, $\Omega \geqslant 0$ are given constants and $\theta\colon \partial\mathcal{V} \to (0, \pi)$ is a continuous function. The third integral represents the contribution of energy given by gravitational and kinetic forces, while the first and second integrals represent the contribution of surface forces.

With $D\varphi_E$ we mean the distributional gradient of the characteristic function φ_E of the set E. We say that E has *finite perimeter* in an open set $D \subset \mathbb{R}^n$ if $D\varphi_E$ is a Radon vector measure with finite total variation in D.

If A is a Borel subset of D we indicate by $\int_A |D\varphi_E|$ the total variation of $D\varphi_E$ over A; when A is open we have the formula

$$\int_A |D\varphi_E| = \sup\left\{\sum_{i=1}^n \int_E D_i\psi_i(x)\, dx,\ \psi_i \in C_0^1(A),\ \sum_{i=1}^n \psi_i^2(x) \leqslant 1\right\}.$$

If the boundary ∂A of A is Lipschitz-continuous, we shall denote respectively the *inner* and *outer* traces of E on ∂A by φ_E^+ and φ_E^- (see [6], [14], [22]). We indicate with H_k the k-dimensional Hausdorff measure in $\mathbb{R}^n$, while

$$\omega_n = H_n(\{x \in \mathbb{R}^n\colon |x| \leqslant 1\}).$$

In the following, for brevity, we will write

$$|E| = H_n(E)$$

for every measurable set $E \subset \mathbb{R}^n$. We seek to minimize the energy functional (1.1) in the class

$$\mathcal{E} = \{E \subset \mho : |E| = 1\} . \tag{1.2}$$

REMARK 1. There exists a constant $\mu > 0$ such that

$$\int_{\partial\mho} \varphi_E \, dH_{n-1} \leqslant \int_{\mho} |D\varphi_E| + \mu |E| \quad \text{for every } E \subset \mho . \tag{1.3}$$

(See [8], [23]).

2. Existence results.

REMARK 2. It is very easy to see that, if $\alpha < 2$, then

$$\inf_{\mathcal{E}} \mathcal{F}_\Omega = -\infty , \qquad \forall \Omega > 0 .$$

The same is true in the case $\alpha = 2$, $g < \Omega$.

In fact, let $E_j \subset \mho$ be the ball of measure 1 tangent to $\partial\mho$ at the point $(j, 0, \ldots, j^2) \in \partial\mho$ and let r be its radious.

We have:

$$\begin{aligned} \mathcal{F}_\Omega(E_j) &= n\omega_n r^{n-1} + g\int_{E_j} z \, dy \, dz - \Omega \int_{E_j} |y|^2 \, dy \, dz \leqslant \\ &\leqslant n\omega_n r^{n-1} + g(j^2 + r)|E_j| - \Omega(j - 2r)^2 |E_j| = \\ &= n\omega_n r^{n-1} + (g - \Omega) j^2 + 4\Omega r j + gr - 4\Omega r^2 = \\ &= (g - \Omega) j^2 + 4\Omega r j + \text{constant} \end{aligned}$$

and this last quantity goes to $-\infty$ as j goes to $+\infty$.

Now, let us prove a first existence theorem:

THEOREM 1. If $\alpha = 2$, $\Omega < g$ and $0 \leqslant \theta < \theta_0 < \pi$, $\forall x \in \partial \mathcal{V}$, then there exists a set $E_\Omega \in \mathcal{E}$ minimizing $\mathcal{F}_\Omega$ in $\mathcal{E}$.

PROOF. At first, note that in this case $\mathcal{F}_\Omega$ is bounded from below in the class $\mathcal{E}$. In fact, from (1.3) it follows that

$$\mathcal{F}_\Omega(\mathcal{E}) \geqslant \int_{\mathcal{V}} |D\varphi_E| - \int_{\mathcal{V}} |D\varphi_E| - \mu|E| + g\int_E z\, dy\, dz - \Omega\int_E |y|^2\, dy\, dz \geqslant$$
$$\geqslant -\mu + g\int_E z\, dy\, dz - \Omega\int_E z\, dy\, dz = -\mu + (g - \Omega)\int_E z\, dy\, dz\,, \qquad \forall E \in \mathcal{E}\,.$$

Then

$$\mathcal{F}_\Omega(E) \geqslant -\mu + (g - \Omega)\int_E z\, dy\, dz\,, \qquad \forall E \in \mathcal{E}\,. \tag{2.1}$$

Now, let $\{E_j\}$ be a minimizing sequence, that is

$$\mathcal{F}_\Omega(E_j) \xrightarrow[j \to +\infty]{} \inf_{\mathcal{E}} \mathcal{F}_\Omega\,,$$

therefore there exists a constant C such that

$$\mathcal{F}_\Omega(E_j) \leqslant C\,, \qquad \forall_j\,.$$

Taking account of inequality (1.3) we then obtain

$$C \geqslant \mathcal{F}_\Omega(E_j) = \int_{\mathcal{V}} |D\varphi_{E_j}| + \int_{\partial\mathcal{V}} \cos\theta \varphi_{E_j}\, dH_{n-1} + \int_{E_j} H(y, z)\, dy\, dz \geqslant$$
$$\geqslant \int_{\mathcal{V}} |D\varphi_{E_j}| - |\cos\theta_0| \int_{\partial\mathcal{V}} \varphi_{E_j}\, dH_{n-1} + (g - \Omega)\int_{E_j} z\, dy\, dz \geqslant$$
$$\geqslant \int_{\mathcal{V}} |D\varphi_{E_j}| - |\cos\theta_0| \Big(\int_{\mathcal{V}} |D\varphi_{E_j}| + \mu \Big) + (g - \Omega)\int_{E_j} z\, dy\, dz =$$
$$= (1 - |\cos\theta_0|)\int_{\mathcal{V}} |D\varphi_{E_j}| - \mu|\cos\theta_0| + (g - \Omega)\int_{E_j} z\, dy\, dz$$

and therefore

$$\int_{\mho} |D\varphi_{E_j}| \leqslant \frac{C + \mu|\cos\theta_0|}{1 - |\cos\theta_0|} = C_1 \tag{2.2}$$

$$\int_{E_j} z\, dy\, dz \leqslant \frac{C + \mu|\cos\theta_0|}{g - \Omega} = C_2\,. \tag{2.3}$$

From (2.2) and a well-known compactness theorem (see [6], [14], [22]) it follows that there exist a set E_Ω and an increasing sequence $j(k)$ such that

$$E_{j(k)} \xrightarrow[k\to+\infty]{} E \tag{2.4}$$

in the $L^1_{\text{loc}}(\mho)$ topology. Now, using the inequality (2.3) we see that such a convergence takes place in the $L^1(\mho)$ sense and thus $E_\Omega \in \mathcal{E}$. The theorem follows now from the lower semicontinuity of $\mathcal{F}_\Omega$ with respect to the $L^1(\mho)$ topology.

REMARK 3. In the limit case $\alpha = 2$, $\Omega = g$ the inequality (2.2) continues to hold, as is immediatly seen. However, the inequality (2.3) fails to hold and we are not able to improve the convergence (2.4).

Let's now prove a second existence theorem:

THEOREM 2. If $\alpha > 2$ and $0 \leqslant \theta < \theta_0 < \pi$, $\forall x \in \partial\mho$, then, $\forall \Omega \geqslant 0$ there exists a set $E_\Omega \in \mathcal{E}$ minimizing $\mathcal{F}_\Omega$ in $\mathcal{E}$.

PROOF. Define $t_0 = t_0(\alpha, \Omega, g) = (g/2\Omega)^{\alpha/(2-\alpha)}$ in the case $\Omega > 0$, otherwise $t_0 = 0$.

Using (1.3) we have, for every $E \in \mathcal{E}$

$$\mathcal{F}_\Omega(E) \geqslant \int_{\mho} |D\varphi_E| - |\cos\theta_0| \int_{\partial\mho} \varphi_E\, dH_{n-1} + \int_E H(y, z)\, dy\, dz \geqslant$$

$$\geqslant \int_{\mho} |D\varphi_E| - |\cos\theta_0| \Big(\int_{\mho} |D\varphi_E| + \mu \Big) + \int_E H(y, z)\, dy\, dz =$$

$$= \int_{\mho} |D\varphi_E| - |\cos\theta_0| \Big(\int_{\mho} |D\varphi_E| + \mu \Big) + \int_{E \cap \{z \leqslant t_0\}} H(y, z)\, dy\, dz + \int_{E \cap \{z > t_0\}} H(y, z)\, dy\, dz \geqslant$$

$$\geqslant \int_{\mho} |D\varphi_E| - |\cos\theta_0| \left(\int_{\mho} |D\varphi_E| + \mu \right) + \int_{E \cap \{z \leqslant t_0\}} H(y, z)\, dy\, dz +$$

$$+ \int_{E \cap \{z > t_0\}} (gz - \Omega z^{2/\alpha})\, dy\, dz =$$

$$= \int_{\mho} |D\varphi_E| - |\cos\theta_0| \left(\int_{\mho} |D\varphi_E| + \mu \right) + \int_{E \cap \{z \leqslant t_0\}} H(y, z)\, dy\, dz +$$

$$+ \int_{E \cap \{z > t_0\}} (gz - \Omega z^{2/\alpha - 1} z)\, dy\, dz \geqslant$$

$$\geqslant \int_{\mho} |D\varphi_E| - |\cos\theta_0| \left(\int_{\mho} |D\varphi_E| + \mu \right) + \int_{E \cap \{z \leqslant t_0\}} H(y, z)\, dy\, dz +$$

$$+ \int_{E \cap \{z > t_0\}} \left(gz - \frac{g}{2} z \right) dy\, dz =$$

$$= \int_{\mho} |D\varphi_E| - |\cos\theta_0| \left(\int_{\mho} |D\varphi_E| + \mu \right) + \int_{E \cap \{z \leqslant t_0\}} H(y, z)\, dy\, dz + \frac{g}{2} \int_{E \cap \{z > t_0\}} z\, dy\, dz\,,$$

that is,

$$\mathcal{F}_\Omega(E) \geqslant \int_{\mho} |D\varphi_E| - |\cos\theta_0| \left(\int_{\mho} |D\varphi_E + \mu \right) + \int_{E \cap \{z \leqslant t_0\}} H(y, z)\, dy\, dz + \frac{g}{2} \int_{E \cap \{z > t_0\}} z\, dy\, dz\,. \tag{2.5}$$

From (2.5) we have

$$\mathcal{F}_\Omega(E) \geqslant C_3\,, \qquad \forall E \in \mathcal{E}\,,$$

where $C_3 = C_3(\theta_0, \mu, \alpha, \Omega, g)$. Therefore

$$\inf_{\mathcal{E}} \mathcal{F}_\Omega \geqslant C_3 > -\infty\,.$$

Let $\{E_j\}$ be a minimizing sequence. Then there exists a constant C

such that

$$C \geqslant \mathcal{F}_\Omega(E_j)\,, \qquad \forall j$$

and therefore, from (2.5) we obtain

$$\int_{\mathcal{V}} |D\varphi_{E_j}| \leqslant \frac{C + \mu|\cos\theta_0| - \int_{E_j\cap\{z\leqslant t_0\}} H(y, z)\, dy\, dz}{1 - |\cos\theta_0|} \leqslant C_4 \tag{2.6}$$

$$\int_{E_j\cap\{z>t_0\}} z\, dy\, dz \leqslant 2\, \frac{C + \mu|\cos\theta_0| - \int_{E_j\cap\{z\leqslant t_0\}} H(y, z)\, dy\, dz}{g} \leqslant C_5 \tag{2.7}$$

where C_4, C_5 depends on $C, \mu, \theta_0, \alpha, \Omega, g$. The theorem follows now arguing as in the proof of theorem 1.

The following sections are devoted to prove the boundedness and regularity of the sets E_Ω in theorems 1 and 2. To this aim it will be useful to prove at first the existence of interior points (in the sense of measure) for the sets E_Ω.

3. Existence of interior points for the sets E_Ω.

Let us consider the case $\alpha > 2$ (the case $\alpha = 2$, $\Omega < g$ goes on in the same way). For the sake of brevity put $E = E_\Omega$.

Choose K so that

$$|E \cap \{(y, z)\colon H(y, z) > K\}| > 0$$

$$|E \cap \{(y, z)\colon H(y, z) < K\}| > 0$$

(we claim that such a K exists because of the continuity of the function $K \to |E \cap \{(y, z)\colon H(y, z) < K\}|$), and because

$$\mathcal{V} \cap \{(y, z)\colon H(y, z) < K\} = \emptyset$$

if $K < K_0 \leqslant 0$ and

$$\lim_{K\to+\infty} |E \cap \{(y, z)\colon H(y, z) < K\}| = 1\,.$$

Hence there exists a ball $B_r(x_0) \subset \{(y, z): H(y, z) > K\}$ such that

$$|B_r(x_0) \cap E| = V_0 > 0 . \tag{3.1}$$

Let's take T such that

$$\mathcal{V}_T = \mathcal{V} \cap \{H(y, z) < T\} \supset B_r(x_0)$$

and put $\tilde{E} = \mathcal{V}_T - E$. The set $\tilde{E}$ minimizes the functional

$$\tilde{\mathcal{F}}_\Omega(F) = \int_{\mathcal{V}_T} |D\varphi_F| - \int_F H(y, z)\, dy\, dz$$

in the family of the sets $F \subset \mathcal{V}_T$ with $\varphi_F|_{\partial\mathcal{V}_T} = (1 - \varphi_E)|_{\partial\mathcal{V}_T}$ and with prescribed volume.

Let $B_\varrho(x_1) \subset \{(y, z): H(y, z) < K\}$ be such that

$$|B_\varrho(x_1)| = \omega_n \varrho^n \leqslant |B_r(x_0) - \tilde{E}| = |B_r(x_0) \cap E| = v_0 .$$

We want to prove that, if $|B_\varrho(x_1) \cap E|$ is « large enough » (with respect to $|B_\varrho(x_1)|$), then $|B_\varrho(x_1) \cap E| = |B_\varrho(x_1)|$ or, equivalently, that if $|B_\varrho(x_1) \cap \tilde{E}|$ is « small enough », then $|B_\varrho(x_1) \cap \tilde{E}| = 0$. To this aim we begin by proving an isoperimetric-type inequality. Let us begin by introducing some notations:

For t_1, t_2, t_3 such that $0 < t_1 < t_2 < t_3 < \varrho$, we put

$$(t_1, t_2) = B_{t_2}(x_1) - B_{t_1}(x_1)$$

$$(t_2, t_3) = B_{t_3}(x_1) - B_{t_2}(x_1)$$

$$\tilde{E}_1 = \tilde{E} \cap (t_1, t_2)$$

$$\tilde{E}_2 = \tilde{E} \cap (t_2, t_3)$$

$$V_1 = |\tilde{E}_1| , \qquad V_2 = |\tilde{E}_2| , \qquad V = V_1 + V_2 .$$

Assume the trace of E on $\partial B_{t_i}(x_1)$ $(i = 1, 2, 3)$ is continuous (i.e.

$\varphi_E^+ = \varphi_E^-$ on $\partial B_{t_i}(x_1)$), and define

$$m = \max_{i=1,2,3} \int_{\partial B_{t_i}} \varphi_{\tilde{E}} \, dH_{n-1} .$$

We have:

LEMMA 1. With the notations given above, there is a constant $C(n)$, depending only on the dimension n, such that

$$V_1 \wedge V_2 \leqslant C(n) \cdot m^N \tag{3.2}$$

where for short $N = n/(n-1)$ and

$$C(n) = (3n\omega_n^{-1/n}(1 - 2^{-1/n})^{-1})^{n/(n-1)} .$$

PROOF. From the isoperimetric property of the sphere (see [7], [19]), we have

$$\int_{\mathbf{R}^n} |D\varphi_{\tilde{E}_i}| = \int_{(t_i, t_{i+1})} |D\varphi_{\tilde{E}_i}| + \int_{\partial B_{t_i}(x_1)} \varphi_{\tilde{E}_i} \, dH_{n-1} + \int_{\partial B_{t_{i+1}}(x_1)} \varphi_{\tilde{E}_i} \, dH_{n-1} \geqslant \tag{3.3}$$
$$\geqslant n\omega_n^{1/n} V_i^{(n-1)/n} \qquad (i = 1, 2) .$$

Let us define a new set F in the following way:

$$F = \begin{cases} \phi & \text{in } (t_1, t_3) \\ \tilde{E} & \text{in } \mho_T - (t_1, t_3) - B_r(x_0) \\ B_{r'}(x_0) \cup \tilde{E} & \text{in } B_r(x_0) \end{cases}$$

where we have chosen r' such that $|\tilde{E}| = F$. From the minimum property of $\tilde{E}$ we have

$$\tilde{\mathcal{F}}_\Omega(\tilde{E}) \leqslant \tilde{\mathcal{F}}_\Omega(F) , \qquad \text{i.e.}$$

$$\int_{(t_1, t_3)} |D\varphi_{\tilde{E}}| + \int_{B_r(x_0)} |D\varphi_{\tilde{E}}| - \int_{\tilde{E} \cap (t_1, t_3)} H(y, z) \, dy \, dz - \int_{\tilde{E} \cap B_r(x_0)} H(y, z) \, dy \, dz \leqslant$$
$$\leqslant \int_{\partial B_{t_1}(x_1)} \varphi_{\tilde{E}} \, dH_{n-1} + \int_{\partial B_{t_3}(x_1)} \varphi_{\tilde{E}} \, dH_{n-1} + \int_{B_r(x_0)} |D\varphi_F| - \int_{F \cap B_r(x_0)} H(y, z) \, dy \, dz .$$

Assume for semplicity the trace of E on $\partial B_{r'}(x_0)$ is continuous; then, we have

$$(3.4)\qquad \int\limits_{(t_1,t_3)}|D\varphi_{\tilde{E}}| + \int\limits_{B_{r'}(x_0)}|D\varphi_{\tilde{E}}| - \int\limits_{\tilde{E}\cap(t_1,t_3)} H(y,z)\,dy\,dz - \int\limits_{\tilde{E}\cap B_{r'}(x_0)} H(y,z)\,dy\,dz \leqslant$$

$$\leqslant \int\limits_{\partial B_{t_1}(x_1)} \varphi_{\tilde{E}}\,dH_{n-1} + \int\limits_{\partial B_{t_3}(x_1)} \varphi_{\tilde{E}}\,dH_{n-1} + n\omega_n r'^{(n-1)} - \int\limits_{\partial B_{r'}(x_0)} \varphi_{\tilde{E}}\,dH_{n-1} - \int\limits_{B_{r'}(x_0)} H(y,z)\,dy\,dz\,.$$

It is easy to see that

$$\int\limits_{B_{r'}(x_0)} H(y,z)\,dy\,dz \geqslant \int\limits_{\tilde{E}\cap B_{r'}(x_0)} H(y,z)\,dy\,dz + \int\limits_{\tilde{E}\cap(t_1,t_3)} H(y,z)\,dy\,dz$$

and therefore, from (3.4) we obtain

$$(3.5)\qquad \int\limits_{(t_1,t_3)}|D\varphi_{\tilde{E}}| + \int\limits_{B_{r'}(x_0)}|D\varphi_{\tilde{E}}| \leqslant \int\limits_{\partial B_{t_1}(x_1)} \varphi_{\tilde{E}}\,dH_{n-1} + \int\limits_{\partial B_{t_3}(x_1)} \varphi_{\tilde{E}}\,dH_{n-1} + $$

$$+ n\omega_n r'^{(n-1)} - \int\limits_{\partial B_{r'}(x_0)} \varphi_{\tilde{E}}\,dH_{n-1}\,.$$

Now, remember that if $B\subset\mathbb{R}^n$ is a ball of radious R and L is a Borel subset of B, it holds

$$(3.6)\qquad \int\limits_{\partial B} \varphi_L\,dH_{n-1} \leqslant \int\limits_{B} |D\varphi_L| + \frac{n}{R}\,|L|$$

(see inequality (1.18) in [23]).

From this inequality we then obtain

$$n\omega_n r'^{(n-1)} - \int\limits_{\partial B_{r'}(x_0)} \varphi_{\tilde{E}}\,dH_{n-1} = \int\limits_{\partial B_{r'}(x_0)} \varphi_{(B_{r'}(x_0)-\tilde{E})}\,dH_{n-1} \leqslant$$

$$\leqslant \int\limits_{B_{r'}(x_0)} |D\varphi_{\tilde{E}}| + \frac{n}{r'}\,|\tilde{E}\cap(t_1,t_3)| = \int\limits_{B_{r'}(x_0)} |D\varphi_{\tilde{E}}| + \frac{n}{r'}\,V\,.$$

Recalling (3.5) we have:

$$\int_{(t_1,t_3)} |D\varphi_{\tilde{E}}| \leqslant \int_{\partial B_{t_1}(x_1)} \varphi_{\tilde{E}}\, dH_{n-1} + \int_{\partial B_{t_3}(x_1)} \varphi_{\tilde{E}}\, dH_{n-1} + \frac{n}{r'} V . \tag{3.7}$$

From $\omega_n r'^n \geqslant V$ we deduce that $nV/r' \leqslant n\omega^{1/n} V^{(n-1)/n}$, that is, combining with (3.7),

$$\int_{(t_1,t_3)} |D\varphi_{\tilde{E}}| \leqslant \int_{\partial B_{t_1}(x_1)} \varphi_{\tilde{E}}\, dH_{n-1} + \int_{\partial B_{t_3}(x_1)} \varphi_{\tilde{E}}\, dH_{n-1} + n\omega^{1/n} V^{(n-1)/n} . \tag{3.8}$$

From (3.3) and (3.8) we have

$$n\omega_n^{1/n}(V_1^{(n-1)/n} + V_2^{(n-1)/n} - V^{(n-1)/n}) \leqslant 2 \sum_{i=1}^{3} \int_{\partial B_{t_i}} \varphi_{\tilde{E}}\, dH_{n-1} . \tag{3.9}$$

Now we assume $V_1 \leqslant V_2$. Then $V_2 = SV_1$ and $V = (1 + S) V_1$ for a suitable $S \geqslant 1$; setting

$$f(s) = 1 + S^{(n-1)/n} - (1 + S)^{(n-1)/n}$$

we obtain from (3.9):

$$2 \sum_{i=1}^{3} \int_{\partial B_{t_i}} \varphi_E dH_{n-1} \geqslant n\omega_1^{n/n} V_1^{(n-1)/n} f(S) \geqslant 2n\omega_n^{1/n}(1 - 2^{-1/n}) V_1^{(n-1)/n} \tag{3.10}$$

since $\min_{S \geqslant 1} f(S) = f(1) = 2(1 - 2)^{-1/n})$.

The same relation (with V_1 replaced by V_2) holds in the opposite case $V_1 > V_2$, so that (3.10) is actually true with V_1 replaced by $\min(V_1, V_2) = V_1 \wedge V_2$ This completes the proof of Lemma 1.

We remark that the only reason to assume the continuity of the trace of E on $\partial B_{t_i}(x_1)$ $(i = 1, 2, 3)$ and on $\partial B_{r'}(x_0)$ was a simplification in the notations, as it is easily verified by considering the inner and outer traces of E.

Now, reasoning in an analogous way as in the proof of Theorem 1 in [18] and Theorem 2.3 in [1], we can prove that, if

$$\frac{|B_\varrho(x_1) \cap \tilde{E}|}{|B_\varrho(x_1)|}$$

is « small enough » (in the sense introduced in Lemma 1 in [18]), then there exists ϱ_1, $0 < \varrho_1 < \varrho$, such that

$$\int_{\partial B_{\varrho_1}(x_1)} \varphi_{\tilde{E}}\, dH_{n-1} = 0\,.$$

But this implies $\int_{B_{\varrho_1}} \varphi_{\tilde{E}}\, dx = 0$, because otherwise, if

$$F = \begin{cases} \tilde{E} & \text{in } \mho_T - B_r(x_0) - B_{\varrho_1}(x_1) \\ \phi & \text{in } B_{\varrho_1}(x_1) \\ \tilde{E} \cup B_{\bar{r}}(x_0) & \text{in } B_r(x_0) \end{cases}$$

where $\bar{r}$ is chosen in order to preserve the volume constraint $|F| = |\tilde{E}|$, we should have $\tilde{\mathcal{F}}_\Omega(F) < \tilde{\mathcal{F}}_\Omega(\tilde{E})$ (which is a contradiction).

In fact:

$$\tilde{\mathcal{F}}_\Omega(\tilde{E}) - \tilde{\mathcal{F}}_\Omega(F) \geqslant \int_{B_{\bar{r}}(x_0)} |D\varphi_{\tilde{E}}| - \int_{\tilde{E} \cap B_{\bar{r}}(x_0)} H(y, z)\, dy\, dz +$$

$$+ \int_{B_{\varrho_1}(x_1)} |D\varphi_{\tilde{E}}| - \int_{\tilde{E} \cap B_{\varrho_1}(x_1)} H(y, z)\, dy\, dz - \int_{B_{\bar{r}}(x_0)} |D\varphi_{\tilde{E}}| -$$

$$- \frac{n}{\bar{r}} |\tilde{E} \cap B_{\varrho_1}(x_1)| + \int_{B_{\bar{r}}(x_0)} H(y, z)\, dy\, dz > \int_{B_{\varrho_1}(x_1)} |D\varphi_{\tilde{E}}| - \frac{n}{\bar{r}} |\tilde{E} \cap B_{\varrho_1}(x_1)| \geqslant$$

$$\geqslant \int_{B_{\varrho_1}(x_1)} |D\varphi_{\tilde{E}}| - n\omega_n^{1/n} |\tilde{E} \cap B_{\varrho_1}(x_1)|^{(n-1)/n} =$$

$$= \int_{\mathbf{R}^n} |D\varphi_{\tilde{E} \cap B_{\varrho_1}(x_1)}| - n\omega_n^{1/n} |\tilde{E} \cap B_{\varrho_1}(x_1)|^{(n-1/n)} \geqslant 0\,.$$

Thus, we conclude that:

LEMMA 2. Let E_Ω be a set minimizing $\mathcal{F}_\Omega$ in $\mathcal{E}$ as in theorems 1, 2. Then there exists a ball $B_\delta(x) \subset \mho$ such that

$$|E_\Omega \cap B_\delta(x)| = |B_\delta(x)|\,.$$

4. Boundedness and regularity of the sets E_Ω.

Let us begin by proving a lemma analogous to Lemma 1. In this proof the existence of interior points to $E = E_\Omega$, as asserted in Lemma 2, will be useful. Let $T^* < T$ be such that

$$|E \cap (\mho - \mho_T)| < \frac{|B_\delta|}{2}$$

and such that there exists a ball $B \subset \mho_{T^*}$ of radious δ with $|B - E| \geqslant |B_\delta|/2$.

For $T \leqslant t_1 < t_2 < t_3$ put

$$E_i = E \cap (t_1, t_{i+1}) \quad \text{where } (t_1, t_{i+1}) = \mho_{t_{i+1}} - \mho_{t_i}, \quad (i = 1, 2)$$

$$V_i = |E_i|, \quad V = V_1 + V_2 .$$

Let us assume the trace of E on $\{H(y, z) = t_i\}$ $(i = 1, 2, 3)$ is continuous and define

$$m = \max_{i=1,2,3} \int_{\{H(y,z)=t_i\}} \varphi_E \, dH_{n-1} .$$

As before, we have

$$(4.1) \qquad \int_{\mathbf{R}^n} |D\varphi_{E_i}| = \int_{(t_i,t_{i+1})} |D\varphi_{E_i}| + \int_{\{H(y,z)=t_i\}} \varphi_{E_i} \, dH_{n-1} + \\ + \int_{\{H(y,z)=t_{i+1}\}} \varphi_{E_i} \, dH_{n-1} + \int_{\partial\mho} \varphi_{E_i} \, dH_{n-1} \geqslant n\omega_n^{1/n} V_i^{(n-1)/n} , \quad (i = 1, 2) .$$

Hence, from (1.3) we deduce

$$(4.2) \qquad n\omega_n^{1/n} V_i^{(n-1)/n} \leqslant 2 \left[\int_{(t_i,t_{i+1})} |D\varphi_{E_i}| + \int_{\{H(y,z)=t_i\}} \varphi_{E_i} \, dH_{n-1} + \int_{\{H(y,z)=t_{i+1}\}} \varphi_{E_i} \, dH_{n-1} \right] + \mu V_i .$$

Let us define a new set F by dropping $E_1 \cup E_2$ and adding a ball $B_\delta(\bar{x}) \subset \mho_{T^*}$ such that $|F| = |E|$ (this is possible by the previous assumptions). Therefore, we have $\mathcal{F}_\Omega(E) \leqslant \mathcal{F}_\Omega(F)$, and recalling (3.6) we obtain

$$\int_{(t_1,t_3)} |D\varphi_E| + \int_{\partial\mho \cap (r_1,t_3)} \cos\theta\varphi_E \, d\dot{H}_{n-1} + \int_{E\cap(t_1,t_3)} H(y,z)\, dy\, dz +$$
$$+ \int_{E\cap B_\delta(\bar{x})} H(y,z)\, dy\, dz \leqslant \int_{\{H(y,z)=t_1\}} \varphi_E \, dH_{n-1} +$$
$$+ \int_{\{H(y,z)=t_3\}} \varphi_E \, dH_{n-1} + \frac{n}{\delta} V + \int_{B_\delta(\bar{x})} H(y,z)\, dy\, dz$$

and, because

$$\int_{E\cap(t_1,t_3)} H(y,z)\, dy\, dz + \int_{E\cap B_\delta(\bar{x})} H(y,z)\, dy\, dz - \int_{B_\delta(\bar{x})} H(y,z)\, dy\, dz \geqslant 0$$

we have

$$\int_{(t_1,t_3)} |D\varphi_E| + \int_{\partial V\cap(t_1,t_3)} \cos\theta\varphi_E \, dH_{n-1} \leqslant \int_{\{H(y,z)=t_1\}} \varphi_E \, dH_{n-1} + \int_{\{H(y,z)=t_3\}} \varphi_E \, dH_{n-1} + \frac{n}{\delta} V . \tag{4.3}$$

From (1.3), recalling that $0 \leqslant \theta < \theta_0 < \pi$, we obtain

$$\int_{(t_1,t_3)} |D\varphi_E| \int_{\partial\mho\cap(t_1,t_3)} \cos\theta\varphi_E \, dH_{n-1} \geqslant \int_{(t_1,t_3)} |D\varphi_E| - |\cos\theta_0| \int_{\partial\mho\cap(t_1,t_3)} \varphi_E \, dH_{n-1} \geqslant \tag{4.4}$$
$$\geqslant (1 - |\cos\theta_0|) \int_{(t_1,t_3)} |D\varphi_E| - |\cos\theta_0| \Big(\int_{\{H(y,z)=t_1\}} \varphi_E \, dH_{n-1} + \int_{\{H(y,z)=t_3\}} \varphi_E \, dH_{n-1} \Big) - \mu V |\cos\theta_0|$$

and therefore, from (4.3), we have

$$(1 - |\cos\theta_0|) \int_{(t_1,t_3)} |D\varphi_E| \leqslant \tag{4.5}$$
$$\leqslant (1 + |\cos\theta_0|)] \Big[\int_{\{H(y,z)=t_1\}} \varphi_E \, dH_{n-1} + \int_{\{H(y,z)=t_3\}} \varphi_E \, dH_{n-1} \Big] + \frac{n}{\delta} V .$$

From (4.2) and (4.5) we obtain:

$$n\omega_n^{1/n}\left(V_1^{(n-1)/n}+V_2^{(n-1/n)}\right)\frac{1-|\cos\theta_0|}{2}\leqslant$$

$$\leqslant(1-|\cos\theta_0|)\left[\int_{(t_1,t_3)}|D\varphi_E|+\int_{\{H(y,z)=t_1\}}\varphi_E\,dH_{n-1}+2\int_{\{H(y,z)=t_2\}}\varphi_E\,dH_{n-1}+\int_{\{H(y,z)=t_3\}}\varphi_E\,dH_{n-1}\right]+$$

$$+\frac{(1-|\cos\theta_0|)\mu V}{2}\leqslant$$

$$\leqslant(1+|\cos\theta_0|)\left[\int_{\{H(y,z)=t_1\}}\varphi_E\,dH_{n-1}+\int_{\{H(y,z)=t_3\}}\varphi_E\,dH_{n-1}\right]+\frac{n}{\delta}V+$$

$$+(1-|\cos\theta_0|)\left[\int_{\{H(y,z)=t_1\}}\varphi_E\,dH_{n-1}+2\int_{\{H(y,z)=t_2\}}\varphi_E\,dH_{n-1}+\int_{\{H(y,z)=t_3\}}\varphi_E\,dH_{n-1}\right]+$$

$$+\frac{(1-|\cos\theta_0|)\mu V}{2}=$$

$$=2\int_{\{H(y,z)=t_1\}}\varphi_E\,dH_{n-1}+2\int_{\{H(y,z)=t_3\}}\varphi_E\,dH_{n-1}+2(1-|\cos\theta_0|)\int_{\{H(y,z)=t_2\}}\varphi_E\,dH_{n-1}+$$

$$+\left(\frac{n}{\delta}+\frac{1-|\cos\theta_0|\mu}{2}\right)V\leqslant 6m+C_5\cdot V\,,$$

that is

$$n\omega_n^{1/n}\left(V_1^{(n-1)/n}+V_2^{(n-1)/n}\right)\leqslant \tag{4.6}$$

$$\leqslant\frac{2}{1-|\cos\theta_0|}\cdot 6m+\frac{2}{1-|\cos\theta_0|}C_5\cdot V=C_6m+C_7V$$

where C_6 depends on θ_0 and C_7 depends on θ_0, δ, μ.

From (4.6) by an analogous proof as in Lemma 1, we obtain the inequality

$$V_1\wedge V_2\leqslant C(n,\theta_0)(m+KV)^N \tag{4.7}$$

where $K=K(\theta_0,\delta,\mu,n)$.

We have thus established the following

LEMMA 3. With the notations given above, we have

$$V_1\wedge V_2\leqslant C(n,\theta_0)(m+KV)^N\,,\qquad K=K(\theta_0,\delta,\mu,n)\,. \tag{4.8}$$

Consider now the strip $\mho_{T+1} - \overline{\mho}_T$. Arguing as in [18], [1], we can prove that, for T «large enough» (and therefore $|E \cap (\mho_{T+1} - \overline{\mho}_T)|$ «small enough»), there exists τ, $T < \tau < T + 1$, such that

$$\int_{\{H(y,z)=\tau\}} \varphi_E \, dH_{n-1} = 0 \,.$$

Let us replace now the set E by a set F obtained by cancelling $E \cap (\mho - \mho_\tau)$ and by adding a ball $B_\delta(x') \subset \mho_{T^*}$ (δ, T^* as in proof of Lemma 3) such that $|E| = |F|$.

Then, from the minimum property of E, and the inequalities (3.6), (1.3), we have:

$$\begin{aligned}\mathcal{F}_\Omega(E) - \mathcal{F}_\Omega(F) &\geqslant \int_{\mho - \overline{\mho}_\tau} |D\varphi_E| + \int_{\partial\mho \cap (\mho - \overline{\mho}_\tau)} \cos\theta \varphi_E \, dH_{n-1} + \\ &+ \int_{E \cap (\mho - \overline{\mho}_T)} H(y, z)\, dy\, dz - \frac{n}{\delta} V - \int_{(B_\delta(x') - E} H(y, z)\, dy\, dz \geqslant \\ &\geqslant \int_{\mho - \overline{\mho}_\tau} |D\varphi_E| - |\cos\theta_0| \int_{\mho - \overline{\mho}_\tau} |D\varphi_E| - \mu V |\cos\theta_0| - \frac{n}{\delta} V = \\ &= (1 - |\cos\theta_0|) \int_{\mho - \overline{\mho}_\tau} |D\varphi_E| - C_8 V \,,\end{aligned}$$

that is

$$\mathcal{F}_\Omega(E) - \mathcal{F}_\Omega(F) \geqslant (1 - |\cos\theta_0|) \int_{\mho - \overline{\mho}_\tau} |D\varphi_E| - C_8 V \tag{4.9}$$

and this last quantity is strictly positive if

$$V = |E \cap (\mho - \overline{\mho}_\tau)|$$

is «small enough».

In fact:

$$\int_{\mho - \overline{\mho}_\tau} |D\varphi_E| \geqslant n \omega_n^{1/n} V^{(n-1)/n} \,. \tag{4.10}$$

On the other hand

$$(4.11)\qquad \int\limits_{\{H(y,z)>\tau\}} |D\varphi_E| = \int\limits_{\mathcal{V}-\mathcal{V}_\tau} |D\varphi_E| + \int\limits_{\partial\mathcal{V}\cap\}\mathcal{V}-\mathcal{V}_\tau)} \varphi_E \, dH_{n-1} \leq 2 \int\limits_{\mathcal{V}-\mathcal{V}_\tau} |D\varphi_E| + \mu V .$$

From (4.10) and (4.11) we have

$$n\omega_n^{1/n} V^{(n-1)/n} \leq 2 \int\limits_{\mathcal{V}-\mathcal{V}_\tau} |D\varphi_E| + \mu V .$$

Hence

$$(1 - |\cos\theta_0|) \int\limits_{\mathcal{V}\cap\{z>\tau\}} |D\varphi_E| - C_8 \cdot V \geq$$

$$\geq (1 - |\cos\theta_0|) \left(\frac{n\omega_n^{1/n} V^{(n-1)/n} - \mu V}{2} \right) - C_8 \cdot V =$$

$$= \frac{1 - |\cos\theta_0|}{2} n\omega_n^{1/n} V^{(n-1)/n} - C_9 \cdot V$$

and this quantity is strictly positive for positive small V.

Thus we have the following

THEOREM 3 (Boundedness of the sets E_Ω). Let E_Ω be a set minimizing $\mathcal{F}_\Omega$ in $\mathcal{E}$ as in theorems 1, 2. Then there exists $\tau = \tau(\alpha, \Omega, g, \theta_0)$ such that

$$|E \cap (\mathcal{V} - \mathcal{V}_\tau)| = 0 .$$

Arguing now as in [18] and [1], we obtain also the following regularity theorem:

THEOREM 4. (Regularity of the sets E_Ω). Let E_Ω be a set minimizing $\mathcal{F}_\Omega$ in $\mathcal{E}$ as in theorems 1, 2. Then ∂E_Ω is an analytic $(n-1)$-dimensional manifold, except possibly for a closed singular set Σ_Ω whose Hausdorff dimension does not exceed $n - 8$.

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