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On Forced Periodic Oscillations in Dissipative Liénard Systems.

FABIO ZANOLIN (*)

1. Introduction.

In this paper we study the problem of the existence of periodic solutions for second order differential systems of the Liénard type:

$$(L) \quad x''(t) + \frac{d}{dt}\varphi(x(t)) + g(x(t)) = h(t) \quad (' = d/dt)$$

where $h(t)$ is a periodic forcing term and φ and g are gradient functions. Our main result considers the case when the amplitude of the dissipative term φ , in the direction of the restoring force g , overcomes the L^1 -norm of h .

Moreover, we suppose the field g satisfies a suitable geometric condition (see (k) below) which generalizes the usual assumptions required in the literature. Then, as a corollary, we obtain an extension to the systems of some classical and recent results like the theorems of Lefschetz [5], Reuter [14], Reissig [12], Mawhin [8], Bebernes-Martelli [1], Ward [17].

Our main tool in the proof is the following theorem by J. Mawhin [9], [10], recalled here in a simpler but less general form (adapted for second order systems) for the reader's convenience.

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LEMMA 1 ([10, Theorem 1]). *Let $f: \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ be continuous and p -periodic in t .*

(α) *Assume that for all possible p -periodic solutions $x(t)$ of class C^2 of*

$$(D_\lambda) \quad x''(t) = \lambda f(t, x(t), x'(t)), \quad \lambda \in (0, 1],$$

there exist two constants K_0 and K_1 , independent of λ and $x(t)$ such that

$$\sup_{[0, p]} |x(t)| < K_0, \quad \sup_{[0, p]} |x'(t)| < K_1.$$

(β) $\deg_B(S, \Omega, 0) \neq 0$ *where*

$$S: z \mapsto \frac{1}{p} \int_0^p f(t, z, 0) dt, \quad z \in \mathbf{R}^n$$

and $\Omega = B(0, r_0)$ for r_0 sufficiently great. Then the differential system (D_1) ($\lambda = 1$) has at least one p -periodic solution of class C^2 .

2. Notations.

In what follows, $|\cdot| = (\cdot|\cdot)^{\frac{1}{2}}$ is the euclidean norm in \mathbf{R}^n induced by the inner product $(\cdot|\cdot)$. If $u, v: \mathbf{R} \rightarrow \mathbf{R}^n$ are continuous and p -periodic functions ($p > 0$), we define $|u|_\infty = \sup_{[0, p]} |u(t)|$, $(u, v)_2 = \int_0^p (u(s)|v(s)) ds$ (the L^2 -scalar product of u and v), $|v|_q = \left(\int_0^p |v(s)|^q ds \right)^{1/q}$, $q \geq 1$ (the L^q -norm of v). Moreover, $Qu = (1/p) \int_0^p u(s) ds$ is the mean value of $u(t)$ in a period. If $x \in \mathbf{R}^n$, $r > 0$, then $B(x, r)$, $\overline{B(x, r)}$ are the open and the closed ball of center x and radius r ; the sphere S^{n-1} is the boundary of $B(0, 1)$. Finally we recall from Krasnosel'skii [4] the following definition: let $V: \mathbf{R}^n \rightarrow \mathbf{R}$ be a C^1 -function; we say that V is nondegenerate with nonzero index, shortly $\gamma(V, \infty) \neq 0$, if there exists $r_0 > 0$ such that: $\text{grad } V(x) \neq 0$ for every $|x| \geq r_0 > 0$ and $\deg_B(\text{grad } V, B(0, r), 0) \neq 0$ for every $r \geq r_0$, where $\deg_B(\cdot, \cdot, \cdot)$ is the Brouwer degree (see [15], [6]).

3. The main result and its consequences.

Let us consider the differential system

$$(L) \quad x''(t) + \frac{d}{dt}\varphi(x(t)) + g(x(t)) = h(t)$$

where it is assumed once for all:

- (i) $\varphi = \text{grad } F$, with $F: \mathbb{R}^n \rightarrow \mathbb{R}$ a C^2 -function;
- (ii) $g = \text{grad } G$, with $G: \mathbb{R}^n \rightarrow \mathbb{R}$ a C^1 -function
- (ii) $h: \mathbb{R} \rightarrow \mathbb{R}^n$ continuous and p -periodic ($p > 0$).

For any $z \in S^{n-1}$, we consider $V_z = \{w \in \mathbb{R}^n: (g(w), z) = 0\}$. Let $M > 0$ be a real number and define

$$W(M) = \{x \in \mathbb{R}^n: \overline{B(x, M)} \cap V_z \neq \emptyset, \text{ for every } z \in S^{n-1}\}.$$

Remark that $x \in W(M)$ if and only if, for every $z \in S^{n-1}$ there exist $|y_z| \leq M$ such that $(g(x + y_z), z) = 0$.

Then the following result holds true.

THEOREM 1. *Let us assume (i), (ii) and (iii). Let $a > 0$, $b \geq 0$, exist such that*

$$(j) \quad (\varphi(x)|g(x)) \geq a|g(x)| - b \quad (x \in \mathbb{R}^n)$$

or

$$(j') \quad (\varphi(x)|g(x)) \leq -a|g(x)| + b \quad (x \in \mathbb{R}^n)$$

holds. Finally, let G be nondegenerate with nonzero index and let us assume

(k) *For any $M > 0$, either $W(M)$ is bounded, or*

$$\lim_{\substack{|x| \rightarrow +\infty \\ x \in W(M)}} |g(x)| = +\infty.$$

Then equation (L) has a p -periodic solution, for any forcing term h ,

such that

$$Qh = 0 \quad \text{and} \quad \sup_t \left| \int_0^t h(s) ds \right| < a.$$

REMARK 1. It is easy to see that (k) is satisfied if

$$(s) \quad \begin{aligned} x_i g_i(x) &> 0 && \text{if } |x_i| \geq R \text{ for } i = 1, \dots, k \\ x_i g_i(x) &< 0 && \text{if } |x_i| \geq R \text{ for } i = k + 1, \dots, n \end{aligned}$$

holds, where $x = (x_1, \dots, x_n)$, $g = (g_1, \dots, g_n)$.

In fact, (s) implies that, for any M , $W(M)$ is bounded: it is sufficient to observe that the set $\{x: \bar{B}(x, M) \cap V_{e_i} \neq \emptyset, i = 1, \dots, n\}$, with $\{e_1, \dots, e_n\}$ the canonical basis in \mathbb{R}^n , is bounded by $(R + M)n^{\frac{1}{2}}$. Moreover, from known results on topological degree, (s) also implies that $\gamma(G, \infty) = \deg_B(g, B(0, r), 0) \neq 0$ for $r > Rn^{\frac{1}{2}}$ ($g = \text{grad } G$) and hence G is nondegenerate with nonzero index.

Condition (s) was used in [11], [7], [19] in order to get periodic solutions in Liénard equations.

Let there exist $N: \mathbb{R}^n \rightarrow \mathbb{R}^n$ continuous and such that $\deg_B(N, B(0, r), 0) \neq 0$, for $r \geq r_0 > 0$. If

$$(w) \quad \text{for any } M > 0, \text{ there exists } r_M > 0 \text{ such that } (g(x + y)|N(x)) > 0, \\ \text{for every } |x| \geq r_M, |y| < M,$$

holds, then it is not difficult to see that $\gamma(G, \infty) \neq 0$ and (k) is satisfied. In fact, through a homothopy, $\gamma(G, \infty) = \deg_B(N, B(0, r), 0) \neq 0$ (for r sufficiently large) and $W(M)$ is bounded because it is contained in the ball $\bar{B}(0, r_M)$. The case $N(x) = x$ (or $= -x$) has been considered in [18] for Rayleigh equations.

Another simple condition on g which ensures (k) is

$$(m) \quad \lim_{|x| \rightarrow +\infty} |g(x)| = +\infty.$$

Observe that if g is a homeomorphism, then (m) is trivially satisfied (g is proper) and G is also nondegenerate with index different from zero (see [6, Th. 3.3.3]).

Moreover, recall (Hadamard Theorem [3]) that (m), together with

$$(m') \quad G \text{ of class } C^2, \det(\text{Hess } G(x)) \neq 0 \text{ for all } x \in \mathbb{R}^n$$

implies that $g = \text{grad } G$ is a homeomorphism (diffeomorphism). Finally, recall from Krasnosel'skii [4, Lemma 6.5] that (m) together with

$$(m'') \quad \lim_{|x| \rightarrow +\infty} G(x) = +\infty \quad (-\infty),$$

implies that G is nondegenerate with nonzero index.

Actually, the degree condition on g holds true if G satisfies (m'') and has a finite number of critical points [2, Th. 2].

The following simple example shows how condition (k) can be easily proved (without make use of (s), (m), (w)).

EXAMPLE. Let $n = 2$, $g(x) = (g_1, g_2)(x) = (x_2 - q(x_1), x_1 - x_2)$, $x = (x_1, x_2)$; where q is a continuously differentiable function such that

$$(e) \quad q'(0) \neq 1;$$

$$(ee) \quad q(s) = s \quad \text{iff } s = 0.$$

Under (e) and (ee), $g = \text{grad } G$ with G nondegenerate with nonzero index and (k) holds.

PROOF. Obviously, g admits a potential G , moreover, by (ee), $g(x) = 0$ only for $x = 0$. Evaluating the jacobian of g at 0, we see that $\det g'(0) = q'(0) - 1$ and therefore (e) implies $\text{deg}(g, B(0, r), 0) \neq 0$ for every $r > 0$ (see [6, p. 3]).

Take $z_1 = (1, 1)/\sqrt{2} \in S^1$. Then $(g(x)|z_1) = 0$ iff $x_1 = q(x_1)$, iff $x_1 = 0$ (for (ee)). Hence $V_{z_1} = \{(0, x_2) : x_2 \in \mathbb{R}\}$ (x_2 - axis).

Take $z_2 = e_2 = (0, 1)$. Then $(g(x)|z_2) = 0$ iff $x_1 = x_2$ and $V_{z_2} = \{(t, t) : t \in \mathbb{R}\}$ ($x_2 = x_1$ line).

Since $w \in W(M)$ implies $\overline{B(w, M)} \cap V_{z_i} \neq \emptyset$ for $i = 1, 2$, one easily realizes that, for any M , $W(M)$ is bounded.

Observe that, in this example, (s) fails and (m) holds only if

$$\lim_{|s| \rightarrow +\infty} |q(s) - s| = +\infty$$

(which is not assumed here). Condition (w) is not so easily reached.

From Theorem 1 we have, as an immediate consequence:

COROLLARY 1. *Let (i), (ii), (iii) hold and let (s), either (w), either (m) and (m'), or (m) and (m'') be satisfied. Finally, let us assume $Qh = 0$*

and

$$(o) \quad \liminf_{|x| \rightarrow +\infty} (\varphi(x)|g(x))/|g(x)| \geq K > \sup_t \left| \int_0^t h(s) ds \right|$$

or

$$(o') \quad \limsup_{|x| \rightarrow +\infty} (\varphi(x)|g(x))/|g(x)| \leq -K < -\sup_t \left| \int_0^t h(s) ds \right|.$$

Then equation (L) has a p -periodic solution.

PROOF OF COROLLARY 1. It is easy to see that (o), (respectively (o')) implies (j) (respectively (j')) with $a = K - \varepsilon$, $\varepsilon > 0$ sufficiently small, and $b = \sup_{|x| \leq r_\varepsilon} a|g(x)| + \sup_{|x| \leq r_\varepsilon} |(\varphi(x)|g(x))|$, for a suitable r_ε depending on ε . Then, from Remark 1 and Theorem 1, the thesis follows. Q.E.D.

Theorem 1 also extends a theorem of Lefschetz [5] to the systems (see [13, Satz 5.3.4, p. 227]); in fact:

COROLLARY 2. Assume (i), (ii) and (iii). Let G be nondegenerate with nonzero index and let

$$(a') \quad \lim_{|x| \rightarrow +\infty} |g(x)|/|x| = +\infty,$$

$$(b') \quad |\varphi(x) - C \cdot g(x)| \leq B|x|, \quad \text{for } |x| \geq R,$$

with C a positive (negative) definite matrix and B, R positive constants. Then equation (L) has a p -periodic solution.

PROOF OF COROLLARY 2. Set $\tilde{G}(x) = G(x) - (Qh|x)$ and $\tilde{h}(t) = h(t) - Qh$. Then equation (L) is equivalent to $x'' + (d/dt)\varphi(x) + \tilde{g}(x) = \tilde{h}$; $\tilde{g} = \text{grad } \tilde{G}$. Clearly, \tilde{G} is nondegenerate with index $\neq 0$ ($\gamma(\tilde{G}, \infty) = \gamma(G, \infty)$) and satisfies (m) of Remark 1. Moreover, from (b') we have:

$$\begin{aligned} -|C \cdot Qh||\tilde{g}(x)| - B|x||\tilde{g}(x)| + (C \cdot \tilde{g}(x))\tilde{g}(x) &\leq (\varphi(x))\tilde{g}(x) \leq \\ &\leq (C \cdot \tilde{g}(x))\tilde{g}(x) + B|x||\tilde{g}(x)| + |C \cdot Qh||\tilde{g}(x)|; \quad \text{for } |x| \geq R \end{aligned}$$

and hence, using (a'), we easily obtain (o) or (o') of Corollary 1, with $K = +\infty$, according to the fact that C be positive or negative definite. Then we apply Theorem 1 and the thesis follows. Q.E.D.

Remark that, for the scalar case, hypothesis (a) in [13, p. 227] implies (a') and also the condition on the index of G .

In the case of the scalar differential equation

$$(L') \quad x''(t) + f(x(t)) \cdot x'(t) + g(x(t)) = h(t),$$

with $f, g, h: \mathbb{R} \rightarrow \mathbb{R}$ continuous functions and h, p -periodic in t , we immediately obtain:

COROLLARY 3 (scalar case). *Let us assume*

$$(s') \quad g(x) \cdot \text{sign } x > 0 \quad (< 0) \quad \text{for } |x| \geq d > 0,$$

$$(s'') \quad \left(\int_0^x f(t) dt \right) \cdot \text{sign } x \geq K \quad (\leq -K) \quad \text{for } |x| \geq d > 0.$$

Then the scalar Liénard equation (L') has a p -periodic solution for any h such that

$$Qh = 0 \quad \text{and} \quad \sup_t \left| \int_0^t h(s) ds \right| < K.$$

PROOF OF COROLLARY 3. Set $\varphi(x) = \int_0^x f(s) ds$ and observe that

$$(\varphi(x)|g(x)) = \varphi(x) \cdot g(x) = (\varphi(x) \text{ sign } x)(g(x) \text{ sign } x)$$

and $|g(x)| = g(x) \text{ sign } x$ (resp. $= -g(x) \text{ sign } x$) for $|x| \geq d$. Then, apply Corollary 1 with sign condition (s). Q.E.D.

REMARK 2. Using a standard perturbation argument based on Ascoli-Arzelà Theorem (see [11, Remark 1]) and the a-priori bounds reached in the proof of Theorem 1 (see the next section), it is not difficult to see that the thesis of Corollary 3 holds true even if in (s') the inequalities are not strict.

Let us observe also that, if

$$\lim_{|x| \rightarrow +\infty} \left(\int_0^x f(t) dt \right) \text{ sign } x = +\infty \quad (-\infty),$$

then no bound is required on $\left| \int_0^t h(s) ds \right|$. If, moreover,

$$\lim_{|x| \rightarrow +\infty} g(x) \cdot \text{sign } x = +\infty (-\infty),$$

then also the assumption $Qh = 0$ can be dropped. In fact, passing to the equivalent equation $x'' + f(x)x' + \tilde{g}(x) = \tilde{h}$, where $\tilde{g}(x) = g(x) - Qh$, $\tilde{h}(t) = h(t) - Qh$, we see that all the hypotheses in Corollary 3 are satisfied.

Therefore, we obtain as corollaries of our result: the theorem of Reuter [14] (see also [16, p. 509], [13, Satz 5.4.3]) in the part concerning existence of periodic solutions, as well as other classical results of the same type (see [16, p. 487-488], [13, Satz 5.3.7, p. 234; Satz 5.3.8-9, p. 236-9] and [12, Satz I]), a theorem by Mawhin [8, Th. 5.4, p. 378] when f and g are polynomials (with even and odd degree respectively) and theorems by Bebernes-Martelli [1, Th. 2, $c \neq 0$] and Ward [17].

4. Proofs.

In the proof of our main result, the following easily established inequality is used.

LEMMA 2. *Let $y = (y_1, \dots, y_n): \mathbb{R} \rightarrow \mathbb{R}^n$ be a p -periodic C^1 -function and let $t_1, \dots, t_n \in [0, p]$. Then, for every $t \in \mathbb{R}$,*

$$|y(t) - w| \leq (p/2)^{\frac{1}{2}} |y'|_2, \quad \text{where } w = \text{col}(y_i(t_i)).$$

Idea of the proof: use Jensen's Inequality on each component.

PROOF OF THEOREM 1. First of all, let us observe that

$$S(z) := \frac{1}{p} \int_0^p \left\{ -\frac{d}{dt} \varphi(z) - g(z) + h(t) \right\} dt = -g(z),$$

for all $z \in \mathbb{R}^n$. Thus, as G is nondegenerate with nonzero index, we immediately see that condition (β) of Mawhin's Lemma is satisfied.

Let $x(t)$ be a p -periodic solution of

$$(L_\lambda) \quad x''(t) + \lambda \frac{d}{dt} \varphi(x(t)) + \lambda g(x(t)) = \lambda h(t), \quad \lambda \in (0, 1].$$

Passing to the mean value in (L_λ) and recalling $Qh = 0$, we obtain (after a division by $\lambda > 0$)

$$(1) \quad \int_0^p g(x(t)) dt = 0$$

and hence it follows that

$$Z(x)(t) := \int_0^t g(x(s)) ds, \quad \text{is } p\text{-periodic.}$$

We take now the L^2 -scalar product of (L_λ) by $Z(x)$ and observe:

$$(x'', Z(x))_2 = - (g(x), x')_2 = G(x(0)) - G(x(p)) = 0,$$

$$\left(\frac{d}{dt} \varphi(x), Z(x) \right)_2 = - (\varphi(x), g(x))_2,$$

$$(g(x), Z(x))_2 = \frac{1}{2} |Z(x)(p)|^2 - \frac{1}{2} |Z(x)(0)|^2 = 0,$$

$$(h, Z(x))_2 = - (H, g(x))_2, \quad \text{where } H(t) := \int_0^t h(s) ds.$$

Thus we immediately obtain (Hölder inequality)

$$(2) \quad |(\varphi(x), g(x))_2| \leq |g(x)|_1 |H|_\infty.$$

Then, from (j) or (j') and (2), a bound for $|g(x)|_1$ is reached:

$$(3) \quad |g(x)|_1 \leq C_1 = bp/(a - |H|_\infty).$$

Moreover, from (3), it easily follows

$$(4) \quad \min_{[0, p]} |g(x(t))| \leq C_1/p.$$

Let $z \in S^{n-1}$ and let us take the inner product of (1) by z . Then, thanks to the mean value theorem, we have that $(g(x(t_z))|z) = 0$ for some $t_z \in [0, p]$.

So, for every $z \in S^{n-1}$, there exists $x_z = x(t_z) \in \mathbb{R}^n$, such that

$$(5) \quad x_z \in V_z.$$

Let us set $u_z(t) = x(t) - x_z$ and take the L^2 -scalar product of (L_λ^-) by u_z . Observe that

$$(x'', u_z)_2 = -|x'|_2^2,$$

$$\left(\frac{d}{dt}\varphi(x), u_z\right)_2 = \left(\frac{d}{dt}\varphi(x), x\right)_2 = -(\varphi(x), x')_2 = F(x(0)) - F(x(p)) = 0$$

and so, with easy computations (using Hölder inequality and $\lambda \leq 1$) one has

$$(6) \quad |x'|_2^2 = |u'_z|_2^2 \leq |g(x)|_1 |u_z|_\infty + |h|_1 |u_z|_\infty.$$

Moreover, taking into account that $|u_z|_\infty^2 \leq (p/2)|u'_z|_2^2$ (Lemma 2) we obtain also (from (6))

$$(7) \quad |u_z|_\infty \leq (p/2)(|g(x)|_1 + |h|_1).$$

Hence, from (7) and (3), we have a bound for $|u_z|_\infty$:

$$(8) \quad |u_z|_\infty = |x - x_z|_\infty \leq M,$$

with $M = (p/2)(C_1 + |h|_1)$.

Finally, from (8) and (6), also a bound for $|x'|_2$ (independent of λ and x) is given:

$$(9) \quad |x'|_2 \leq C_2.$$

Observe that (5) and (8) mean that

$$(10) \quad x(t) \in W(M) \quad \text{for every } t \in [0, p].$$

Then, condition (k), together with (4) and (10), implies that there ex-

ists a constant C_3 (independent of λ and x) such that

$$(11) \quad |x(t)| \leq C_3 \quad \text{for some } t \in [0, p].$$

From (9) and (11) we easily obtain a bound for $|x|_\infty$:

$$(12) \quad |x|_\infty < K_0.$$

From (9) and (12) and equation (L_λ) we see that $|x''|_2$ is also bounded (in virtue of the continuity of g and $\varphi' = \text{Hess } F$) and so, with easy inequality (Lemma 2), we obtain

$$(13) \quad |x'|_\infty < K_1.$$

Therefore, condition (α) in Mawhin's Lemma holds true and the thesis follows. Q.E.D.

At last, we observe that (from the proof) it is clear that the assumption $|H|_\infty < a$ can be changed into $|\bar{H}|_\infty < a$, where \bar{H} is any other primitive of h ($\bar{H}'(t) = h(t)$).

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