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Group Algebras of Abelian Groups.

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1. Introduction.

Let R be a ring and G a multiplicative group. The group ring RG is the R-algebra of all formal finite sums $\sum r_g g$, where $g \in G$ and $r_g \in R$, with component-wise addition and with multiplication defined using the group multiplication of G. A problem of some interest is that of deducing information about G from the group algebra RG. Of particular interest are conditions under which RG determines G, or equivalently, conditions under which the isomorphism of the R-algebras RG and RH implies that of the groups G and H. Our concern is with the case when R is commutative and G is abelian, and we assume this throughout the rest of this paper.

Let Z be the ring of integers, G_t the torsion subgroup of G, and G_p the p-primary part of G. The principal known results, which may be found in [MAY1] or [BERM], are these:

1) ZG determines G;
2) If F is a field, then FG determines G/G_t;
3) If F is a field of characteristic p, then $F(G/G_p)$, the divisible part of G_p, and the Ulm p-invariants of G are determined by FG.

In particular, if F is a field of characteristic p and G is a countable p-group or, more generally, a totally projective p-group, then G is determined by FG.

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There are some sobering negative results. For example, there exist countable groups G and H which are not isomorphic, yet FG and FH are isomorphic for any field F [MAY3, Example 2]. If G and H are any countable infinite p-groups and F is algebraically closed of characteristic different from p, then FG is isomorphic to FH [MAY3, Example 1, BERM].

The positive results above, except for FG determining G/G_t, follow immediately from our main theorem (Theorem 3.1). In addition there are a number of large classes of abelian groups G for which the p-socle $G[p] = \{g \in G: g^p = 1\}$, together with the p-heights of the elements of $G[p]$ as computed in G, determine G. This is expressed by saying that the valuated vector space $G[p]$ determines G. For such classes, FG determines G if it determines $G[p]$ as a valuated subgroup of G. Our main theorem addresses this issue as well.

Two significant classes of abelian groups, both containing the class of totally projective p-groups, and admitting complete sets of numerical invariants, are Warfield groups and S-groups. For such groups G, if the pertinent numerical invariants can be extracted from FG, then so can G. We do this for Warfield groups (Theorem 4.1) and for S-groups (Theorem 4.3). A byproduct of this is a new characterization of the invariants for S-groups (Theorem 4.2).

2. Valuations and other preliminaries.

A fundamental notion in our treatment is that of a *valuation* [RW]. Let S be a commutative semigroup and p a prime. Define $S^p = \{s^p: s \in S\}$, set $S_0 = S$, and for any ordinal α set $S_\alpha = \bigcap_{\beta<\alpha} (S_\beta)^p$, setting $S_\infty = \bigcap_\alpha S_\alpha$. The elements of $S_\beta \setminus S_{\beta+1}$ are the elements of *p-height* β.

The S_α satisfy:

(1) if $\alpha < \beta$, then $S_\alpha \supset S_\beta$;

(2) if α is a limit ordinal, then $S_\alpha = \bigcap_{\beta<\alpha} S_\beta$;

(3) $(S_\alpha)^p \subset S_{\alpha+1}$.

Any family of subsets of S, indexed by the ordinals and ∞, that satisfies (1), (2) and (3) is called a *p-filtration* and gives rise to a function v from S to the ordinals and ∞ by setting $v(s)$ equal to the largest α such that $s \in S_\alpha$. The function v is called a *p-valuation* on S. If v

is a p-valuation on S, then the restriction of v to any subsemigroup T of S is a p-valuation on T. If v is a p-valuation on S, and w is a p-valuation on a subsemigroup T of S, then T is *p-isotype* in S if v agrees with w on T. Note that p-height is a p-valuation; in fact it is the smallest p-valuation.

We will use these notions when S is a multiplicative abelian group G and when $S = RG$ where R is a ring of prime characteristic p. The resulting filtrations on RG and G are denoted by $\{(RG)_\alpha : \alpha\}$ and $\{G_\alpha : \alpha\}$, respectively. If G is additive we often write $p^\alpha G$ for G_α. In pertinent situations, one is able to recover from RG not just certain subgroups of G, but those subgroups along with the p-valuations on them that are inherited from G. Since in several significant cases the socle $G[p]$, together with its p-valuation as inherited from G, determines G, it is of interest to recover this more complicated structure from RG.

If S is an R-algebra, then the R-algebra maps from RG to S are in one-to-one correspondence with the group maps from G to the group of units of S. The *augmentation map* $\varepsilon: RG \to R$ is given by setting $\varepsilon(g) = 1$. If $\varphi: RG \to R$ is an R-algebra map, then there is an automorphism α of RG given by $\alpha(\sum r_g) g = \sum r_g(\varphi(g) g)$ such that $\varphi = \varepsilon\alpha$. Thus any R-algebra map from RG to R may serve in lieu of ε, so we may assume that ε can be retrieved from RG. We let M denote the kernel of ε (the augmentation ideal).

The group of units U of RG is the direct product of the group R^* of units of R, and $U' = \{u \in U : \varepsilon u = 1\}$. The group G is embedded as a subgroup of U', an element g of G corresponding to the sum $\sum r_h h$ where $r_g = 1$ and $r_h = 0$ for $h \neq g$.

3. The main theorem and its corollaries.

Our main theorem is an isomorphism between tensor products of R with certain subgroups of G, and entities which can often be retrieved from RG.

THEOREM 3.1. If $\lambda_R: R \to R$ is an injective ring homomorphism and $\lambda_G: G \to G$ is a group homomorphism, then $\lambda: RG \to RG$ defined by

$$\lambda(\sum r_g g) = \sum \lambda_R(r_g)\, \lambda_G(g)$$

is a ring homomorphism, and

$$R \otimes \operatorname{Ker} \lambda_G \cong \operatorname{Ker} \lambda / (M \operatorname{Ker} \lambda)$$

as R-modules. If R is a perfect field of characteristic p, then the isomorphism above induces, for each α, an isomorphism

$$R\otimes(\operatorname{Ker}\lambda_G)_\alpha \cong \big((\operatorname{Ker}\lambda)_\alpha + M\operatorname{Ker}\lambda\big)/(M\operatorname{Ker}\lambda)\,.$$

PROOF. Clearly λ is a ring homomorphism. Consider the map φ from $R\times\operatorname{Ker}\lambda_G$ to the R-module $\operatorname{Ker}\lambda/(M\operatorname{Ker}\lambda)$ given by taking (r,g) to $r(g-1)+M\operatorname{Ker}\lambda$. Clearly φ is linear in R, and

$$\varphi(r,gh) = r(gh-1)+M\operatorname{Ker}\lambda = r(g-1)+r(h-1)+ \\ + r(g-1)(h-1)+M\operatorname{Ker}\lambda\,.$$

Thus φ is a bilinear map, and hence induces a homomorphism φ from $R\otimes\operatorname{Ker}\lambda_G$ to $\operatorname{Ker}\lambda/(M\operatorname{Ker}\lambda)$ given by $\varphi(r\otimes g)=r(g-1)+M\operatorname{Ker}\lambda$.

Let $\sum r_g g$ be in $\operatorname{Ker}\lambda$. Let C be the set of cosets of $\operatorname{Ker}\lambda_G$ and $g_c\in c$ for each $c\in C$. Then

$$\sum r_g g = \sum_{c\in C}\sum_{g\in c} r_g g\,,$$

and since λ_R is one-to-one, $\sum_{g\in c} r_g = 0$ for each $c\in C$. Define $\psi\colon \operatorname{Ker}\lambda\to R\otimes\operatorname{Ker}\lambda_G$ by sending $\sum r_g g$ to $\sum_{c\in C}\sum_{g\in c} r_g\otimes(g-g_c)$. Since $\sum_{g\in c} r_g=0$, this is independent of the choice of g_c's. Now

$$\psi\big((h-1)\sum r_g g\big) = \psi\big(\sum r_g hg\big)-\psi\big(\sum r_g g\big) = \\ = \sum_{c\in C}\sum_{g\in c} r_g\otimes\big(h+g-(h+g_c)\big) - \sum_{c\in C}\sum_{g\in c} r_g\otimes(g-g_c) = 0\,.$$

Thus ψ induces an R-homomorphism from $M/(M\operatorname{Ker}\lambda)$ to $R\otimes\operatorname{Ker}\lambda_G$, and it is easy to check that φ and ψ are inverses of each other.

Let R be a perfect field of characteristic p. It is clear that φ induces a map from $R\otimes(\operatorname{Ker}\lambda_G)_\alpha$ into $\big((\operatorname{Ker}\lambda)_\alpha+M\operatorname{Ker}\lambda\big)/(M\operatorname{Ker}\lambda)$. An element of $(\operatorname{Ker}\lambda)_\alpha$ is a sum of elements of the form $\sum_{g\in c} r_g g$ such that $\sum_{g\in c} r_g = 0$, and $g\in G_\alpha$ whenever $r_g\neq 0$. Noting that

$$\sum r_g(gg_c^{-1}-1)-\sum r_g g = \big(\sum r_g g\big)(g_c^{-1}-1)\in M\operatorname{Ker}\lambda\,,$$

we conclude that $\varphi\big(\sum r_g(gg_c^{-1})\big)=\sum r_g g + M\operatorname{Ker}\lambda$. □

This theorem will be applied by choosing $\lambda = \varepsilon$ or $\lambda x = x^p$. Throughout, F_p will denote the field with p elements.

COROLLARY 3.2. The R-modules $R \otimes G$ and M/M^2 are isomorphic. Thus the group G can be retrieved from ZG, and if $pR = 0$, the group G/G^p can be retrieved from RG.

PROOF. Apply Theorem 3.1 with $\lambda = \varepsilon$ and use the fact that $Z \otimes G \cong G$, and the $R \otimes G \cong R \otimes G/G^p$ is a free R-module of the same rank as G/G^p if $pR = 0$. □

COROLLARY 3.3. If R is an integral domain with $pR = 0$, then the vector space $G[p]$ can be retrieved from RG.

PROOF. Application of Theorem 3.1 with $\lambda x = x^p$ yields $R \otimes G[p] \cong \operatorname{Ker} \lambda/(M \operatorname{Ker} \lambda)$. Then note that the rank of the free R-module $R \otimes G[p]$ is the dimension of the vector space $G[p]$. □

LEMMA 3.4. Let F be a perfect field of characteristic p and G an abelian group. Then $(FG)_\beta = FG_\beta$, that is, G is p-isotype in FG.

PROOF. We proceed by induction on β. The proof follows from the equations:

$$(FG)_\beta = \bigcap_{\alpha<\beta} \big((FG)_\alpha\big)^p = \bigcap_{\alpha<\beta} (FG_\alpha)^p = \bigcap_{\alpha<\beta} (FG_{\alpha+1}) = FG_\beta .$$

The first equality follows from the definition; the second is the induction hypothesis; the third uses perfectness of the field. □

COROLLARY 3.5. If F is a perfect field of characteristic p, then FG_β and $F(G/G_\beta)$ can be retrieved from FG.

PROOF. Apply Lemma 3.4 to get $(FG)_\beta = FG_\beta$. Then use the fact [PASS, Lemma 1.8] that $FG(M \cap FG_\beta)$ is the kernel of $FG \to F(G/G_\beta)$ to conclude that $FG/FG(M \cap FG_\beta) \cong F(G/G_\beta)$.

COROLLARY 3.6. If F is a field of characteristic p, then $(G_p)_\infty$ can be retrieved from FG. If $G_q^1 = 0$ when $q \neq p$, then G_∞ can be retrieved from FG.

PROOF. We may assume that F is perfect. In Corollary 3.5, we showed that FG_∞ can be obtained from FG. Next, application of Theorem 3.1 with $\alpha = \infty$ and $\lambda x = x^p$ shows that $F \otimes G_\infty[p]$ may be gotten from FG_∞. But $F \otimes G_\infty[p]$ is a vector space over F of dimension equal to the dimension of the vector space $G_\infty[p]$. Thus we may retrieve $(G_p)_\infty$ from FG. If $G_q = 0$ for $q \neq p$, we use the fact that $G_\infty/(G_\infty)_t$ can be obtained from FG_∞ together with the identity $G_\infty \cong (G_p)_\infty \oplus G_\infty/(G_\infty)_t$ to finish the proof. □

COROLLARY 3.7. If F is a perfect field of characteristic p, then $F \otimes G[p]$, as a filtered vector space with filtration given by the spaces $F \otimes G_\alpha[p]$, can be obtained from FG. Thus the Ulm p-invariants for G may be gotten from FG, and if $F = F_p$, then $G[p]$ itself can be obtained as a valuated vector space.

PROOF. By Theorem 3.1, the filtered vector space $F \otimes G[p]$ is isomorphic to the vector space $\operatorname{Ker}\lambda/(M \operatorname{Ker}\lambda)$ filtered by the subspaces $((\operatorname{Ker}\lambda)_\alpha + M \operatorname{Ker}\lambda)/(M \operatorname{Ker}\lambda)$. We get the Ulm p-invariants for G by observing that $F \otimes (G_\alpha[p]/G_{\alpha+1}[p])$ is isomorphic to $(F \otimes G_\alpha[p])/(F \otimes G_{\alpha+1}[p])$, and that $F \otimes (G_\alpha[p]/G_{\alpha+1}[p])$ has F-rank equal to the dimension of the vector space $G_\alpha[p]/G_{\alpha+1}[p]$. Finally, we use the fact that $F_p \otimes G_\alpha[p] \cong G_\alpha[p]$ to obtain $G[p]$ as a valuated vector space from F_pG. □

This last corollary raises the question as to how much information is lost in passing from the filtered F_p-space $G[p]$ to the filtered F-space $F \otimes G[p]$. The associated graded spaces are essentially the same, so we don't lose the Ulm invariants. But can we, for example, look at $F \otimes G[p]$ and tell whether or not $G[p]$ is complete?

We now look at specific classes of groups and fields for which the group algebra FG determines G.

THEOREM 3.8. If F is a field of characteristic p, and if G is a totally projective p-group, then FG determines G.

PROOF. We may assume that F is perfect. By Corollary 3.7, the Ulm p-invariants for G may be obtained from FG. Since totally projective p-groups are characterized by their Ulm p-invariants, the proof is complete. □

THEOREM 3.9. If G belongs to a class of groups which are distinguished by their valuated p-socles, then F_pG determines G. In particular, F_pG determines G when G belongs to any one of the following classes of groups:

(i) the class of direct sums of torsion-complete p-groups;

(ii) the class of p-groups G such that $p^{\omega+1}G = 0$ and $G/p^{\omega}G$ is torsion-complete;

(iii) the class of $p^{\omega+1}$-projective p-groups.

PROOF. From Corollary 3.7, $G[p]$ may be recovered as a valuated vector space from F_pG. Within each class stated in the theorem, two groups are distinguished by the valuated structure of their socles [HILL], [RICH, Corollary 1], [FUCHS2, Theorem 3]. □

A group is *p-local* if it is a module over the ring of integers localized at p.

THEOREM 3.10. The group algebra F_pG determines G when G is a direct sum of p-local algebraically compact groups.

PROOF. Let G be a direct sum of p-local algebraically compact groups. By Corollary 3.5 and Corollary 3.6 we may assume that G is reduced. Then G may be written as $A \oplus F$, where A is a direct sum $\sum A_i$ of adjusted algebraically compact groups and F is torsion-free. From Corollary 3.8, F_pG determines the structure of $G[p]$, and so of $A[p]$, as a valuated vector space. The problem here is to show that A is determined by the structure of $A[p]$ as a valuated vector space. Let $T = A_t$ and $T_i = (A_i)_t$. Then $A_i = \mathrm{Ext}\,(Q/Z, T_i) = \tilde{T}_i$ and each T_i is torsion-complete. It is clear that $T = \sum T_i$. We can recover T from $A[p]$ [HILL]. Suppose $T = \sum S_j$, where the S_j's are torsion-complete. The two decompositions $\sum T_i$ and $\sum S_j$ have isomorphic refinements, with $T_i = \sum T_{ik}$ and $S_j = \sum S_{jm}$. For each i, there exists $n(i)$ such that $p^{n(i)}T_{ij} = 0$, for all but finitely many j. Therefore, $\tilde{T}_i = \sum_k \tilde{T}_{ik}$ and similarly for the S_{jm}'s. Thus $A = \sum \tilde{T}_{ik} = \sum \tilde{T}_i = \sum \tilde{S}_j = \sum \tilde{S}_{jm}$. But there exists a one-to-one correspondence between the T_{ik}'s and the S_{jm}'s such that corresponding T_{ik}'s and S_{jm}'s are isomorphic. Therefore, $A = \sum \tilde{S}_j$.

Furthermore, since $G/G_t \cong A/G_t \oplus F$ may be gotten from F_pG, and since A/G_t is the maximal divisible subgroup of G/G_t, the group algebra F_pG determines G. □

Note that the fact that F is algebraically compact was not used, only that F was torsion-free reduced. Thus the group algebra F_pG determines G when $G = A \oplus F$, with A a direct sum of algebraically compact adjusted p-local groups and F torsion-free reduced.

4. Warfield groups.

Local Warfield groups were introduced by Warfield in [WARF1], where he gave a complete set of invariants for them. A complete and leisurely account of their theory can be found in [HRW]. These groups are distinguished by their Ulm invariants and their Warfield invariants. For our purposes, it is enough to know that if B is an isotype subgroup of A, and A/B is torsion, then A and B have the same Warfield invariants.

THEOREM 4.1. If F is a field of characteristic p and G is a p-local group, then FG determines the Warfield invariants of G. Hence, if G is a p-local Warfield group, then FG determines G.

PROOF. We may assume F is perfect. Let $U(FG)$ be the group of units of FG, and let H be a torsion-free subgroup of G such that G/H is torsion. If $x = \sum r_g g$ is in $U(FG)$, then there exists a positive integer $q = p^n$ such that $x^q \in U(FH)$. Now $U(FG) = F^* \oplus U'$, where $U' = \{u \in U(FG) \colon \varepsilon u = 1\}$. Since H is torsion-free, the units of FH are trivial, so $U(FH) \simeq F^* \oplus H$. Therefore U'/H, and hence U'/G, is torsion. But G is isotype in FG by Lemma 3.4, and so is isotype in U'. Thus G and U' have the same Warfield invariants. Corollary 3.7 guarantees that the Ulm p-invariants for G may be gotten from FG, and since U' may be gotten from FG, then FG determines G when G is a p-local Warfield group. □

Torsion subgroups of Warfield groups are called *S-groups*. An S-group G is characterized by its Ulm invariants and an invariant $k(\lambda, G)$ defined for each limit ordinal λ not cofinal with ω [WARF2, page 158]. To show that we can recover an S-group G from its group ring $F_p G$ we first need a new characterization of $k(\lambda, G)$.

THEOREM 4.2. If G is an S-group and λ is a limit ordinal that is not cofinal with ω, then $k(\lambda, G)$ is the codimension of $(G/p^\lambda G)[p]$ in its λ-completion.

PROOF. As taking λ-completions commutes with direct sums, it suffices to show that the codimension of the socle of a λ-elementary S-group H in its λ-completion is 1. Let $H = K_t$ where K is a λ-elementary balanced projective. For details, see [WARF2]. Then $K/p^\lambda K$

is simply presented, so $P = (K/p^\lambda K)[p]$ is λ-complete. But $H[p] = K[p]$ is λ-dense in P and $P/K[p] \cong p^\lambda K/p^{\lambda+1}K$ has dimension 1. □

THEOREM 4.3. If G is an S-group, then $k(\lambda, G)$ can be recovered from $F_p G$. Hence, if G is an S-group, then FG determines G.

PROOF. By Theorem 4.2 it suffices to recover the valuated vector space $(G/p^\lambda G)[p]$ for limit ordinals λ that are not cofinal with ω. □

5. Splitting.

We close on a note about the splitting out of G as a summand of the group of units of FG. A subsemigroup T of a semigroup S is *nice* if the set $\{v(st): t \in T\}$ has a maximum element for each s in S.

LEMMA 5.1. If F is a perfect field of characteristic p, then G is nice in FG.

PROOF. Let $x = \sum r_g g$. We need $\{x \cdot h: h \in G\}$ to have an element of maximum height. If the height of h is greater than the minimium of the heights of the g's, then xh has the same height as x. Therefore if one of the g's equals 1, then the height of x cannot be raised by multiplication by an element of G. Hence multiplying $x = \sum r_g g$ by any g^{-1} yields an element of maximum height in the set $\{x \cdot h: h \in G\}$, and thus G is nice in FG. □

THEOREM 5.2. Let F be a perfect field of characteristic p. If U'/G is a totally projective p-group, then G is a summand of U, In particular, if G is a countable p-group and FG is countable, then G is a summand of U.

PROOF. The group G is nice and isotype in FG, and hence in U By [FUCHS1, Theorem 81.9], G is a summand of U', and hence of U. □

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