RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

DONNA BEERS

FRED RICHMAN

ELBERT A. WALKER

Group algebras of abelian groups

Rendiconti del Seminario Matematico della Università di Padova, tome 69 (1983), p. 41-50

<http://www.numdam.org/item?id=RSMUP_1983__69__41_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 69 (1983)

Group Algebras of Abelian Groups.

DONNA BEERS - FRED RICHMAN - ELBERT A. WALKER (*)

1. Introduction.

Let R be a ring and G a multiplicative group. The group ring RG is the R-algebra of all formal finite sums $\sum r_g g$, where $g \in G$ and $r_g \in R$, with component-wise addition and with multiplication defined using the group multiplication of G. A problem of some interest is that of deducing information about G from the group algebra RG. Of particular interest are conditions under which RG determines G, or equivalently, conditions under which the isomorphism of the R-algebras RG and RH implies that of the groups G and H. Our concern is with the case when R is commutative and G is abelian, and we assume this throughout the rest of this paper.

Let Z be the ring of integers, G_t the torsion subgroup of G, and G_p the p-primary part of G. The principal known results, which may be found in [MAY1] or [BERM], are these:

1) ZG determines G;
2) If F is a field, then FG determines G/G_t;
3) If F is a field of characteristic p, then $F(G/G_p)$, the divisible part of G_p, and the Ulm p-invariants of G are determined by FG.

In particular, if F is a field of characteristic p and G is a countable p-group or, more generally, a totally projective p-group, then G is determined by FG.

(*) Indirizzo degli AA.: Wellesley College and New Mexico State University.
Research partially supported by NSF MCS-8003060.

There are some sobering negative results. For example, there exist countable groups G and H which are not isomorphic, yet FG and FH are isomorphic for any field F [MAY3, Example 2]. If G and H are any countable infinite p-groups and F is algebraically closed of characteristic different from p, then FG is isomorphic to FH [MAY3, Example 1, BERM].

The positive results above, except for FG determining G/G_t, follow immediately from our main theorem (Theorem 3.1). In addition there are a number of large classes of abelian groups G for which the p-socle $G[p] = \{g \in G: g^p = 1\}$, together with the p-heights of the elements of $G[p]$ as computed in G, determine G. This is expressed by saying that the valuated vector space $G[p]$ determines G. For such classes, FG determines G if it determines $G[p]$ as a valuated subgroup of G. Our main theorem addresses this issue as well.

Two significant classes of abelian groups, both containing the class of totally projective p-groups, and admitting complete sets of numerical invariants, are Warfield groups and S-groups. For such groups G, if the pertinent numerical invariants can be extracted from FG, then so can G. We do this for Warfield groups (Theorem 4.1) and for S-groups (Theorem 4.3). A byproduct of this is a new characterization of the invariants for S-groups (Theorem 4.2).

2. Valuations and other preliminaries.

A fundamental notion in our treatment is that of a *valuation* [RW]. Let S be a commutative semigroup and p a prime. Define $S^p = \{s^p: s \in S\}$, set $S_0 = S$, and for any ordinal α set $S_\alpha = \bigcap_{\beta<\alpha} (S_\beta)^p$, setting $S_\infty = \bigcap_\alpha S_\alpha$. The elements of $S_\beta \setminus S_{\beta+1}$ are the elements of *p-height* β.

The S_α satisfy:

(1) if $\alpha < \beta$, then $S_\alpha \supset S_\beta$;

(2) if α is a limit ordinal, then $S_\alpha = \bigcap_{\beta<\alpha} S_\beta$;

(3) $(S_\alpha)^p \subset S_{\alpha+1}$.

Any family of subsets of S, indexed by the ordinals and ∞, that satisfies (1), (2) and (3) is called a *p-filtration* and gives rise to a function v from S to the ordinals and ∞ by setting $v(s)$ equal to the largest α such that $s \in S_\alpha$. The function v is called a *p-valuation* on S. If v

is a p-valuation on S, then the restriction of v to any subsemigroup T of S is a p-valuation on T. If v is a p-valuation on S, and w is a p-valuation on a subsemigroup T of S, then T is *p-isotype* in S if v agrees with w on T. Note that p-height is a p-valuation; in fact it is the smallest p-valuation.

We will use these notions when S is a multiplicative abelian group G and when $S = RG$ where R is a ring of prime characteristic p. The resulting filtrations on RG and G are denoted by $\{(RG)_\alpha : \alpha\}$ and $\{G_\alpha : \alpha\}$, respectively. If G is additive we often write $p^\alpha G$ for G_α. In pertinent situations, one is able to recover from RG not just certain subgroups of G, but those subgroups along with the p-valuations on them that are inherited from G. Since in several significant cases the socle $G[p]$, together with its p-valuation as inherited from G, determines G, it is of interest to recover this more complicated structure from RG.

If S is an R-algebra, then the R-algebra maps from RG to S are in one-to-one correspondence with the group maps from G to the group of units of S. The *augmentation map* $\varepsilon: RG \to R$ is given by setting $\varepsilon(g) = 1$. If $\varphi: RG \to R$ is an R-algebra map, then there is an automorphism α of RG given by $\alpha(\sum r_g) g = \sum r_g(\varphi(g) g)$ such that $\varphi = \varepsilon\alpha$. Thus any R-algebra map from RG to R may serve in lieu of ε, so we may assume that ε can be retrieved from RG. We let M denote the kernel of ε (the augmentation ideal).

The group of units U of RG is the direct product of the group R^* of units of R, and $U' = \{u \in U : \varepsilon u = 1\}$. The group G is embedded as a subgroup of U', an element g of G corresponding to the sum $\sum r_h h$ where $r_g = 1$ and $r_h = 0$ for $h \neq g$.

3. The main theorem and its corollaries.

Our main theorem is an isomorphism between tensor products of R with certain subgroups of G, and entities which can often be retrieved from RG.

THEOREM 3.1. If $\lambda_R: R \to R$ is an injective ring homomorphism and $\lambda_G: G \to G$ is a group homomorphism, then $\lambda: RG \to RG$ defined by

$$\lambda(\sum r_g g) = \sum \lambda_R(r_g)\,\lambda_G(g)$$

is a ring homomorphism, and

$$R \otimes \operatorname{Ker} \lambda_G \simeq \operatorname{Ker} \lambda / (M \operatorname{Ker} \lambda)$$

as R-modules. If R is a perfect field of characteristic p, then the isomorphism above induces, for each α, an isomorphism

$$R\otimes(\operatorname{Ker}\lambda_G)_\alpha \cong \big((\operatorname{Ker}\lambda)_\alpha + M\operatorname{Ker}\lambda\big)/(M\operatorname{Ker}\lambda)\,.$$

PROOF. Clearly λ is a ring homomorphism. Consider the map φ from $R\times\operatorname{Ker}\lambda_G$ to the R-module $\operatorname{Ker}\lambda/(M\operatorname{Ker}\lambda)$ given by taking (r, g) to $r(g-1)+M\operatorname{Ker}\lambda$. Clearly φ is linear in R, and

$$\varphi(r, gh) = r(gh-1) + M\operatorname{Ker}\lambda = r(g-1) + r(h-1) + {}$$
$$+ r(g-1)(h-1) + M\operatorname{Ker}\lambda\,.$$

Thus φ is a bilinear map, and hence induces a homomorphism φ from $R\otimes\operatorname{Ker}\lambda_G$ to $\operatorname{Ker}\lambda/(M\operatorname{Ker}\lambda)$ given by $\varphi(r\otimes g) = r(g-1)+M\operatorname{Ker}\lambda$.

Let $\sum r_g g$ be in $\operatorname{Ker}\lambda$. Let C be the set of cosets of $\operatorname{Ker}\lambda_G$ and $g_c\in c$ for each $c\in C$. Then

$$\sum r_g g = \sum_{c\in C}\sum_{g\in c} r_g g\,,$$

and since λ_R is one-to-one, $\sum_{g\in c} r_g = 0$ for each $c\in C$. Define $\psi\colon \operatorname{Ker}\lambda \to R\otimes\operatorname{Ker}\lambda_G$ by sending $\sum r_g g$ to $\sum_{c\in C}\sum_{g\in c} r_g\otimes(g-g_c)$. Since $\sum_{g\in c} r_g = 0$, this is independent of the choice of g_c's. Now

$$\psi\big((h-1)\sum r_g g\big) = \psi\big(\sum r_g hg\big) - \psi\big(\sum r_g g\big) =$$
$$= \sum_{c\in C}\sum_{g\in c} r_g\otimes\big(h+g-(h+g_c)\big) - \sum_{c\in C}\sum_{g\in c} r_g\otimes(g-g_c) = 0\,.$$

Thus ψ induces an R-homomorphism from $M/(M\operatorname{Ker}\lambda)$ to $R\otimes\operatorname{Ker}\lambda_G$, and it is easy to check that φ and ψ are inverses of each other.

Let R be a perfect field of characteristic p. It is clear that φ induces a map from $R\otimes(\operatorname{Ker}\lambda_G)_\alpha$ into $\big((\operatorname{Ker}\lambda)_\alpha + M\operatorname{Ker}\lambda\big)/(M\operatorname{Ker}\lambda)$. An element of $(\operatorname{Ker}\lambda)_\alpha$ is a sum of elements of the form $\sum_{g\in c} r_g g$ such that $\sum_{g\in c} r_g = 0$, and $g\in G_\alpha$ whenever $r_g\neq 0$. Noting that

$$\sum r_g(gg_c^{-1}-1) - \sum r_g g = \big(\sum r_g g\big)(g_c^{-1}-1) \in M\operatorname{Ker}\lambda\,,$$

we conclude that $\varphi\big(\sum r_g(gg_c^{-1})\big) = \sum r_g g + M\operatorname{Ker}\lambda$. □

This theorem will be applied by choosing $\lambda = \varepsilon$ or $\lambda x = x^p$. Throughout, F_p will denote the field with p elements.

COROLLARY 3.2. The R-modules $R \otimes G$ and M/M^2 are isomorphic. Thus the group G can be retrieved from ZG, and if $pR = 0$, the group G/G^p can be retrieved from RG.

PROOF. Apply Theorem 3.1 with $\lambda = \varepsilon$ and use the fact that $Z \otimes G \cong G$, and the $R \otimes G \cong R \otimes G/G^p$ is a free R-module of the same rank as G/G^p if $pR = 0$. □

COROLLARY 3.3. If R is an integral domain with $pR = 0$, then the vector space $G[p]$ can be retrieved from RG.

PROOF. Application of Theorem 3.1 with $\lambda x = x^p$ yields $R \otimes G[p] \cong \operatorname{Ker} \lambda/(M \operatorname{Ker} \lambda)$. Then note that the rank of the free R-module $R \otimes G[p]$ is the dimension of the vector space $G[p]$. □

LEMMA 3.4. Let F be a perfect field of characteristic p and G an abelian group. Then $(FG)_\beta = FG_\beta$, that is, G is p-isotype in FG.

PROOF. We proceed by induction on β. The proof follows from the equations:

$$(FG)_\beta = \bigcap_{\alpha<\beta} \left((FG)_\alpha\right)^p = \bigcap_{\alpha<\beta} (FG_\alpha)^p = \bigcap_{\alpha<\beta} (FG_{\alpha+1}) = FG_\beta \,.$$

The first equality follows from the definition; the second is the induction hypothesis; the third uses perfectness of the field. □

COROLLARY 3.5. If F is a perfect field of characteristic p, then FG_β and $F(G/G_\beta)$ can be retrieved from FG.

PROOF. Apply Lemma 3.4 to get $(FG)_\beta = FG_\beta$. Then use the fact [PASS, Lemma 1.8] that $FG(M \cap FG_\beta)$ is the kernel of $FG \to F(G/G_\beta)$ to conclude that $FG/FG(M \cap FG_\beta) \cong F(G/G_\beta)$.

COROLLARY 3.6. If F is a field of characteristic p, then $(G_p)_\infty$ can be retrieved from FG. If $G_q^1 = 0$ when $q \neq p$, then G_∞ can be retrieved from FG.

PROOF. We may assume that F is perfect. In Corollary 3.5, we showed that FG_∞ can be obtained from FG. Next, application of Theorem 3.1 with $\alpha = \infty$ and $\lambda x = x^p$ shows that $F \otimes G_\infty[p]$ may be gotten from FG_∞. But $F\otimes G_\infty[p]$ is a vector space over F of dimension equal to the dimension of the vector space $G_\infty[p]$. Thus we may retrieve $(G_p)_\infty$ from FG. If $G_q = 0$ for $q \neq p$, we use the fact that $G_\infty/(G_\infty)_t$ can be obtained from FG_∞ together with the identity $G_\infty \cong (G_p)_\infty \oplus G_\infty/(G_\infty)_t$ to finish the proof. $\square$

COROLLARY 3.7. *If F is a perfect field of characteristic p, then $F\otimes G[p]$, as a filtered vector space with filtration given by the spaces $F\otimes G_\alpha[p]$, can be obtained from FG. Thus the Ulm p-invariants for G may be gotten from FG, and if $F = F_p$, then $G[p]$ itself can be obtained as a valuated vector space.*

PROOF. By Theorem 3.1, the filtered vector space $F\otimes G[p]$ is isomorphic to the vector space $\operatorname{Ker}\lambda/(M\operatorname{Ker}\lambda)$ filtered by the subspaces $((\operatorname{Ker}\lambda)_\alpha + M\operatorname{Ker}\lambda)/(M\operatorname{Ker}\lambda)$. We get the Ulm p-invariants for G by observing that $F\otimes(G_\alpha[p]/G_{\alpha+1}[p])$ is isomorphic to $(F\otimes G_\alpha[p])/(F\otimes G_{\alpha+1}[p])$, and that $F\otimes(G_\alpha[p]/G_{\alpha+1}[p])$ has F-rank equal to the dimension of the vector space $G_\alpha[p]/G_{\alpha+1}[p]$. Finally, we use the fact that $F_p\otimes G_\alpha[p] \cong G_\alpha[p]$ to obtain $G[p]$ as a valuated vector space from F_pG. $\square$

This last corollary raises the question as to how much information is lost in passing from the filtered F_p-space $G[p]$ to the filtered F-space $F\otimes G[p]$. The associated graded spaces are essentially the same, so we don't lose the Ulm invariants. But can we, for example, look at $F\otimes G[p]$ and tell whether or not $G[p]$ is complete?

We now look at specific classes of groups and fields for which the group algebra FG determines G.

THEOREM 3.8. *If F is a field of characteristic p, and if G is a totally projective p-group, then FG determines G.*

PROOF. We may assume that F is perfect. By Corollary 3.7, the Ulm p-invariants for G may be obtained from FG. Since totally projective p-groups are characterized by their Ulm p-invariants, the proof is complete. $\square$

THEOREM 3.9. *If G belongs to a class of groups which are distinguished by their valuated p-socles, then F_pG determines G. In particular, F_pG determines G when G belongs to any one of the following classes of groups:*

(i) *the class of direct sums of torsion-complete p-groups;*

(ii) the class of p-groups G such that $p^{\omega+1}G = 0$ and $G/p^{\omega}G$ is torsion-complete;

(iii) the class of $p^{\omega+1}$-projective p-groups.

PROOF. From Corollary 3.7, $G[p]$ may be recovered as a valuated vector space from F_pG. Within each class stated in the theorem, two groups are distinguished by the valuated structure of their socles [HILL], [RICH, Corollary 1], [FUCHS2, Theorem 3]. □

A group is *p-local* if it is a module over the ring of integers localized at p.

THEOREM 3.10. The group algebra F_pG determines G when G is a direct sum of p-local algebraically compact groups.

PROOF. Let G be a direct sum of p-local algebraically compact groups. By Corollary 3.5 and Corollary 3.6 we may assume that G is reduced. Then G may be written as $A \oplus F$, where A is a direct sum $\sum A_i$ of adjusted algebraically compact groups and F is torsion-free. From Corollary 3.8, F_pG determines the structure of $G[p]$, and so of $A[p]$, as a valuated vector space. The problem here is to show that A is determined by the structure of $A[p]$ as a valuated vector space. Let $T = A_t$ and $T_i = (A_i)_t$. Then $A_i = \mathrm{Ext}\,(Q/Z, T_i) = \tilde{T}_i$ and each T_i is torsion-complete. It is clear that $T = \sum T_i$. We can recover T from $A[p]$ [HILL]. Suppose $T = \sum S_j$, where the S_j's are torsion-complete. The two decompositions $\sum T_i$ and $\sum S_j$ have isomorphic refinements, with $T_i = \sum T_{ik}$ and $S_j = \sum S_{jm}$. For each i, there exists $n(i)$ such that $p^{n(i)}T_{ij} = 0$, for all but finitely many j. Therefore, $\tilde{T}_i = \sum_k \tilde{T}_{ik}$ and similarly for the S_{jm}'s. Thus $A = \sum \tilde{T}_{ik} = \sum \tilde{T}_i = \sum \tilde{S}_j = \sum \tilde{S}_{jm}$. But there exists a one-to-one correspondence between the T_{ik}'s and the S_{jm}'s such that corresponding T_{ik}'s and S_{jm}'s are isomorphic. Therefore, $A = \sum \tilde{S}_j$.

Furthermore, since $G/G_t \cong A/G_t \oplus F$ may be gotten from F_pG, and since A/G_t is the maximal divisible subgroup of G/G_t, the group algebra F_pG determines G. □

Note that the fact that F is algebraically compact was not used, only that F was torsion-free reduced. Thus the group algebra F_pG determines G when $G = A \oplus F$, with A a direct sum of algebraically compact adjusted p-local groups and F torsion-free reduced.

4. Warfield groups.

Local Warfield groups were introduced by Warfield in [WARF1], where he gave a complete set of invariants for them. A complete and leisurely account of their theory can be found in [HRW]. These groups are distinguished by their Ulm invariants and their Warfield invariants. For our purposes, it is enough to know that if B is an isotype subgroup of A, and A/B is torsion, then A and B have the same Warfield invariants.

THEOREM 4.1. *If F is a field of characteristic p and G is a p-local group, then FG determines the Warfield invariants of G. Hence, if G is a p-local Warfield group, then FG determines G.*

PROOF. We may assume F is perfect. Let $U(FG)$ be the group of units of FG, and let H be a torsion-free subgroup of G such that G/H is torsion. If $x = \sum r_g g$ is in $U(FG)$, then there exists a positive integer $q = p^n$ such that $x^q \in U(FH)$. Now $U(FG) = F^* \oplus U'$, where $U' = \{u \in U(FG) : \varepsilon u = 1\}$. Since H is torsion-free, the units of FH are trivial, so $U(FH) \cong F^* \oplus H$. Therefore U'/H, and hence U'/G, is torsion. But G is isotype in FG by Lemma 3.4, and so is isotype in U'. Thus G and U' have the same Warfield invariants. Corollary 3.7 guarantees that the Ulm p-invariants for G may be gotten from FG, and since U' may be gotten from FG, then FG determines G when G is a p-local Warfield group. $\square$

Torsion subgroups of Warfield groups are called *S-groups*. An S-group G is characterized by its Ulm invariants and an invariant $k(\lambda, G)$ defined for each limit ordinal λ not cofinal with ω [WARF2, page 158]. To show that we can recover an S-group G from its group ring $F_p G$ we first need a new characterization of $k(\lambda, G)$.

THEOREM 4.2. *If G is an S-group and λ is a limit ordinal that is not cofinal with ω, then $k(\lambda, G)$ is the codimension of $(G/p^\lambda G)[p]$ in its λ-completion.*

PROOF. As taking λ-completions commutes with direct sums, it suffices to show that the codimension of the socle of a λ-elementary S-group H in its λ-completion is 1. Let $H = K_t$ where K is a λ-elementary balanced projective. For details, see [WARF2]. Then $K/p^\lambda K$

is simply presented, so $P = (K/p^\lambda K)[p]$ is λ-complete. But $H[p] = K[p]$ is λ-dense in P and $P/K[p] \simeq p^\lambda K/p^{\lambda+1}K$ has dimension 1. □

THEOREM 4.3. If G is an S-group, then $k(\lambda, G)$ can be recovered from F_pG. Hence, if G is an S-group, then FG determines G.

PROOF. By Theorem 4.2 it suffices to recover the valuated vector space $(G/p^\lambda G)[p]$ for limit ordinals λ that are not cofinal with ω. □

5. Splitting.

We close on a note about the splitting out of G as a summand of the group of units of FG. A subsemigroup T of a semigroup S is *nice* if the set $\{v(st): t \in T\}$ has a maximum element for each s in S.

LEMMA 5.1. If F is a perfect field of characteristic p, then G is nice in FG.

PROOF. Let $x = \sum r_g g$. We need $\{x \cdot h: h \in G\}$ to have an element of maximum height. If the height of h is greater than the minimium of the heights of the g's, then xh has the same height as x. Therefore if one of the g's equals 1, then the height of x cannot be raised by multiplication by an element of G. Hence multiplying $x = \sum r_g g$ by any g^{-1} yields an element of maximum height in the set $\{x \cdot h: h \in G\}$, and thus G is nice in FG. □

THEOREM 5.2. Let F be a perfect field of characteristic p. If U'/G is a totally projective p-group, then G is a summand of U, In particular, if G is a countable p-group and FG is countable, then G is a summand of U.

PROOF. The group G is nice and isotype in FG, and hence in U By [FUCHS1, Theorem 81.9], G is a summand of U', and hence of U. □

REFERENCES

[BERM] S. D. BERMAN, *Group algebras of countable abelian p-groups*, Publ. Math. Debrecen, **14** (1967), pp. 364-405.

[FUCHS1] L. FUCHS, *Abelian Groups*, vol. 2, Academic Press, New York (1973).

[FUCHS2] L. Fuchs - J. M. Irwin, *On $p^{\omega+1}$-projective p-groups*, Proc. London Math. Soc., **30** (1975), pp. 459-470.

[HILL] P. Hill, *A classification of direct sums of closed groups*, Acta Math. Acad. Sci. Hungar., **17** (1966), pp. 263-266.

[HRW] R. Hunter - F. Richman - E. Walker, *Warfield Modules, Abelian Group Theory*, Proc. 2nd New Mexico State Univ. Conf. (1976), Lecture Notes in Math., vol. 616, Springer-Verlag, Berlin and New York (1977), pp. 87-123.

[MAY1] W. May - *Commutative group algebras*, Trans. Amer. Math. Soc., **136** (1969), pp. 139-149.

[MAY2] W. May - *Invariants for commutative group algebras*, Ill. J. Math., **15** (1971), pp. 525-531.

[MAY3] W. May - *Isomorphism of group algebras*, J. Alg., **40** (1976), pp. 10-18.

[PASS] D. S. Passman, *The algebraic structure of group rings*, Wiley and Sons, New York (1977).

[RICH] F. Richman, *Extensions of p-bounded groups*, Arch. der Math., **21** (1970), pp. 449-454.

[RW] F. Richman - E. A. Walker, *Valuated groups*, J. Algebra, **56** (1979), pp. 145-167.

[WARF1] R. B. Warfield Jr., *Invariants and a classification theorem for modules over a discrete valuation ring*, preprint (1971).

[WARF2] R. B. Warfield Jr., *A classification theorem for abelian p-groups*, Trans. Amer. Math. Soc., **210** (1975), pp. 149-168.

Manoscritto pervenuto in redazione il 22 settembre 1981.