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On the Uniform Paracompactness of the Product of Two Uniform Spaces.

UMBERTO MARCONI (*)

SUMMARY - A necessary and sufficient condition for the product of two metric spaces to be uniformly paracompact, recently given by A. Hohti, is proved to be necessary and sufficient when the uniform spaces are (topologically) first countable. Examples are given to show that the condition is in general not necessary. The methods used here are quite different from Hohti's..

1. Introduction.

Throughout this paper uX will denote a uniform space and X the associated topological space. If X is a uniformizable topological space, we denote by fX the uniform space equipped by the fine uniformity, in particular $f\mathbb{N}$ will denote the discrete uniformity on the discrete space of natural-numbers.

Following M. D. Rice [R], we say that a uniform space uX is uniformly paracompact if every open cover \mathcal{A} of X admits a open refinement \mathcal{B} which is uniformly locally finite that is, there exists a uniform cover \mathcal{U} such that the set $\{B \in \mathcal{B} : B \cap U \neq \emptyset\}$ is finite for every $U \in \mathcal{U}$.

In [R], it is proved that uX is uniformly paracompact if and only if for every open cover \mathcal{A} , the cover \mathcal{A}_f consisting of all the finite unions of elements of \mathcal{A} is uniform. Using this, it is easy to see that a locally

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compact uniform space is uniformly paracompact if and only if it is uniformly locally compact, that is, there exists a uniform cover made of compact sets (see [R], Thm. 4).

2. Results.

It is not difficult to extend the *sufficiency* of the conditions given in [H], Thm. 2.2.1, for uniform paracompactness of a product, to an arbitrary pair of uniform spaces.

THEOREM 1.

- i) Let uX, vY be uniformly paracompact and locally compact uniform spaces; then $uX \times vY$ is uniformly paracompact.
- ii) Let uX be a uniformly paracompact space and vY a compact space, then $uX \times vY$ is uniformly paracompact.

PROOF. i) Easy, using uniform local compactness quoted above.

ii) Let \mathcal{A} be an open cover of $X \times Y$, $\mathcal{A} = \{U_{(x,y)} \times V_{(x,y)} : (x, y) \in X \times Y\}$ where, for each $(x, y) \in X \times Y$, $U_{(x,y)}, V_{(x,y)}$ are open neighborhood of x and y respectively.

Take $\bar{x} \in X$. Since Y is compact the cover $V_{(\bar{x},y)}$ has a finite subcover, say, $\{V_{(\bar{x},y)} : y \in F_{\bar{x}}\}$ where $F_{\bar{x}}$ is a finite subset of Y . Put $U_{\bar{x}} = \bigcap_{y \in F_{\bar{x}}} U_{(\bar{x},y)}$.

Let

$$\mathfrak{U} = \{U_{\bar{x}} \times V_{(\bar{x},y)} : \bar{x} \in X, y \in F_{\bar{x}}\}.$$

Since \mathfrak{U} refines \mathcal{A} , if \mathfrak{U}_f is uniform, then \mathcal{A}_f is uniform. But \mathfrak{U}_f is refined by \mathfrak{U}_f , where $\mathfrak{U} = \{U_{\bar{x}} \times Y : \bar{x} \in X\}$, and \mathfrak{U}_f is uniform because vX is uniformly paracompact.

REMARK. Part (ii) of the theorem is proved, by another argument, in [FL] (see also [S]). There exist various examples of uniformly paracompact (even metric) spaces whose product with $f\mathbb{N}$ is not uniformly paracompact (example 1).

The product of two uniform spaces can be uniformly paracompact even if neither condition of theorem is satisfied.

In fact if X and Y are Lindelöf P -spaces, one can prove that $X \times Y$ is still a Lindelöf, therefore paracompact, P -space. Moreover $f(X \times Y) = fX \times fY$ ([I] ch. VII Thm. 35) and so $fX \times fY$ is uniformly paracompact. For instance if S is the space of [GJ], problem 4N, we have that S is not locally compact and $fS \times fS$ and $fS \times f\mathbb{N}$ are uniformly paracompact.

Later, example 2 yields a non locally compact sequential space T such that $fT \times fS$ (and consequently $fT \times f\mathbb{N}$) is uniformly paracompact.

This shows that in general the conditions of Theorem 1 fail to be necessary; however:

THEOREM 2. *Let uX, vY be uniformly paracompact spaces which are topologically first countable.*

Then $uX \times vY$ is uniformly paracompact if and only if one of the following holds:

- i) *both spaces are locally compact;*
- ii) *at least one of them is compact.*

PROOF. Sufficiency is theorem 1. Necessity is an immediate consequence of the following lemmas.

Recall that, given a point p in a topological space X , the neighborhood weight, or point character, of X at p is the smallest cardinality of a basis at p .

LEMMA 1. *Let uX, vY be uniform spaces. Assume that $p \in X$ has no totally bounded neighborhood, and that the weight of X at p is α (necessarily infinite), and that vY has a uniformly discrete subset of cardinality α . Then $uX \times vY$ is not uniformly paracompact.*

PROOF. Let $\{V_i: i \in \alpha\}$ be a basis neighborhood at p . Since V_i is not totally bounded, it contains a uniformly discrete infinite subset N_i ; we may assume $p \in N_i$. Let $\{y_i: i \in \alpha\}$ be a uniformly discrete subset of Y of cardinality α .

Put

$$F = \bigcup_{i \in \alpha} N_i \times \{y_i\}.$$

Clearly F is a topologically discrete closed subspace of $X \times Y$. Uniform paracompactness being inherited by closed subspaces, it is enough to show that F is not uniformly paracompact. We do this proving that there is no uniform cover of F consisting of finite sets. If \mathcal{U} is

an open uniform cover of $uX \times vY$, then there exists a neighborhood U of p in X such that, for every $i \in \alpha$

$$U \times \{y_i\} \cap F$$

is contained in some member of \mathcal{U} .

Hence there exists an index $j \in \alpha$ such that

$$U \times \{y_i\} \cap F \supseteq V_j \times \{y_j\} \cap F \supseteq N_j \times \{y_j\}$$

and this set is infinite.

LEMMA 2. *Let uX, vY be uniform spaces such that there exists a point $p \in X$ with no precompact neighborhood and countable point character. If $uX \times vY$ is uniformly paracompact, then vY is compact.*

PROOF. By the previous lemma, vY has no infinite uniformly discrete subset; if $uX \times vY$ is uniformly paracompact, then vY is, too; hence vY is complete, hence compact.

EXAMPLE 1. If uX is any of the following paracompact spaces (equipped with any compatible uniformity which makes it uniformly paracompact): the Sorgenfrey line, the Michael line [M], a metric hedgehog with infinitely many spines, then a product $uX \times vY$ is uniformly paracompact if and only if vY is compact: this is an obvious consequence of Lemma 2. In particular $uX \times fN$ is not uniformly paracompact.

For every infinite cardinal χ , denote by T_χ the quotient space obtained by identifying all points 0 in χ -many disjoint copies of $I = [0, 1]$; in short, T_χ is a «non uniform» hedgehog with χ spines; we denote by p the «vertex», by I_α the α -th spine, for every $\alpha \in \chi$ (thus I_α is a copy of $[0, 1]$). Clearly T_χ is sequential, paracompact, and the only point without compact neighborhoods is p ; the point character at p is clearly uncountable.

EXAMPLE 2. We prove that $fT_{\aleph_0} \times fS$ is uniformly paracompact (S is the uncountable P -space with a unique non isolated point σ , described in [GJ, 4N]). Let \mathcal{A} be an open cover of $T_{\aleph_0} \times S$,

$$\mathcal{A} = \{U_{(t,s)} \times V_{(t,s)} : t \in T, s \in S\}.$$

Using the fact that T is Lindelöf and that σ is a P -point, we find a countable open cover $\mathcal{C} = \{T_j : j \in J\}$ of T and a neighborhood V of σ such that every set of the form $T_j \times V$ is contained in some element of \mathcal{A} .

Now consider $T \times S \setminus T \times V = T \times (S \setminus V)$.

We refine \mathcal{A} by the cover

$$\mathcal{B} = \{T_j \times V, U_{(t,n)} \times \{n\} : j \in J, n \in S \setminus V\}.$$

Let U be the neighborhood of p define as follows:

$$U \cap I_\alpha = \bigcap_{n=0}^{\alpha} U_{(\rho,n)} \cap I_\alpha \quad \text{for every } \alpha \in \aleph_0.$$

Since the points of U which do not belong to $U_{(\rho,n)}$ belong at most to the first n spines, then every $U \times \{n\}$ is covered by a finite union of elements of \mathcal{B} .

Consider the open cover of T

$$\mathcal{C} = \{U, I_\alpha, \alpha \in \aleph_0\}.$$

For the compactness of $I_\alpha \cup \{p\}$ we get easily that every element of $\mathcal{C} \times \{\{n\}, n \in S \setminus V\}$ is contained in some finite union of elements of \mathcal{B} ,

If \mathcal{D} is a common refinement of \mathcal{C} and \mathcal{B} , and $\mathcal{E} = \{\{n\}, V, n \in S \setminus V\}$. then $\mathcal{D} \times \mathcal{E}$ is a uniform refinement of \mathcal{B} .

From Lemma 1 and example 2, one might conjecture that $fX \times f\mathbb{N}$ is uniformly paracompact, whenever the punctual character of the points of X without precompact neighborhood is uncountable.

The answer is negative, as proved by the following example.

EXAMPLE 3. We prove that if χ is uncountable then $fT_\chi \times f\mathbb{N}$ is not uniformly paracompact.

For $n \in \mathbb{N}$, let U_n be the neighborhood of p which meets every I_α in $[0, 1/n]$.

Let

$$\mathcal{A}_n = \{U_n, I_\alpha \setminus U_{n+1} : \alpha \in \chi\}.$$

We consider on $fT \times f\mathbb{N}$ the open cover

$$\mathcal{A} = \{A \times \{n\} : n \in \mathbb{N}, A \in \mathcal{A}_n\}.$$

Let \mathfrak{U} an open cover of T and let U a element of \mathfrak{U} containing p . The set $U \cap I_\alpha$ contains a set of the form $]0, \varepsilon_\alpha]$, with $\varepsilon_\alpha > 0$ for every $\alpha \in \chi$.

There exists $n \in \mathbb{N}$ such that the set

$$\{\alpha \in \chi : \varepsilon_\alpha > 1/n\} \text{ is infinite.}$$

For such n there is no finite subset F of χ for which

$$U \subseteq U_n \cup \left(\bigcup_{\alpha \in F} (I_\alpha \setminus U_{n+1}) \right).$$

Therefore the cover consisting of all the finite unions of elements of \mathcal{A} is not uniform.

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