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## On Weighted Estimated for Some Systems of Partial Differential Operators.

MAURO NACINOVICH (\*)

### Introduction.

Let  $\Omega$  be an open set in  $\mathbb{R}^n$  and let  $A(x, D): \mathcal{E}^p(\Omega) \rightarrow \mathcal{E}^q(\Omega)$  be a linear partial differential operator with smooth coefficients in  $\Omega$ .

We want to solve the equation

$$(1) \quad u \in \mathcal{E}^p(\Omega), \quad A(x, D)u = f \quad \text{on } \Omega$$

when the right hand side  $f \in \mathcal{E}^q(\Omega)$  satisfies suitable integrability conditions, that we assume to be of the form

$$(2) \quad B(x, D)f = 0$$

for a differential operator

$$B(x, D): \mathcal{E}^q(\Omega) \rightarrow \mathcal{E}^r(\Omega) \quad \text{with } B(x, D) \circ A(x, D) = 0.$$

This problem generalizes that of the integrability of closed exterior differential forms on a differentiable manifold or of closed anti-holomorphic forms on a complex manifold.

This last problem in particular (Dolbeault complex), related to the solution of E. E. Levi problem, motivated many researches on

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overdetermined systems. In 1952 Garabedian and Spencer [6] introduced the  $\bar{\partial}$ -Neumann problem, a non-elliptic boundary value problem that by a regularity theorem of Kohn and Nirenberg [10] yielded solvability of (1), (2) for  $\bar{\partial}$  in strictly pseudoconvex domains. This kind of approach was pursued in full generality, in the context of the theory of pseudodifferential operators, by Hörmander in [9].

In this paper I want to outline the extension to general complexes of an alternative method, also developed for the study of  $\bar{\partial}$ , but not implying solving the  $\bar{\partial}$ -Neumann problem. It consists in the use of a priori estimates involving weight functions, that are related to a method developed by Carleman [5] to prove uniqueness for solutions of the Cauchy problem. The idea of using this method was suggested to Andreotti and Vesentini [3], [4] by the observation that problem (1), (2) is easily dealt with in the case of compact manifolds without boundary and then a next reasonable step was to investigate manifolds endowed with a complete metric (the weight function played an essential role for the completeness of the metric). For the use of weight functions for  $\bar{\partial}$ , cf. also Hörmander [7] and [8].

While the two methods are giving equivalent results for  $\bar{\partial}$ , it turns out that the first, having stronger implications (regularity up to the boundary) requires a priori estimates more difficult to establish, while it cannot be applied directly on domains either unbounded or with non smooth boundaries.

## 1. Sobolev spaces with weights and regularity theorems.

a. Let  $\Omega$  be an open set in  $\mathbb{R}^n$  and let  $\psi: \Omega \rightarrow \mathbb{R}$  be a  $C^\infty$ -function. We set

$$\langle \psi \rangle = (1 + |\text{grad } \psi|^2)^{1/2}.$$

If  $m$  is a nonnegative integer, we denote by  $W^m(\Omega, \psi)$  the space of functions  $u$  in  $W_{loc}^m(\Omega)$  (= space of functions that are locally square summable with all weak derivatives up to order  $m$ ) for which is finite the norm:

$$\|u\|_{m,\psi} = \left( \sum_{|\alpha| \leq m} \int_{\Omega} \langle \psi \rangle^{2(m-|\alpha|)} |D^\alpha u|^2 e^{-\psi} dx \right)^{1/2}$$

This is the norm associated to the scalar product

$$(u, v)_{m, \psi} = \sum_{|\alpha| \leq m} \int_{\Omega} \langle \psi \rangle^{2(m-|\alpha|)} D^\alpha u \overline{D^\alpha v} e^{-\psi} dx,$$

that gives to  $W^m(\Omega, \psi)$  a structure of Hilbert space.

We also set  $W^\infty(\Omega, \psi) = \varprojlim_m W^m(\Omega, \psi)$  with the Fréchet topology of inverse limit of a sequence of Hilbert spaces.

We will restrict our consideration to  $\sigma$ -smooth open subsets of  $\mathbb{R}^n$ , i.e. such that there exists a  $C^\infty$  function  $\psi: \Omega \rightarrow \mathbb{R}$  with the properties:

- (3)  $\forall c \in \mathbb{R}$  the set  $\Omega_c = \{x \in \Omega \mid \psi(x) < c\}$  is relatively compact in  $\Omega$ ;
  - (4) the set  $\{x \in \Omega \mid d\psi(x) = 0\}$  is a compact subset of  $\Omega$ ;
- and to the class  $\mathcal{P}(\Omega)$  of weight functions  $\psi$  that satisfy (3), (4) and moreover
- (5)  $\forall$  integer  $m \geq 0$  and real  $\varepsilon > 0$  we can find a constant  $c(m, \varepsilon)$  such that

$$\sum_{|\alpha| \leq m} |D^\alpha \psi(x)| \leq c(m, \varepsilon) \langle \psi \rangle^{1+\varepsilon} \quad \text{on } \Omega.$$

The following lemma is fundamental for the use of weight functions:

**LEMMA 1.** *Assume that  $\Omega$  is  $\sigma$ -smooth and let  $\varphi \in C^\infty(\Omega, \mathbb{R})$  satisfy (3) and (4).*

*Then for every upper semicontinuous function  $\lambda: \Omega \rightarrow \mathbb{R}$  we can find a  $C^\infty$  function  $h: \Omega \rightarrow \mathbb{R}$  such that*

$$\psi = h(\varphi) \in \mathcal{P}(\Omega) \quad \text{and} \quad \psi \geq \lambda \quad \text{on } \Omega.$$

Let  $m$  be either a nonnegative integer or  $+\infty$ . From the previous lemma we obtain the following:

**PROPOSITION 1.** *If  $\Omega$  is  $\sigma$ -smooth and  $\varphi \in C^\infty(\Omega, \mathbb{R})$  satisfies (3) and (4), then for any sequence  $\{f_n\}$  in  $W_{\text{loc}}^m(\Omega)$  we can find  $h \in C^\infty(\Omega, \mathbb{R})$  such that  $\psi = h(\varphi) \in \mathcal{P}(\Omega)$  and  $f_n \in W^m(\Omega, \psi), \forall n$ . If moreover  $f_n \rightarrow g$  in  $W_{\text{loc}}^m(\Omega)$ , then we can choose  $h$  in such a way that  $f_n \rightarrow g$  in  $W^m(\Omega, \psi)$ .*

This proposition implies in particular that  $W_{\text{loc}}^m(\Omega)$  is the direct limit of the spaces  $W^m(\Omega, \psi)$  for  $\psi$  in  $\mathcal{P}(\Omega)$ .

Having fixed  $\psi$  in  $\mathcal{P}(\Omega)$ , we will also consider for non negative integers  $m$  and real  $\delta$ , the spaces  $W^{m,\delta}(\Omega, \psi) = W^m(\Omega, \psi + \delta \ln \langle \psi \rangle)$ .

By linear interpolation we consider also the spaces  $W^{s,\delta}(\Omega, \psi)$  for  $s$  real  $\geq 0$ . After identifying the dual of  $W^0(\Omega, \psi)$  with itself by Riesz isomorphism, we define the space  $W^{s,\delta}(\Omega, \psi)$  for  $s < 0$  as the dual of  $W^{-s,-\delta}(\Omega, \psi)$ ; as the Riesz isomorphism yields natural inclusions  $W^{s,\delta}(\Omega, \psi) \hookrightarrow \mathcal{D}'(\Omega)$ , we identify all these spaces to spaces of distributions. We denote by

$$\|u\|_{s,\delta}$$

a continuous norm in  $W^{s,\delta}(\Omega, \psi)$ , ( $s, \delta \in \mathbb{R}$ ).

The spaces we have introduced have the following properties:

PROPOSITION 2. *For every  $s, \delta \in \mathbb{R}$  and  $\psi \in \mathcal{P}(\Omega)$ , the space  $\mathcal{D}(\Omega)$  of  $C^\infty$  functions with compact support in  $\Omega$  is dense in  $W^{s,\delta}(\Omega, \psi)$ .*

*If  $s, s', \delta, \delta' \in \mathbb{R}$  and  $s \leq s', \delta \leq \delta' + s' - s$ , then we have a continuous inclusion*

$$W^{s',\delta'}(\Omega, \psi) \rightarrow W^{s,\delta}(\Omega, \psi).$$

*If  $s < s'$  and  $\delta < \delta' + s' - s$ , then the inclusion is compact.*

Let  $P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$  be a linear differential operator of order  $\leq m$ .

We say that  $P(x, D)$  has type  $(m, \delta)$  with respect to  $\psi \in \mathcal{P}(\Omega)$  if for every multiindex  $\beta$  and real  $\varepsilon > 0$  we can find a constant  $c(\beta, \varepsilon) > 0$  such that

$$|D^\beta a_\alpha| \leq c(\beta, \varepsilon) \langle \psi \rangle^{m-|\alpha|+\delta+\varepsilon|\beta|} \quad \forall |\alpha| \leq m.$$

We denote by  $P_\psi^*(x, D)$  the formal adjoint of  $P(x, D)$  for the scalar product of  $W^0(\Omega, \psi)$ , characterized by:

$$(P(x, D)u, v)_{0,\psi} = (u, P_\psi^*(x, D)v)_{0,\psi} \quad \forall u, v \in \mathcal{D}(\Omega).$$

If  $Q(x, D)$  is another differential operator with smooth coefficients on  $\Omega$ , we denote by  $[P, Q] = P \circ Q - Q \circ P$  the commutator of  $P$  and  $Q$ . Then we have:

PROPOSITION 3. a) If  $P(x, D)$  is of type  $(m, \delta)$  with respect to  $\psi \in \Psi(\Omega)$ , then for every  $s, \sigma \in \mathbb{R}$  it defines a continuous linear map

$$P(x, D): W^{s, \sigma}(\Omega, \psi) \rightarrow W^{s-m, \sigma-\delta}(\Omega, \psi).$$

b) The operator  $P_{\psi}^*(x, D)$  is also of type  $(m, \delta)$ .

c) If  $Q(x, D)$  is of type  $(k, \sigma)$ , then the commutator  $[P, Q]$  is of type  $(m + k - 1, \lambda)$  for every  $\lambda > \delta + \sigma$ .

If  $s = (s_1, \dots, s_p) \in \mathbb{R}^p$  and  $\delta \in \mathbb{R}$ , we will write  $W^{s, \delta}(\Omega, \psi)$  for  $W^{s_1, \delta}(\Omega, \psi) \times \dots \times W^{s_p, \delta}(\Omega, \psi)$ . We will also use the notations

$$(u, v)_{s, \psi, \delta} = (u^1, v^1)_{s_1, \psi, \delta} + \dots + (u^p, v^p)_{s_p, \psi, \delta}$$

for the scalar product on  $W^{s, \delta}(\Omega, \psi)$  if  $u = (u^1, \dots, u^p)$ ,  $v = (v^1, \dots, v^p)$  and for each  $j = 1, \dots, p$ , we denoted by  $(\cdot, \cdot)_{s_j, \psi, \delta}$  a continuous scalar product in  $W^{s_j, \delta}(\Omega, \psi)$ ; we set also

$$\|u\|_{s, \psi, \delta} = (u, u)_{s, \psi, \delta}^{\frac{1}{2}}.$$

For  $s \in \mathbb{R}^p$  and  $t \in \mathbb{R}$ , we set also  $\mathbf{t} = (t, \dots, t) \in \mathbb{R}^p$  and  $s + \mathbf{t} = (s_1 + t, \dots, s_p + t)$ .

An operator  $A(x, D) = (A_{ij}(x, D))_{i=1, \dots, q; j=1, \dots, p}$  is said to be of type  $(m, k, \delta)$  for a  $p$ -uple of integers  $m = (m_1, \dots, m_p)$ , a  $q$ -uple of integers  $k = (k_1, \dots, k_q)$  and a real  $\delta$  with respect to  $\psi \in \Psi(\Omega)$  if for every pair of indices  $i, j$  the operator  $A_{ij}(x, D)$  is of type  $(m_j - k_i, \delta)$ . Such an operator defines a linear and continuous map

$$A(x, D): W^{m+\mathbf{t}, \sigma}(\Omega, \psi) \rightarrow W^{k+\mathbf{t}, \sigma-\delta}(\Omega, \psi)$$

for all real  $t, \sigma$ .

b. Let  $m = (m_1, \dots, m_p)$  be a  $p$ -uple of nonnegative integers and let  $\psi \in \Psi(\Omega)$ . A differential operator with smooth coefficients

$$E(x, D): \mathcal{E}^p(\Omega) \rightarrow \mathcal{E}^N(\Omega)$$

will be said to be  $W^m(\Omega, \psi)$ -elliptic if it is of type  $(m, \mathbf{0}; 0)$  and there is a constant  $c > 0$  such that

$$\|E(x, D)u\|_{\mathbf{0}, \psi}^2 \geq c \|u\|_{m, \psi}^2 \quad \forall u \in \mathcal{D}^p(\Omega).$$

We have the following:

**PROPOSITION 4.** *If  $E(x, D)$  is  $W^m(\Omega, \psi)$ -elliptic, then for every  $s, \delta \in \mathbb{R}$   $L(x, D) = E_\psi^*(x, D) \circ E(x, D): W^{m+s, \delta}(\Omega, \psi) \rightarrow W^{s-m, \delta}(\Omega, \psi)$*

*is an isomorphism.*

As an example of such an operator  $L(x, D)$ , we can consider the operator  $\Delta_{m, \psi}: \mathcal{E}^p(\Omega) \rightarrow \mathcal{E}^p(\Omega)$  characterized by the identity:

$$(\Delta_{m, \psi} u, v)_{0, \psi} = (u, v)_{m, \psi} \quad \forall u, v \in \mathcal{D}^p(\Omega).$$

Let now  $0 < \delta \leq 1$  be fixed. We say that  $E(x, D): \mathcal{E}^p(\Omega) \rightarrow \mathcal{E}^N(\Omega)$  is  $W^{m-1, \delta}(\Omega, \psi)$ -coercive if  $E(x, D)$  is of type  $(m, \mathbf{0}; 0)$  and there are constants  $c > 0$  and  $\lambda \geq 0$  such that

$$(6) \quad c \|u\|_{m-1, \psi, \delta}^2 \leq \|Eu\|_{\mathbf{0}, \psi}^2 + \lambda \|u\|_{\mathbf{0}, \psi}^2 \quad \forall u \in \mathcal{D}^p(\Omega).$$

Note that, while  $W^m(\Omega, \psi)$ -ellipticity implies that  $E_\psi^*(x, D) \circ E(x, D)$  is in  $\Omega$  an elliptic operator in the sense of Douglis and Nirenberg (cf. [12]), neither ellipticity nor sub-ellipticity are implied by  $W^{m-1, \delta}(\Omega, \psi)$ -coerciveness. Thus we shall need also the following assumption:

$$(7) \quad E(x, D) \text{ is sub-elliptic, i.e. there is a real number } \sigma, \text{ with } 0 \leq \sigma < 1, \text{ such that every distribution } u \in \mathcal{D}'(\Omega^p) \text{ for which } E(x, D)u \in (L_{\text{loc}}^2(\Omega))^N \text{ belongs to } W_{\text{loc}}^{m-\sigma}(\Omega).$$

(For  $\sigma = 1/2$  necessary and sufficient conditions for subellipticity have been studied by Hörmander in [9]).

We have the following:

**PROPOSITION 5 (Regularity Theorem).** *Let us assume that (6) and (7) hold. Then, if  $f \in W^{s+1-m, \sigma}(\Omega, \psi)$  with  $s \geq 0$  and  $s + \sigma + \delta \geq 0$  and  $u \in W^{m-1, \delta}(\Omega, \psi)$  with  $E(x, D)u \in W^0(\Omega, \psi)$  solves*

$$(8) \quad (E(x, D)u, E(x, D)v)_{0, \psi} = f(e^{-\psi} \bar{v}) \quad \forall v \in \mathcal{D}^p(\Omega);$$

*we have*

$$u \in W^{m+s-1, 2\delta+\sigma}(\Omega, \psi) \quad \text{and} \quad E(x, D)u \in W^{s, \delta+\sigma}(\Omega, \psi).$$

This is the key result for the application of estimates involving

weight functions, and plays here a role analogous of the regularization method of Kohn and Nirenberg for the  $\bar{\partial}$ -Neumann problem. The proof is done by elliptic regularization.

**2. Application to complexes of partial differential operators.**

Let us consider a complex

$$(9) \quad \mathfrak{E}^p(\Omega) \xrightarrow{A(x,D)} \mathfrak{E}^q(\Omega) \xrightarrow{B(x,D)} \mathfrak{E}^r(\Omega)$$

of differential operators with smooth coefficients on  $\Omega$  ( $B(x, D) \circ \circ A(x, D) = 0$ ).

We assume that for  $s \in \mathbb{Z}^p, m \in \mathbb{Z}^q, t \in \mathbb{Z}^r$  and  $\psi \in \Psi(\Omega)$  the operator  $A(x, D)$  is of type  $(s, m; 0)$  and the operator  $B(x, D)$  is of type  $(m, t; 0)$ .

Let us choose  $\lambda < \inf m_i$  and an operator  $F(x, D): \mathfrak{E}^q(\Omega) \rightarrow \mathfrak{E}^s(\Omega)$   $W^{m-\lambda}(\Omega, \psi)$ -elliptic. Then we choose an integer  $l$  in such a way that  $l + s$  and  $l + 2\lambda - t$  have all components  $\geq 0$  and we define  $E_\psi(x, D)$  by

$$(E_\psi(x, D)u, E_\psi(x, D)v)_{0,\psi} = (A_\psi^*(x, D)u, A_\psi^*(x, D)v)_{s+l,\psi} + (B(x, D)F_\psi^*Fu, B(x, D)F_\psi^*Fv)_{l+2\lambda-t},$$

for every  $u, v \in \mathfrak{D}^q(\Omega)$ .

Then  $E_\psi(x, D)$  is of type  $(m + l, \mathbf{0}; 0)$  with respect to  $\psi$ .

We have the following:

**PROPOSITION 6.** *The properties of  $E_\psi(x, D)$  of being either subelliptic or  $W^{m+l-1,\delta}(\Omega, \psi)$ -coercive for some  $0 < \delta \leq 1$  are independent of the choice of  $\lambda, l$  and  $F$ .*

From the regularity theorem (Proposition 5) we obtain:

**PROPOSITION 7.** *Assume that the operator  $E_\psi$  defined above satisfies (6) and (7). If  $\delta_1 > -\delta$  and  $f \in W^{-m+\lambda,\delta_1}(\Omega, \psi)$  satisfies  $B(x, D)f = 0$ , then for any solution  $u \in W^{m+l-1,\delta}(\Omega, \psi)$  with  $E_\psi(x, D)u \in W^0(\Omega, \psi)$  of*

$$(E_\psi(x, D)u, E_\psi(x, D)v)_{0,\psi} = f(e^{-\psi}v) \quad \forall v \in \mathfrak{D}^q(\Omega)$$

we have

$$w = \Delta_{s+l} \circ A_\psi^*(x, D)u \in W^{\lambda-s-1,\delta+\delta_1}(\Omega, \psi) \quad \text{and} \quad A(x, D)w = f \text{ on } \Omega.$$

Moreover,

$$B_{\psi}^*(x, D) \circ \Delta_{l+2\lambda-l} B(x, D) \circ F_{\psi}^*(x, D) \circ F(x, D) u = 0 \quad \text{on } \Omega.$$

Let us denote by  $A(x, D): W^{s+h, \sigma}(\Omega, \psi) \dots \rightarrow W^{h+m+1, \sigma-\delta}(\Omega, \psi)$  the closed densely defined linear operator obtained by considering the differential operator  $A(x, D)$  on the domain

$$D(A) = \{u \in W^{h+s, \sigma}(\Omega, \psi) \mid A(x, D) u \in W^{h+m+1, \sigma-\delta}(\Omega, \psi)\}$$

and analogously for  $B(x, D)$ ,  $A_{\psi}^*(x, D)$  and  $B_{\psi}^*(x, D)$ . Then we set

$$\begin{aligned} H(h, \sigma; \Omega, \psi) &= \\ &= \frac{\text{Ker}(B(x, D): W^{h+m+1, \sigma-\delta}(\Omega, \psi) \dots \rightarrow W^{h+l+2, \sigma-2\delta}(\Omega, \psi))}{\text{Image}(A(x, D): W^{h+s, \sigma}(\Omega, \psi) \dots \rightarrow W^{h+m+1, \sigma-\delta}(\Omega, \psi))}. \end{aligned}$$

$$N(\Omega, \psi) = \left\{ u \in (W^{\infty}(\Omega, \psi))^q \mid A_{\psi}^*(x, D) u = 0, \right. \\ \left. B(x, D) \circ F_{\psi}^*(x, D) \circ F(x, D) u = 0 \right\}.$$

Then from the regularity theorem we obtain the following:

PROPOSITION 8. *Under the same assumptions of Proposition 7:*

- (a)  $\dim_{\mathbf{C}} N(\Omega, \psi) = d < \infty$
- (b)  $\forall h, \sigma \in \mathbf{R}, \dim_{\mathbf{C}} H(h, \sigma; \Omega, \psi) = d.$

If  $A(x, D)$  and  $B(x, D)$  are differential operators with coefficients bounded with all derivatives in  $\Omega$ , then all operators  $E_{\psi}(x, D)$  obtained as explained above from different weight functions  $\psi \in \Psi(\Omega)$  are of type  $(m + l, 0; 0)$ . Then we obtain the following:

PROPOSITION 9. *Assume that for every upper semicontinuous function  $\varphi: \Omega \rightarrow \mathbf{R}$  there is  $\psi \in \Psi(\Omega)$  such that  $\psi \geq \varphi$  and the operator  $E_{\psi}(x, D)$  satisfies (6) and (7), then the space*

$$H(\Omega) = \frac{\text{Ker}(B(x, D): \mathcal{E}^{\alpha}(\Omega) \rightarrow \mathcal{E}^{\tau}(\Omega))}{\text{Image}(A(x, D): \mathcal{E}^{\rho}(\Omega) \rightarrow \mathcal{E}^{\alpha}(\Omega))}$$

*is finite dimensional.*

### 3. Localization of the estimates.

Let us consider now a stronger coerciveness estimate: namely we assume that for the operator  $E(x, D): \mathfrak{E}^p(\Omega) \rightarrow \mathfrak{E}^N(\Omega)$  of type  $(m, \mathbf{0}; \mathbf{0})$  we have

$$(10) \quad \|u\|_{m-1/2, \psi}^2 \leq c\{\|E(x, D)u\|_{0, \psi}^2 + \|u\|_{0, \psi}^2\} \quad \forall u \in \mathcal{D}^p(\Omega).$$

For

$$E(x, D) = (E_{ij}(x, D)) \quad \text{with} \quad E_{ij}(x, D) = \sum_{|\alpha| \leq m_j} E_{ij}^\alpha(x) D^\alpha,$$

we set

$$\hat{E}_{ij}(x, \xi) = \sum_{|\alpha| = m_j} E_{ij}^\alpha(x) \xi^\alpha \quad \text{and} \quad \hat{E}(x, \xi) = (\hat{E}_{ij}(x, \xi)).$$

Then the following theorem holds:

**PROPOSITION 10.** *A necessary and sufficient condition in order that estimate (10) holds, is that there exist a constant  $C > 0$  such that*

$$(11) \quad \sum_{j=1}^p (\langle \psi(x) \rangle + |\xi|)^{2m_j-1} \int |v_j(y)|^2 dy \\ \leq C \left\{ \int |\hat{E}(x, i\xi)v(y) + \langle \psi(x) \rangle^{-1/2} \sum_h \partial \hat{E}(x, i\xi) / \partial x_h \cdot y_h \cdot v(y) \right. \\ \left. + \langle \psi(x) \rangle^{1/2} \sum_h \partial \hat{E}(x, i\xi) / \partial \xi_h \cdot \partial v / \partial y_h \right|^2 dy \\ \left. + \sum_{j=1}^p \sum_{|\alpha| \leq m_j} (\langle \psi(x) \rangle + |\xi|)^{2m-2} \int |D^\alpha v_j(y)|^2 dy \right\} \\ \forall v \in \mathcal{D}^p(B(0, 1)), \quad \forall x \in \Omega, \quad \forall \xi \in \mathbf{R}^n.$$

The proof of this statement is similar to that of the analogous statement in Hörmander [9].

We also note that, if  $\Omega$  is relatively compact and  $E(x, D)$  is sub-elliptic with  $\sigma = 1/2$  on a neighborhood of the closure of  $\Omega$ , then (10) is a consequence of (6) with  $\delta = 1/2$ , while (6) cannot be easily localized.

**4. An application to the case of complexes differential operators with constant coefficients.**

Let  $\mathcal{F}$  denote the ring of polynomials in  $n$  indeterminates  $\xi_1, \dots, \xi_n$ , filtered by the degree. Given a  $\mathcal{F}$ -module  $M$  of finite type, we choose a filtration

$$0 = M_{-1} \subset M_0 \subset M_1 \subset M_2 \subset \dots \quad \text{of } M$$

compatible with that of  $\mathcal{F}$  and we denote by  $M^0$  the associated graded ring:

$$M^0 = \bigoplus M_j / M_{j-1}.$$

To any Hilbert resolution of  $M^0$  by homogeneous matrices of polynomials

$$0 \leftarrow M^0 \leftarrow \mathcal{F}^{p_0} \xleftarrow{A_0} \mathcal{F}^{p_1} \xleftarrow{A_1} \mathcal{F}^{p_2} \leftarrow \dots \leftarrow \mathcal{F}^{p_a} \leftarrow 0$$

corresponds a resolution of  $M$

$$0 \leftarrow M \leftarrow \mathcal{F}^{p_0} \xleftarrow{A_0} \mathcal{F}^{p_1} \xleftarrow{A_1} \mathcal{F}^{p_2} \leftarrow \dots \leftarrow \mathcal{F}^{p_a} \leftarrow 0$$

where, for a suitable choice of multigraduations, the  $\hat{A}_j$ 's can be considered as the homogeneous parts of higher degree of the  $A_j$ 's. (Cf. [2]).

The modules  $\text{Ext}^i(M, \mathcal{E}(\Omega))$  (where  $\mathcal{E}(\Omega)$  is considered as a left- $\mathcal{F}$ -module by  $p(\xi) \cdot f = p(D)f$ ) are isomorphic to the cohomology groups of the complex of differential operators with constant coefficients:

$$\mathcal{E}^{p_0}(\Omega) \xrightarrow{A_0(D)} \mathcal{E}^{p_1}(\Omega) \xrightarrow{A_1(D)} \mathcal{E}^{p_2}(\Omega) \rightarrow \dots \rightarrow \mathcal{E}^{p_a}(\Omega) \rightarrow 0.$$

For  $\xi^0 \in \mathbb{C}^n$ , we denote by  $L_{\xi^0}$  the localization at  $\xi^0$  of  $\mathcal{F}$ , i.e. the ring of fractions  $p/q$  for  $p, q \in \mathcal{F}$  and  $q(\xi^0) \neq 0$ .

We say that  $M^0$  is simple of principal type if the characteristic variety  $V(M^0) = \{\xi \in \mathbb{C}^n | M^0 \otimes_{\mathcal{F}} \mathcal{F}/\mathfrak{m}_{\xi} \neq 0\}$  (where  $\mathfrak{m}_{\xi}$  is the ideal of polynomials vanishing at  $\xi$ ) is smooth outside 0 and  $\forall \xi^0 \in V(M^0) - \{0\}$ , having chosen  $p_1, \dots, p_k$  such that  $V(M^0)$  is defined by  $p_1 = \dots = p_k = 0$

near  $\xi^0$ , with  $dp_1 \wedge \dots \wedge dp_k \neq 0$  at  $\xi^0$ , we have

$$M^0 \otimes_{\mathcal{F}} L_{\xi_0} \cong L_{\xi_0} / (p_1, \dots, p_k)$$

where  $(p_1, \dots, p_k)$  is the ideal of  $L_{\xi_0}$  generated by  $p_1, \dots, p_k$ .

The following proposition, that is a consequence of the results of the preceding sections, is a generalization of the vanishing theorems for  $\bar{\partial}$  on strictly pseudoconvex domains of  $\mathbb{C}^n$  for  $M^0$  simple of principal type:

**PROPOSITION 11.** *Let us assume that  $V(M^0) \cap \mathbb{R}^n \subset \{0\}$ , and let  $\Omega$  be a  $\sigma$ -smooth open set of  $\mathbb{R}^n$  with a  $C^\infty$  function  $\varphi: \Omega \rightarrow \mathbb{R}$  satisfying (3) and (4) and the following convexity assumption:*

*There is a compact set  $K \subset \Omega$  such that*

$$\forall x \in \Omega - K, \quad \forall \xi \in \mathbb{R}^n \quad \text{such that} \quad \zeta = i\xi + \text{grad } \varphi(x) \in V(M^0),$$

*the quadratic form*

$$\sum \partial^2 \varphi(x) / \partial x_k \partial x_k \cdot v^h \cdot \bar{v}^k$$

*restricted to the complex linear space  $H$  of vectors*

$$v = \text{grad } p(\zeta) \quad \text{where } p \in \mathcal{F} \text{ vanishes on } V(M^0) \text{ and}$$

$$\langle \text{grad } \varphi(x), \text{grad } p(\zeta) \rangle = 0$$

*has either at least  $j$  negative or at least  $\dim_{\mathbb{C}} H - j + 1$  positive eigenvalues. Then  $\text{Ext}^j(M, \mathcal{E}(\Omega))$  is finite dimensional over  $\mathbb{C}$ .*

*If moreover  $K$  is contained in a convex open subset of  $\Omega$ ,*

$$\text{Ext}^j(M, \mathcal{E}(\Omega)) = 0.$$

## 5. Concluding remarks.

The results of the preceding paragraphs apply also to complexes of linear partial differential operators with variable coefficients; for instance we can study the Cauchy-Riemann complex induced on a generic real submanifold of  $\mathbb{C}^n$ . However we will not discuss these applications here. We hope also to develop by means of the result of § 4 a «function theory» for some complexes of p.d.e. with con-

stant coefficients that could be of help in the study of analytic hypoellipticity and propagation of analytic singularities (cf. Schapira [14]).

We also want to note that the results of sections 1, 2, 3 can be extended to the case of linear differential operators between vector bundles over a complete,  $\sigma$ -smooth Riemannian manifold, endowed with affine connections. (cf. [2]).

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