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On Weighted Estimated for Some Systems of Partial Differential Operators.

MAURO NACINOVICH (*)

Introduction.

Let Ω be an open set in $\mathbb{R}^n$ and let $A(x, D): \mathcal{E}^p(\Omega) \to \mathcal{E}^q(\Omega)$ be a linear partial differential operator with smooth coefficients in Ω.

We want to solve the equation

$$(1) \qquad u \in \mathcal{E}^p(\Omega), \quad A(x, D)u = f \quad \text{on } \Omega$$

when the right hand side $f \in \mathcal{E}^q(\Omega)$ satisfies suitable integrability conditions, that we assume to be of the form

$$(2) \qquad B(x, D)f = 0$$

for a differential operator

$$B(x, D): \mathcal{E}^q(\Omega) \to \mathcal{E}^r(\Omega) \quad \text{with } B(x, D)\circ A(x, D) = 0 .$$

This problem generalizes that of the integrability of closed exterior differential forms on a differentiable manifold or of closed antiholomorphic forms on a complex manifold.

This last problem in particular (Dolbeault complex), related to the solution of E. E. Levi problem, motivated many researches on

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overdetermined systems. In 1952 Garabedian and Spencer [6] introduced the $\bar{\partial}$-Neumann problem, a non-elliptic boundary value problem that by a regularity theorem of Kohn and Nirenberg [10] yielded solvability of (1), (2) for $\bar{\partial}$ in strictly pseudoconvex domains. This kind of approach was pursued in full generality, in the context of the theory of pseudodifferential operators, by Hörmander in [9].

In this paper I want to outline the extension to general complexes of an alternative method, also developed for the study of $\bar{\partial}$, but not implying solving the $\bar{\partial}$-Neumann problem. It consists in the use of a priori estimates involving weight functions, that are related to a method developed by Carleman [5] to prove uniqueness for solutions of the Cauchy problem. The idea of using this method was suggested to Andreotti and Vesentini [3], [4] by the observation that problem (1), (2) is easily dealt with in the case of compact manifolds without boundary and then a next reasonable step was to investigate manifolds endowed with a complete metric (the weight function played an essential role for the completeness of the metric). For the use of weight functions for $\bar{\partial}$, cf. also Hörmander [7] and [8].

While the two methods are giving equivalent results for $\bar{\partial}$, it turns out that the first, having stronger implications (regularity up to the boundary) requires a priori estimates more difficult to establish, while it cannot be applied directly on domains either unbounded or with non smooth boundaries.

1. Sobolev spaces with weights and regularity theorems.

a. Let Ω be an open set in $\mathbb{R}^n$ and let $\psi\colon \Omega \to \mathbb{R}$ be a C^∞-function. We set

$$\langle\psi\rangle = (1 + |\text{grad}\,\psi|^2)^{1/2}.$$

If m is a nonnegative integer, we denote by $W^m(\Omega, \psi)$ the space of functions u in $W^m_{loc}(\Omega)$ (= space of functions that are locally square summable with all weak derivatives up to order m) for which is finite the norm:

$$\|u\|_{m,\psi} = \Big(\sum_{|\alpha|\leqslant m} \int_\Omega \langle\psi\rangle^{2(m-|\alpha|)} |D^\alpha u|^2 e^{-\psi}\, dx\Big)$$

This is the norm associated to the scalar product

$$(u, v)_{m,\psi} = \sum_{|\alpha|\leqslant m} \int_{\Omega} \langle\psi\rangle^{2(m-|\alpha|)} D^\alpha u \overline{D^\alpha v} e^{-\psi} dx ,$$

that gives to $W^m(\Omega, \psi)$ a structure of Hilbert space.

We also set $W^\infty(\Omega, \psi) = \varprojlim_m W^m(\Omega, \psi)$ with the Fréchet topology of inverse limit of a sequence of Hilbert spaces.

We will restrict our consideration to σ-smooth open subsets of $\mathbf{R}^n$, i.e. such that there exists a C^∞ function $\psi: \Omega \to \mathbf{R}$ with the properties:

(3) $\forall c \in \mathbf{R}$ the set $\Omega_c = \{x \in \Omega | \psi(x) < c\}$ is relatively compact in Ω;

(4) the set $\{x \in \Omega | d\psi(x) = 0\}$ is a compact subset of Ω;

and to the class $\Psi(\Omega)$ of weight functions ψ that satisfy (3), (4) and moreover

(5) $\forall$ integer $m \geqslant 0$ and real $\varepsilon > 0$ we can find a constant $c(m, \varepsilon)$ such that

$$\sum_{|\alpha|\leqslant m} |D^\alpha \psi(x)| \leqslant c(m, \varepsilon) \langle\psi\rangle^{1+\varepsilon} \quad \text{on } \Omega .$$

The following lemma is fundamental for the use of weight functions:

Lemma 1. *Assume that Ω is σ-smooth and let $\varphi \in C^\infty(\Omega, \mathbf{R})$ satisfy* (3) *and* (4).

Then for every upper semicontinuous function $\lambda: \Omega \to \mathbf{R}$ we can find a C^∞ function $h: R \to R$ such that

$$\psi = h(\varphi) \in \Psi(\Omega) \quad \textit{and} \quad \psi \geqslant \lambda \quad \textit{on } \Omega .$$

Let m be either a nonnegative integer or $+\infty$. From the previous lemma we obtain the following:

Proposition 1. *If Ω is σ-smooth and $\varphi \in C^\infty(\Omega, \mathbf{R})$ satisfies* (3) *and* (4), *then for any sequence $\{f_n\}$ in $W^m_{\text{loc}}(\Omega)$ we can find $h \in C^\infty(\mathbf{R}, \mathbf{R})$ such that $\psi = h(\varphi) \in \Psi(\Omega)$ and $f_n \in W^m(\Omega, \psi)$, $\forall n$. If moreover $f_n \to g$ in $W^m_{\text{loc}}(\Omega)$, then we can choose h in such a way that $f_n \to g$ in $W^m(\Omega, \psi)$.*

This proposition implies in particular that $W^m_{\text{loc}}(\Omega)$ is the direct limit of the spaces $W^m(\Omega, \psi)$ for ψ in $\Psi(\Omega)$.

Having fixed ψ in $\Psi(\Omega)$, we will also consider for non negative integers m and real δ, the spaces $W^{m,\delta}(\Omega,\psi) = W^m(\Omega,\psi + \delta \ln\langle\psi\rangle)$.

By linear interpolation we consider also the spaces $W^{s,\delta}(\Omega,\psi)$ for s real $\geqslant 0$. After identifying the dual of $W^0(\Omega,\psi)$ with itself by Riesz isomorphism, we define the space $W^{s,\delta}(\Omega,\psi)$ for $s<0$ as the dual of $W^{-s,-\delta}(\Omega,\psi)$; as the Riesz isomorphism yields natural inclusions $W^{s,\delta}(\Omega,\psi)\hookrightarrow \mathfrak{D}'(\Omega)$, we identify all these spaces to spaces of distributions. We denote by

$$\|u\|_{s,\psi,\delta}$$

a continuous norm in $W^{s,\delta}(\Omega,\psi)$, $(s,\delta\in R)$.

The spaces we have introduced have the following properties:

PROPOSITION 2. *For every $s,\delta\in\mathbb{R}$ and $\psi\in\Psi(\Omega)$, the space $\mathfrak{D}(\Omega)$ of C^∞ functions with compact support in Ω is dense in $W^{s,\delta}(\Omega,\psi)$.*

If $s, s', \delta, \delta'\in R$ and $s\leqslant s'$, $\delta\leqslant\delta'+s'-s$, then we have a continuous inclusion

$$W^{s',\delta'}(\Omega,\psi)\to W^{s,\delta}(\Omega,\psi)\,.$$

If $s<s'$ and $\delta<\delta'+s'-s$, then the inclusion is compact.

Let $P(x,D)=\sum_{|\alpha|\leqslant m} a_\alpha(x)D^\alpha$ be a linear differential operator of order $\leqslant m$.

We say that $P(x,D)$ has type (m,δ) with respect to $\psi\in\Psi(\Omega)$ if for every multiindex β and real $\varepsilon>0$ we can find a constant $c(\beta,\varepsilon)>0$ such that

$$|D^\beta a_\alpha|\leqslant c(\beta,\varepsilon)\langle\psi\rangle^{m-|\alpha|+\delta+\varepsilon|\beta|}\qquad \forall\,|\alpha|\leqslant m\,.$$

We denote by $P^*_\psi(x,D)$ the formal adjoint of $P(x,D)$ for the scalar product of $W^0(\Omega,\psi)$, characterized by:

$$\big(P(x,D)u,v\big)_{0,\psi}=\big(u,P^*_\psi(x,D)v\big)_{0,\psi}\qquad \forall u,v\in\mathfrak{D}(\Omega)\,.$$

If $Q(x,D)$ is another differential operator with smooth coefficients on Ω, we denote by $[P,Q]=P\circ Q-Q\circ P$ the commutator of P and Q.

Then we have:

PROPOSITION 3. *a) If $P(x, D)$ is of type (m, δ) with respect to $\psi \in \Psi(\Omega)$, then for every $s, \sigma \in \mathbf{R}$ it defines a continuous linear map*

$$P(x, D)\colon W^{s,\sigma}(\Omega, \psi) \to W^{s-m,\sigma-\delta}(\Omega, \psi)\,.$$

b) The operator $P^*_\psi(x, D)$ is also of type (m, δ).

c) If $Q(x, D)$ is of type (k, σ), then the commutator $[P, Q]$ is of type $(m + k - 1, \lambda)$ for every $\lambda > \delta + \sigma$.

If $s = (s_1, \ldots, s_p) \in \mathbf{R}^p$ and $\delta \in \mathbf{R}$, we will write $W^{s,\delta}(\Omega, \psi)$ for $W^{s_1,\delta}(\Omega, \psi)\times \ldots \times W^{s_p,\delta}(\Omega, \psi)$. We will also use the notations

$$(u, v)_{s,\psi,\delta} = (u^1, v^1)_{s_1,\psi,\delta} + \ldots + (u^p, v^p)_{s_p,\psi,\delta}$$

for the scalar product on $W^{s,\delta}(\Omega, \psi)$ if $u = (u^1, \ldots, u^p)$, $v = (v^1, \ldots, v^p)$ and for each $j = 1, \ldots, p$, we denoted by $(\cdot,\cdot)_{s_j,\psi,\delta}$ a continuous scalar product in $W^{s_j,\delta}(\Omega, \psi)$; we set also

$$\|u\|_{s,\psi,\delta} = (u, u)^{\frac{1}{2}}_{s,\psi,\delta}\,.$$

For $s \in \mathbf{R}^p$ and $t \in \mathbf{R}$, we set also $\boldsymbol{t} = (t, \ldots, t) \in \mathbf{R}^p$ and $s + \boldsymbol{t} = (s_1 + t, \ldots, s_p + t)$.

An operator $A(x, D) = \big(A_{ij}(x, D)\big)_{i=1,\ldots,q;\, j=1,\ldots,p}$ is said to be of type (m, k, δ) for a p-uple of integers $m = (m_1, \ldots, m_p)$, a q-uple of integers $k = (k_1, \ldots, k_q)$ and a real δ with respect to $\psi \in \Psi(\Omega)$ if for every pair of indices i, j the operator $A_{ij}(x, D)$ is of type $(m_j - k_i, \delta)$. Such an operator defines a linear and continuous map

$$A(x, D)\colon W^{m+\boldsymbol{t},\sigma}(\Omega, \psi) \to W^{k+\boldsymbol{t},\sigma-\delta}(\Omega, \psi)$$

for all real t, σ.

b. Let $m = (m_1, \ldots, m_p)$ be a p-uple of nonnegative integers and let $\psi \in \Psi(\Omega)$. A differential operator with smooth coefficients

$$E(x, D)\colon \mathcal{E}^p(\Omega) \to \mathcal{E}^N(\Omega)$$

will be said to be $W^m(\Omega, \psi)$-elliptic if it is of type $(m, \mathbf{0}; 0)$ and there is a constant $c > 0$ such that

$$\|E(x, D)u\|^2_{\mathbf{0},\psi} \geq c\|u\|^2_{m,\psi} \qquad \forall u \in \mathcal{D}^p(\Omega)\,.$$

We have the following:

PROPOSITION 4. *If $E(x, D)$ is $W^m(\Omega, \psi)$-elliptic, then for every $s, \delta \in \mathbb{R}$ $L(x, D) = E^*_\psi(x, D) \circ E(x, D)\colon W^{m+s,\delta}(\Omega, \psi) \to W^{s-m,\delta}(\Omega, \psi)$*

is an isomorphism.

As an example of such an operator $L(x, D)$, we can consider the operator $\mathbf{\Delta}_{m,\psi}\colon \mathcal{E}^p(\Omega) \to \mathcal{E}^p(\Omega)$ characterized by the identity:

$$(\mathbf{\Delta}_{m,\psi} u, v)_{0,\psi} = (u, v)_{m,\psi} \qquad \forall u, v \in \mathfrak{D}^p(\Omega)\,.$$

Let now $0 < \delta \leqslant 1$ be fixed. We say that $E(x, D)\colon \mathcal{E}^p(\Omega) \to \mathcal{E}^N(\Omega)$ is $W^{m-1,\delta}(\Omega, \psi)$-coercive if $E(x, D)$ is of type $(m, \mathbf{0}; 0)$ and there are constants $c > 0$ and $\lambda \geqslant 0$ such that

$$c\|u\|^2_{m-1,\psi,\delta} \leqslant \|Eu\|^2_{0,\psi} + \lambda\|u\|^2_{0,\psi} \qquad \forall u \in \mathfrak{D}^p(\Omega)\,. \tag{6}$$

Note that, while $W^m(\Omega, \psi)$-ellipticity implies that $E^*_\psi(x, D) \circ E(x, D)$ is in Ω an elliptic operator in the sense of Douglis and Nirenberg (cf. [12]), neither ellipticity nor sub-ellipticity are implied by $W^{m-1,\delta}(\Omega, \psi)$-coerciveness. Thus we shall need also the following assumption:

(7) *$E(x, D)$ is sub-elliptic, i.e. there is a real number σ, with $0 \leqslant \sigma < 1$, such that every distribution $u \in \mathfrak{D}'(\Omega^p)$ for which $E(x, D)u \in (L^2_{\mathrm{loc}}(\Omega))^N$ belongs to $W^{m-\sigma}_{\mathrm{loc}}(\Omega)$.*

(For $\sigma = 1/2$ necessary and sufficient conditions for subellipticity have been studied by Hörmander in [9]).

We have the following:

PROPOSITION 5 (*Regularity Theorem*). *Let us assume that* (6) *and* (7) *hold. Then, if $f \in W^{s+1-m,\sigma}(\Omega, \psi)$ with $s \geqslant 0$ and $s + \sigma + \delta \geqslant 0$ and $u \in W^{m-1,\delta}(\Omega, \psi)$ with $E(x, D)u \in W^0(\Omega, \psi)$ solves*

$$\big(E(x, D)u, E(x, D)v\big)_{0,\psi} = f(e^{-\psi}\bar{v}) \qquad \forall v \in \mathfrak{D}^p(\Omega)\,; \tag{8}$$

we have

$$u \in W^{m+s-1,2\delta+\sigma}(\Omega, \psi) \qquad \text{and} \qquad E(x, D)u \in W^{s,\delta+\sigma}(\Omega, \psi)\,.$$

This is the key result for the application of estimates involving weight functions, and plays here a role analogous of the regularization method of Kohn and Nirenberg for the $\bar{\partial}$-Neumann problem. The proof is done by elliptic regularization.

2. Application to complexes of partial differential operators.

Let us consider a complex

$$\mathcal{E}^p(\Omega) \xrightarrow{A(x,D)} \mathcal{E}^q(\Omega) \xrightarrow{B(x,D)} \mathcal{E}^r(\Omega) \tag{9}$$

of differential operators with smooth coefficients on Ω $\big(B(x, D)\circ \circ A(x, D) = 0\big)$.

We assume that for $s \in \mathbb{Z}^p$, $m \in \mathbb{Z}^q$, $t \in \mathbb{Z}^r$ and $\psi \in \Psi(\Omega)$ the operator $A(x, D)$ is of type $(s, m; 0)$ and the operator $B(x, D)$ is of type $(m, t; 0)$.

Let us choose $\lambda \leqslant \inf m_i$ and an operator $F(x, D)\colon \mathcal{E}^q(\Omega) \to \mathcal{E}^N(\Omega)$ $W^{m-\boldsymbol{\lambda}}(\Omega, \psi)$-elliptic. Then we choose an integer l in such a way that $\boldsymbol{l} + s$ and $\boldsymbol{l} + 2\boldsymbol{\lambda} - t$ have all components $\geqslant 0$ and we define $E_\psi(x, D)$ by

$$\big(E_\psi(x, D)u, E_\psi(x, D)v\big)_{0,\psi} = \big(A_\psi^*(x, D)u, A_\psi^*(x, D)v\big)_{s+\boldsymbol{l},\psi} + \\ + \big(B(x, D)F_\psi^*Fu, B(x, D)F_\psi^*\circ Fv\big)_{\boldsymbol{l}+2\boldsymbol{\lambda}-t},$$

for every $u, v \in \mathfrak{D}^q(\Omega)$.

Then $E_\psi(x, D)$ is of type $(m + \boldsymbol{l}, \boldsymbol{0}; 0)$ with respect to ψ.

We have the following:

PROPOSITION 6. *The properties of $E_\psi(x, D)$ of being either subelliptic or $W^{m+\boldsymbol{l}-\boldsymbol{1},\delta}(\Omega, \psi)$-coercive for some $0 < \delta \leqslant 1$ are independent of the choice of λ, l and F.*

From the regularity theorem (Proposition 5) we obtain:

PROPOSITION 7. *Assume that the operator E_ψ defined above satisfies* (6) *and* (7). *If $\delta_1 > -\delta$ and $f \in W^{-m+\boldsymbol{\lambda},\delta_1}(\Omega, \psi)$ satisfies $B(x, D)f = 0$, then for any solution $u \in W^{m+\boldsymbol{l}-\boldsymbol{1},\delta}(\Omega, \psi)$ with $E_\psi(x, D)u \in W^{\boldsymbol{0}}(\Omega, \psi)$ of*

$$\big(E_\psi(x, D)u, E_\psi(x, D)v\big)_{0,\psi} = f(e^{-\psi}\bar{v}) \qquad \forall v \in \mathfrak{D}^q(\Omega)$$

we have

$$w = \boldsymbol{\Delta}_{s+\boldsymbol{l}}\circ A_\psi^*(x, D)u \in W^{\boldsymbol{\lambda}-s-\boldsymbol{1},\delta+\delta_1}(\Omega, \psi) \qquad \textit{and} \qquad A(x, D)w = f \textit{ on } \Omega.$$

Moreover,

$$B^*_\psi(x, D)\circ\Delta_{l+2\lambda-t}B(x, D)\circ F^*_\psi(x, D)\circ F(x, D)u = 0 \quad \text{on } \Omega .$$

Let us denote by $A(x, D)\colon W^{s+h,\sigma}(\Omega, \psi) \dots \to W^{h+m+1,\sigma-\delta}(\Omega, \psi)$ the closed densely defined linear operator obtained by considering the differential operator $A(x, D)$ on the domain

$$D(A) = \{u \in W^{h+s,\sigma}(\Omega, \psi) \,|\, A(x, D)u \in W^{h+m+1,\sigma-\delta}(\Omega, \psi)\}$$

and analogously for $B(x, D)$, $A^*_\psi(x, D)$ and $B^*_\psi(x, D)$. Then we set

$$H(h, \sigma; \Omega, \psi) = \\ = \frac{\operatorname{Ker}\big(B(x, D)\colon W^{h+m+1,\sigma-\delta}(\Omega, \psi) \dots \to W^{h+t+2,\sigma-2\delta}(\Omega, \psi)\big)}{\operatorname{Image}\big(A(x, D)\colon W^{h+s,\sigma}(\Omega, \psi) \dots \to W^{h+m+1,\sigma-\delta}(\Omega, \psi)\big)} .$$

$$N(\Omega, \psi) = \Big\{u \in \big(W^\infty(\Omega, \psi)\big)^q \,|\, A^*_\psi(x, D)u = 0, \\ B(x, D)\circ F^*_\psi(x, D)\circ F(x, D)u = 0\Big\} .$$

Then from the regularity theorem we obtain the following:

PROPOSITION 8. *Under the same assumptions of Proposition* 7:

(*a*) $\dim_{\mathbb{C}} N(\Omega, \psi) = d < \infty$

(*b*) $\forall h, \sigma \in \mathbb{R}$, $\dim_{\mathbb{C}} H(h, \sigma; \Omega, \psi) = d$.

If $A(x, D)$ and $B(x, D)$ are differential operators with coefficients bounded with all derivatives in Ω, then all operators $E_\psi(x, D)$ obtained as explained above from different weight functions $\psi \in \Psi(\Omega)$ are of type $(m + l, 0; 0)$. Then we obtain the following:

PROPOSITION 9. *Assume that for every upper semicontinuous function* $\varphi\colon \Omega \to \mathbb{R}$ *there is* $\psi \in \Psi(\Omega)$ *such that* $\psi \geqslant \varphi$ *and the operator* $E_\psi(x, D)$ *satisfies* (6) *and* (7), *then the space*

$$H(\Omega) = \frac{\operatorname{Ker}\big(B(x, D)\colon \mathcal{E}^q(\Omega) \to \mathcal{E}^r(\Omega)\big)}{\operatorname{Image}\big(A(x, D)\colon \mathcal{E}^p(\Omega) \to \mathcal{E}^q(\Omega)\big)}$$

is finite dimensional.

3. Localization of the estimates.

Let us consider now a stronger coerciveness estimate: namely we assume that for the operator $E(x, D)\colon \mathcal{E}^p(\Omega) \to \mathcal{E}^N(\Omega)$ of type $(m, \mathbf{0}; 0)$ we have

$$\|u\|^2_{m-1/2,\psi} \leqslant c\{\|E(x, D)u\|^2_{0,\psi} + \|u\|^2_{0,\psi}\} \qquad \forall u \in \mathcal{D}^p(\Omega)\,. \tag{10}$$

For

$$E(x, D) = \big(E_{ij}(x, D)\big) \qquad \text{with} \qquad E_{ij}(x, D) = \sum_{|\alpha| \leqslant m_j} E^\alpha_{ij}(x) D^\alpha\,,$$

we set

$$\hat{E}_{ij}(x, \xi) = \sum_{|\alpha| = m_j} E^\alpha_{ij}(x)\xi^\alpha \qquad \text{and} \qquad \hat{E}(x, \xi) = \big(\hat{E}_{ij}(x, \xi)\big)\,.$$

Then the following theorem holds:

PROPOSITION 10. *A necessary and sufficient condition in order that estimate* (10) *holds, is that there exist a constant* $C > 0$ *such that*

$$\begin{aligned} &\sum_{j=1}^{p} (\langle\psi(x)\rangle + |\xi|)^{2m_j - 1} \int |v_j(y)|^2 dy \\ &\quad \leqslant C\Big\{\int \Big|\hat{E}(x, i\xi)v(y) + \langle\psi(x)\rangle^{-1/2} \sum_h \partial\hat{E}(x, i\xi)/\partial x_h \cdot y_h \cdot v(y) \\ &\quad + \langle\psi(x)\rangle^{1/2} \sum_h \partial\hat{E}(x, i\xi)/\partial\xi_h \cdot \partial v/\partial y_h\Big|^2 dy \\ &\quad + \sum_{j=1}^{p} \sum_{|\alpha| \leqslant m_j} (\langle\psi(x)\rangle + |\xi|)^{2m-2} \int |D^\alpha v_j(y)|^2 dy\Big\} \\ &\qquad\qquad \forall v \in \mathcal{D}^p\big(B(0, 1)\big),\ \forall x \in \Omega,\ \forall \xi \in \mathbf{R}^n\,. \end{aligned} \tag{11}$$

The proof of this statement is similar to that of the analogous statement in Hörmander [9].

We also note that, if Ω is relatively compact and $E(x, D)$ is subelliptic with $\sigma = 1/2$ on a neighborhood of the closure of Ω, then (10) is a consequence of (6) with $\delta = 1/2$, while (6) cannot be easily localized.

4. An application to the case of complexes differential operators with constant coefficients.

Let $\mathcal{T}$ denote the ring of polynomials in n indeterminates $\xi_1, \ldots, \xi_n$, filtered by the degree. Given a $\mathcal{T}$-module M of finite type, we choose a filtration

$$0 = M_{-1} \subset M_0 \subset M_1 \subset M_2 \subset \ldots \quad \text{of } M$$

compatible with that of $\mathcal{T}$ and we denote by M^0 the associated graded ring:

$$M^0 = \oplus\, M_j/M_{j-1}\,.$$

To any Hilbert resolution of M^0 by homogeneous matrices of polynomials

$$0 \leftarrow M^0 \leftarrow \mathcal{T}^{p_0} \xleftarrow{\;{}^t\hat{A}_0\;} \mathcal{T}^{p_1} \xleftarrow{\;{}^t\hat{A}_1\;} \mathcal{T}^{p_2} \leftarrow \ldots \leftarrow \mathcal{T}^{p_d} \leftarrow 0$$

corresponds a resolution of M

$$0 \leftarrow M \leftarrow \mathcal{T}^{p_0} \xleftarrow{\;{}^tA_0\;} \mathcal{T}^{p_1} \xleftarrow{\;{}^tA_1\;} \mathcal{T}^{p_2} \leftarrow \ldots \leftarrow \mathcal{T}^{p_d} \leftarrow 0$$

where, for a suitable choice of multigraduations, the $\hat{A}_j$'s can be considered as the homogeneous parts of higher degree of the A_j's. (Cf. [2]).

The modules $\operatorname{Ext}^j(M, \mathcal{E}(\Omega))$ (where $\mathcal{E}(\Omega)$ is considered as a left-$\mathcal{T}$-module by $p(\xi)\cdot f = p(D)f$) are isomorphic to the cohomology groups of the complex of differential operators with constant coefficients:

$$\mathcal{E}^{p_0}(\Omega) \xrightarrow{\;A_0(D)\;} \mathcal{E}^{p_1}(\Omega) \xrightarrow{\;A_1(D)\;} \mathcal{E}^{p_2}(\Omega) \to \ldots \to \mathcal{E}^{p_d}(\Omega) \to 0\,.$$

For $\xi^0 \in \mathbb{C}^n$, we denote by L_{ξ^0} the localization at ξ^0 of $\mathcal{T}$, i.e. the ring of fractions p/q for $p, q \in \mathcal{T}$ and $q(\xi^0) \neq 0$.

We say that M^0 is simple of principal type if the characteristic variety $V(M^0) = \{\xi \in \mathbb{C}^n | M^0 \otimes_{\mathcal{T}} \mathcal{T}/\mathfrak{m}_\xi \neq 0\}$ (where $\mathfrak{m}_\xi$ is the ideal of polynomials vanishing at ξ) is smooth outside 0 and $\forall \xi^0 \in V(M^0) - \{0\}$, having chosen $p_1, \ldots, p_k$ such that $V(M^0)$ is defined by $p_1 = \ldots = p_k = 0$

near ξ^0, with $dp_1\wedge\ldots\wedge dp_k \neq 0$ at ξ^0, we have

$$M^0 \otimes_{\mathcal{T}} L_{\xi_0} \cong L_{\xi_0}/(p_1, \ldots, p_k)$$

where $(p_1, \ldots, p_k)$ is the ideal of L_{ξ^0} generated by $p_1, \ldots, p_k$.

The following proposition, that is a consequence of the results of the preceding sections, is a generalization of the vanishing theorems for $\bar{\partial}$ on strictly pseudoconvex domains of $\mathbb{C}^n$ for M^0 simple of principal type:

PROPOSITION 11. *Let us assume that* $V(M^0) \cap \mathbb{R}^n \subset \{0\}$, *and let* Ω *be a* σ*-smooth open set of* $\mathbb{R}^n$ *with a* C^∞ *function* $\varphi\colon \Omega \to \mathbb{R}$ *satisfying* (3) *and* (4) *and the following convexity assumption*:

There is a compact set $K \subset \Omega$ *such that*

$$\forall x \in \Omega - K\,, \quad \forall \xi \in \mathbb{R}^n \quad \text{such that} \quad \zeta = i\xi + \operatorname{grad}\varphi(x) \in V(M^0)\,,$$

the quadratic form

$$\sum \partial^2\varphi(x)/\partial x_k\,\partial x_k \cdot v^h \cdot \bar{v}^k$$

restricted to the complex linear space H *of vectors*

$$v = \operatorname{grad} p(\zeta) \quad \text{where } p \in \mathcal{T} \text{ vanishes on } V(M^0) \text{ and}$$

$$\langle \operatorname{grad}\varphi(x), \operatorname{grad} p(\zeta)\rangle = 0$$

has either at least j *negative or at least* $\dim_{\mathbb{C}} H - j + 1$ *positive eigenvalues. Then* $\operatorname{Ext}^j(M, \mathcal{E}(\Omega))$ *is finite dimensional over* $\mathbb{C}$.

If moreover K *is contained in a convex open subset of* Ω,

$$\operatorname{Ext}^j(M, \mathcal{E}(\Omega)) = 0\,.$$

5. Concluding remarks.

The results of the preceding paragraphs apply also to complexes of linear partial differential operators with variable coefficients; for instance we can study the Cauchy-Riemann complex induced on a generic real submanifold of $\mathbb{C}^n$. However we will not discuss these applications here. We hope also to develop by means of the result of § 4 a « function theory » for some complexes of p.d.e. with constant coefficients that could be of help in the study of analytic hypoellipticity and propagation of analytic singularities (cf. Schapira [14]).

We also want to note that the results of sections 1, 2, 3 can be extended to the case of linear differential operators between vector bundles over a complete, σ-smooth Riemannian manifold, endowed with affine connections. (cf. [2]).

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