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On a Characterization of Reflexive Banach Spaces.

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0. Introduction.

Let X be a Banach space. Consider the Cauchy problem

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0 \tag{1}$$

where $x, x_0 \in X$ and $f: \mathbb{R} \times X \to X$ (here the symbol «$\cdot$» denotes the strong derivative).

Let C be the space of all strongly continuous functions from $\mathbb{R} \times X$ into X, with supremum norm. It is well known that, when X is finite dimensional, (1) has a solution for every $f \in C$ and $x_0 \in X$.

Dieudonné ([3]) remarked that in the case $X = c_0$ the existence of solutions is not guaranted for every continuous function f.

Recent results assure that in the infinite dimensional case the set of all $f \in C$ for which problem (1) has no solution is a non-empty, dense, of first category subset of C (see [7], [5], [8], [4]).

In this paper we are interested in nonexistence of weak solutions of problem (1). Let $\mathcal{C}$ be the set of all continuous functions from $\mathbb{R} \times (X, \tau)$ into (X, τ), where τ denotes the weak topology of X. For $f \in \mathcal{C}$ and $(t_0, x_0) \in \mathbb{R} \times X$ denote by $[f; t_0, x_0]$ the weak version of problem (1). Let $\mathfrak{C}$ the set of all $f \in \mathcal{C}$ for which the problem $[f; t_0, x_0]$ has no weak solution.

It is well known that in reflexive Banach spaces the problem $[f; t_0, x_0]$ has a weak solution for every $f \in \mathcal{C}$ ([9]). Moreover, it has

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been shown that for every non reflexive retractive Banach space the set $\mathcal{C}$ is non empty.

In this paper we state that the non reflexivity is in itself sufficient to imply $\mathcal{C} \neq \emptyset$, and hence that the existence of a weak solution for every $f \in \mathfrak{C}$ is a characterization of the reflexive Banach spaces. Furthermore we prove that $\mathcal{C}$ and $\mathfrak{C} \setminus \mathcal{C}$ are dense in $\mathfrak{C}$.

1. Definitions.

Let X be a Banach space, τ the weak topology of X and Ω an open subset of $\mathbb{R} \times (X, \tau)$. We recall some definitions which we shall use in the following.

DEFINITION 1. We shall call τ-neighbourhood of $\bar{x} \in X$ the set

$$U = \{x \in X \colon |x^*(x - \bar{x})| < \varepsilon, \, x^* \in Y^*\}$$

where $\varepsilon > 0$ and Y^* is a finite subset of the dual space X^* of X.

DEFINITION 2. The map $f \colon \mathbb{R} \times X \to X$ is continuous from Ω into (X, τ) if for every $(t', x') \in \Omega$ and arbitrary τ-neighbourhood U of the point $f(t', x')$, there exist $\delta > 0$ and a τ-neighbourhood V of x' such that $f(t, x) \in U$ if $x \in V$, $|t - t'| < \delta$ and $(t, x) \in \Omega$.

Denote by $\mathfrak{C}$ the set of all continuous functions from $\mathbb{R} \times (X, \tau)$ into (X, τ).

DEFINITION 3. $x \colon \mathbb{R} \to X$ is weakly continuous at t_0 if $t \to t_0$ implies

$$x^*\big(x(t)\big) \to x^*\big(x(t_0)\big) \qquad \text{for every } x^* \in X^*.$$

DEFINITION 4. $x \colon \mathbb{R} \to X$ is weakly differentiable at t_0 if there exists $y \in X$ such that $t \to t_0$ implies

$$\frac{x^*\big(x(t) - x(t_0)\big)}{t - t_0} \to x^*(y) \qquad \text{for every } x^* \in X^*.$$

We shall say that y is the weak derivative of x at t_0 and we shall denote it by $\dot{x}(t_0)$.

DEFINITION 5. $x\colon \mathbb{R} \to X$ is weakly integrable in $[a, b]$ if the function $x^*x\colon \mathbb{R} \to \mathbb{R}$, defined by $x^*x(t) = x^*(x(t))$, is Riemann-integrable in $[a, b]$ for every $x^* \in X^*$ and there exists $\bar{x} \in X$ such that

$$x^*(\bar{x}) = \int_a^b x^*(x(t))\, dt \qquad \text{for every } x^* \in X^* .$$

DEFINITION 6. $x\colon \mathbb{R} \to X$ is a weak solution of $[f; t_0, x_0]$ if there exists $\delta > 0$ such that for every $t \in [t_0, t_0 + \delta]$

a) $(t, x(t)) \in \Omega$,

b) x is weakly differentiable at t,

c) $\dot{x}(t) = f(t, x(t))$ in the sense of definition 4,

d) $x(t_0) = x_0$.

Hence if x is a weak solution of $[f; t_0, x_0]$, this implies that, for every $x^* \in X^*$

$$x^*(x(t)) = x^*(x_0) + \int_{t_0}^{t} x^*\big(f(s, x(s))\big)\, ds , \qquad t \in [t_0, t_0 + \delta] . \tag{2}$$

2. Nonexistence of weak solutions.

THEOREM 1. *Let X be a nonreflexive Banach space with norm $\|\cdot\|$. Given $a \in X$ and $(t_0, y_0) \in R \times X$, there exists $f \in \mathcal{C}$ such that $f(t_0, y_0) = a$ and the problem $[f; t_0, y_0]$ has no τ-solution.*

REMARK 1. In the case of X retractive, the result has been obtained in [10]: the additional hypotesis guarantees the crucial fact, required by the tecnique used there, that a continuous function defined on a subspace of (X, τ) can be extended to the whole space. The strategy here is, instead, the direct construction of a family of continuous functions defined on the whole space and, from this, of the f which satisfies the statement.

Proof of Theorem 1. Let B be the closed ball with center at y_0 and radius 1. Let S^* be the boundary of the unite ball of X^*. From the James' characterization of reflexivity ([2]) there exists $v \in S^*$ such that $|v(x - y_0)| < 1$ for every $x \in B$.

If $v(y_0) = b$, by definition of the norm of v, we can get a sequence $\{x_n\}$ in the boundary of B, with $x_1 = y_0 + x/\|x\|$ ($x \in \ker v$), such that $v(x_n) < v(x_{n+1})$, $v(x_n) \to 1 + b$.

Let ϑ be the pseudonormed topology generated by v ($|x|_\vartheta = |v(x)|$). Let $B_\vartheta = \{x \in X: |x - y_0|_\vartheta < 1\}$ and $x_0 = 2y_0 - x_2$.

Note that the sets

$$O_n = \{x \in X: 2v(x_{n-1}) < v(x) + 1 + b < 2v(x_{n+1})\}, \qquad n \in \mathbb{N}$$

are non-empty (for every $\bar{x}$, with $v(\bar{x}) = 1$, $x = 2x_n - y_0 - \bar{x} \in O_n$), ϑ-open, and their union is a point finite cover of B_ϑ. In fact it is not difficult to check that every point $x \in B_\vartheta$ belongs to at most two O_n and moreover that $B_\vartheta = \bigcup_{n \geqslant 2} O_n$.

Define

$$\varphi_0(x) = 1 - \frac{v(x) + 1 + b - 2v(x_0)}{2v(x_1 - x_0)},$$

and

$$\varphi_n(x) = \begin{cases} 0, & x \notin O_n; \\ 1 - \varphi_{n-1}(x), & x \in O_n \setminus O_{n+1}; \\ 1 - \dfrac{v(x) + 1 + b - 2v(x_n)}{2v(x_{n+1} - x_n)}, & x \in O_n \cap O_{n+1}. \end{cases}$$

It is easy to prove that the functions $\varphi_0, \varphi_n: X \to \mathbb{R}$ have the following properties:

a) for every $n \in \mathbb{N}$ φ_n is continuous from (X, ϑ) into $[0, 1]$;

b) φ_0 is continuous on (X, ϑ) and $\varphi_0(x) \in [0, 1)$ if $x \in O_1 \setminus O_2$;

c) $\sum_{n=1}^{\infty} \varphi_n(x) \leqslant 1$ for every $x \in X$ and in particular $\sum_{n=1}^{\infty} \varphi_n(x) = 1$ if $x \in B_\vartheta$.

Consider the function $g: X \to X$ given by

$$g(x) = y_0 + \sum_{n=1}^{\infty} \frac{\varphi_n(x)}{2v(x_{n+1} - y_0)} (x_{n+1} - y_0)\big(v(x) + 1 - b\big).$$

g being continuous from (X, ϑ) into X, g is in fact continuous from (X, τ) into (X, τ) because the topology ϑ is weaker than the τ-topology.

For every $x \in X$, $g(x) \in B$. In fact:

if $x \notin \cup O_n$ then $g(x) = y_0$;

if $x \in O_1 \setminus B_\vartheta$ then

$$\|g(x) - y_0\| = \frac{\|\varphi_1(x)(x_2 - y_0)(v(x) + 1 - b)\|}{2v(x_2 - y_0)} < 1 ;$$

if $x \in B_\vartheta$ and n' is such that x belongs at most to $O_{n'-1}$ and $O_{n'}$, then

$$\|g(x) - y_0\| = \varphi_{n'-1}(x) \|x_{n'} - y_0\| \frac{|v(x) + 1 - b|}{2v(x_{n'} - y_0)} + \\ + \varphi_{n'}(x) \|x_{n'+1} - y_0\| \frac{|v(x) + 1 - b|}{2v(x_{n'+1} - y_0)} < \varphi_{n'-1}(x) + \varphi_{n'}(x) = 1 .$$

Moreover, for every $x \in B_\vartheta$,

$$(3) \qquad v(g(x)) = b + \frac{v(x) + 1 - b}{2} \sum_{n=1}^{\infty} \varphi_n(x) = \frac{v(x) + 1 + b}{2} .$$

Let

$$h(t, x) = \frac{x - y_0 - a(t - t_0)}{(t - t_0)^2} .$$

We claim that the function $f: \mathbf{R} \times X \to X$ given by

$$f(t, x) = \begin{cases} 2(t - t_0) g(h(t, x)) + a , & t \neq t_0; \\ a & t = t_0, \end{cases}$$

satisfies the statement of the theorem. Indeed, for every $t \neq t_0$, $g(h) \in \mathfrak{C}$, and for every $x^* \in S^*$ we have

$$|x^*[(f(t, x)) - a]| \leqslant 2|t - t_0| \|g(h(t, x))\| \leqslant 2|t - t_0|(1 + \|y_0\|) .$$

So $f \in \mathfrak{C}$. Furthermore, if $y: \mathbf{R} \to X$ is a weak solution of $[f; t_0, y_0]$, then there exists $\delta > 0$ such that for any $t \in [t_0, t_0 + \delta]$, $x^* \in S^*$:

(see [2])

$$|x^*[h(t, y(t)) - y_0]| \leqslant \frac{1}{(t-t_0)^2} \int_{t_0}^{t} | x^*[f(s, y(s)) - a - 2(s-t_0) y_0]| \, ds \leqslant$$

$$\leqslant \frac{1}{(t-t_0)^2} \int_{t_0}^{t} 2(s-t_0) \left| x^*\Big(g\big(h(s, y(s))\big) - y_0\Big)\right| ds \leqslant \frac{1}{(t-t_0)^2} \int_{t_0}^{t} 2(s-t_0) \, ds = 1$$

hence $h(t, y(t)) \in B$. Consequently (see [3]),

$$v\Big(g\big(h(t, y(t))\big)\Big) = \frac{v\big(h(t, y(t))\big) + 1 + b}{2}$$

and so

$$\dot{v}\big(y(t)\big) = v\big(\dot{y}(t)\big) = v\big(f(t, y(t))\big) = \frac{v\big(y(t)\big) - b}{(t-t_0)} + (t-t_0)(1+b)\,.$$

Since the only solution of

$$\dot{\eta} = \frac{\eta - b}{(t-t_0)} + (t-t_0)(1+b)\,,$$

such that

$$\left| \frac{\eta - b - v(a)(t-t_0)}{(t-t_0)^2} - b \right| \leqslant 1$$

and $\eta(t_0) = b$, is given by

$$\eta(t) = (b+1)(t-t_0)^2 + v(a)(t-t_0) + b\,,$$

then

$$v\big(h(t, y(t))\big) = \frac{1}{(t-t_0)^2} [\eta(t) - b - v(a)(t-t_0)] = b + 1\,.$$

But this is a contradiction as $h(t, x(t)) \in B$. This completes the proof.

3. Density result.

LEMMA 1. *Let X be a Banach space and U a τ-neighbourhood of the origin of X. If $F, F' \in \mathcal{C}$ are such that $F(t_0, y_0) = F'(t_0, y_0)$, where (t_0, y_0) is any fixed point of $\mathbb{R} \times X$, then there exist $\alpha > 0$, $F'' \in \mathcal{C}$ and* *a τ-neighbourhood V of y_0 such that*

$$F''(t,x)\in F(t,x)+U\,,\qquad (t,x)\in\mathbb{R}\times X\,;$$
$$F''(t,x)=F'(t,x)\,,\qquad x\in V \text{ and } |t-t_0|\leqslant\alpha\,.$$

PROOF. Let $U=\{x\in X\colon |x_i^*(x)|<\varepsilon,\ x_i^*\in X^*,\ i=1,\dots,n\}$. Since $(F-F')\in\mathcal{C}$, there exist $\delta>0$, $\sigma>0$ and a finite subset Y^* of X^* such that $F(t,x)-F'(t,x)\in U$ if $|t-t_0|<\delta$ and $|x^*(x-y_0)|<\sigma$ for every $x^*\in Y^*$.

Set $K(x)=\max\limits_{x^*\in Y^*}|x^*(x-y_0)|$. Clearly K is a continuous function from (X,τ) into $\mathbb{R}$ and moreover, if $|t-t_0|<\delta$ and $K(x)<\sigma$, then $F(t,x)-F'(t,x)\in U$.

Let

$$I=\left[t_0-\frac{\delta}{2},\quad t_0+\frac{\delta}{2}\right],$$
$$A=\{(t,x)\colon t\in I,\ K(x)<\sigma/2\}\,,$$
$$B=\{(t,x)\colon t\in I,\ \sigma/2\leqslant K(x)\leqslant(2/3)\sigma\}\,,$$
$$C=\{(t,x)\colon t\in I,\ K(x)>(2/3)\sigma\}\,,$$

and consider the function $G\colon I\times X\to X$ given by

$$G(t,x)=\begin{cases} F'(t,x)\,, & (t,x)\in A\,;\\[1ex] F(t,x)+\dfrac{4\sigma-6K(x)}{\sigma}\,[F'(t,x)-F(t,x)]\,, & (t,x)\in B\,;\\[1ex] F(t,x) & (t,x)\in C\,.\end{cases}$$

We claim that $G(t,x)\in F(t,x)+U$ for every $(t,x)\in I\times X$. Indeed, if $(t,x)\in A\cup C$ it is obvious; if $(t,x)\in B$ then

$$|x_i^*(F(t,x)-G(t,x))|=\frac{|4\sigma-6K(x)|}{\sigma}\,|x_i^*[F(t,x)-F'(t,x)]|<\varepsilon\,,\qquad (i=1,\dots,n)\,.$$

Moreover G is continuous from $I\times(X,\tau)$ into (X,τ).

Define a function $\gamma\colon I\times X\to U$ by

$$\gamma(t,x)=G(t,x)-F(t,x)\,.$$

Let r be a continuous function from $\mathbb{R}$ into I such that $r(t) = t$ if $t \in I$.

The function $F'': \mathbb{R} \times X \to X$ given by

$$F''(t, x) = F(t, x) + \gamma(r(t), x)$$

is the required function, provided that $\alpha = \delta/2$ and $V = \{x \in X: |x^*(x - y_0)| < \sigma/2, x^* \in Y^*\}$. In fact, $F''(t, x) - F(t, x) = \gamma(r(t), x) \in U$ for every $(t, x) \in \mathbb{R} \times X$. In addition, if $(t, x) \in A$ then $\gamma(r(t), x) = \gamma(t, x)$ and so $F''(t, x) = G(t, x) = F'(t, x)$. This completes the proof.

DEFINITION 7. A subset $\mathcal{A}$ of $\mathcal{C}$ is said to be τ-dense in $\mathcal{C}$ if, for every $F \in \mathcal{C}$ and for every τ-neighbourhood U of the origin of X, there exists $f \in \mathcal{C}$ such that $f(t, x) \in F(t, x) + U$ for every $(t, x) \in \mathbb{R} \times X$.

Let $\mathfrak{C} = \{f \in \mathcal{C}: [f; t_0, y_0]$ has no weak solution$\}$.

REMARK 2. Given $a \in X$ and $(t_0, y_0) \in \mathbb{R} \times X$, there exists $\xi \in \mathcal{C} \setminus \mathfrak{C}$ such that $\xi(t_0, y_0) = a$; in fact the function $y(t) = a(e^{(t-t_0)} - 1) + y_0$ is a weak solution of the problem $[\xi; t_0, y_0]$ with $\xi(t, x) = x - y_0 + a$.

THEOREM 2. *In nonreflexive Banach spaces,* $\mathfrak{C}$ *is* τ*-dense in* $\mathcal{C}$.

PROOF. Given $F \in \mathcal{C}$, $(t_0, y_0) \in \mathbb{R} \times X$ and an arbitrary τ-neighbourhood U of the origin of X, by Theorem 1, there exists $f \in \mathfrak{C}$ such that $f(t_0, y_0) = F(t_0, y_0)$. Thence, by Lemma 1, there exist $\alpha > 0$, $F'' \in \mathcal{C}$ and $V \subset X$ such that $F''(t, x) \in F(t, x) + U$ for $(t, x) \in \mathbb{R} \times X$ and $F''(t, x) = f(t, x)$ for $x \in V$ and $|t - t_0| < \alpha$.

Suppose $F'' \notin \mathfrak{C}$. Then there exist $\delta' > 0$ and $y: \mathbb{R} \to X$ such that (see [2]) for every $x^* \in X^*$,

$$x^*(y(t)) = x^*(y_0) + \int_{t_0}^{t} x^*[F''(s, y(s))]\, ds, \qquad t \in [t_0, t_0 + \delta'].$$

Since y is weakly continuous, there exists $\delta'' > 0$ such that $y(t) \in V$ for $|t - t_0| < \delta''$. Hence $F''(s, y(s)) = f(s, y(s))$ for $|s - t| < \min(\alpha, \delta', \delta'')$ and so $y(t)$ is a weak solution of $[f; t_0, y_0]$: a contradiction. The theorem is proved.

THEOREM 3. *In non reflexive Banach spaces the set* $\mathcal{C} \setminus \mathfrak{C}$ *is* τ*-dense in* $\mathcal{C}$.

PROOF. Given $F \in \mathcal{C}$, $(t_0, y_0) \in \mathbb{R} \times X$ and an arbitrary τ-neighbourhood U of the origin of X, by Remark 2 there exists $\xi \in \mathcal{C} \setminus \mathfrak{C}$

such that $\xi(t_0, y_0) = F(t_0, y_0)$. By Lemma 1 there exist $\alpha > 0$, $F'' \in \mathcal{C}$ and $V \subset X$ such that $F''(t, x) \in F(t, x) + U$ for $(t, x) \in \mathbb{R} \times X$ and $F''(t, x) = \xi(t, x)$ for $x \in V$ and $|t - t_0| < \alpha$. Since $\xi \in \mathcal{C} \setminus \mathcal{C}$, there exist $\delta' > 0$ and a weak solution y of $[\xi; t_0, y_0]$, defined in $[t_0, t_0 + \delta']$, which is weakly continuous. Consequently there exists $\delta'' > 0$ such that $y(t) \in V$ if $|t - t_0| < \delta''$. Set $\delta = \min(\alpha, \delta', \delta'')$. Then, for every $x^* \in X^*$, we have (see [2])

$$x^*(y(t)) - x^*(y_0) = \int_{t_0}^{t} x^*[\xi(s, y(s))]\, ds = \int_{t_0}^{t} x^*[F''(s, y(s))]\, ds, \qquad t \in [t_0, t_0 + \delta].$$

So $F'' \in \mathcal{C} \setminus \mathcal{C}$ and the proof is complete.

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