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## Majorizing-Injectivity in Abelian Lattice-Ordered Groups.

ANTHONY W. HAGER - JAMES J. MADDEN (\*)

### Introduction.

$\mathcal{LA}$  stands for the category of abelian  $l$ -groups with  $l$ -homomorphisms and Arch the full subcategory with archimedean objects. It is well-known that neither  $\mathcal{LA}$  nor Arch has any nontrivial injectives. In contrast, we shall describe here a quite nice theory of «majorizing-injectivity», as follows (with most definitions given in the text):

The main theorem is that, in  $\mathcal{LA}$ ,  $K$  is majorizing-injective iff  $K$  is (conditionally Dedekind) complete and divisible. The necessity in this theorem is new here; the proof is *via* a simple ultrapower construction. The sufficiency was known, but the proof presented below is new and most natural, being a model of Banaschewski's abstract scheme of relative injectivity; the existential element of the proof is provided by Pierce's theorem that  $\mathcal{LA}$  has the Amalgamation Property.

For Arch, the Amalgamation Property fails (also shown by Pierce), but the theorem above implies that Arch is «majorizing-injectively complete», and it follows that majorizing embeddings amalgamate.

We note here (in the introduction only) that the results specialize to the category, say  $\mathcal{U}$ , of archimedean  $l$ -groups with distinguished strong order unit and unit-preserving  $l$ -homomorphisms: Here, every morphism is majorizing, and the theorem yields Popa's result that  $\mathcal{U}$ -injective = complete divisible; so  $\mathcal{U}$  is injectively-complete, and the Amalgamation Property follows.

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### 1. The $\mathcal{M}$ -injectives.

All objects and morphisms will be in  $\mathfrak{L}\mathcal{A}$ , the category of abelian  $l$ -groups with  $l$ -group homomorphisms. This section, containing the main theorem, takes place « within  $\mathfrak{L}\mathcal{A}$  » and is reasonably self-contained. Most of what we need about  $\mathfrak{L}\mathcal{A}$  is standard, and can be found in [BKW].

A morphism  $A \xrightarrow{\mu} B$  is called: *majorizing* if  $\forall b \in B \exists a \in A$  with  $|b| \leq \mu(a)$ ; *large* if each ideal  $I \neq (0)$  of  $B$  has  $I \cap \mu(A) \neq (0)$ ; an *embedding* if  $\mu$  is one-to-one; an *essential* morphism if  $\mu$  is a large embedding.

Let  $\mathcal{M}$  be the class of majorizing embeddings. An  $\mathcal{M}$ -essential morphism is an essential morphism which is in  $\mathcal{M}$ —a large majorizing embedding. An object  $K \in |\mathfrak{L}\mathcal{A}|$  is called  $\mathcal{M}$ -injective if given  $G \xrightarrow{\varphi} K$  and  $G \xrightarrow{\mu} H \in \mathcal{M}$  there is  $H \xrightarrow{\epsilon} K$  with  $\epsilon\mu = \varphi$ .

1.1 THEOREM.  $K$  is  $\mathcal{M}$ -injective iff  $K$  is complete and divisible.

1.1 is the main theorem. The proof will use (c) of the following.

1.2 THEOREM. These are equivalent conditions on  $K$ .

- (a)  $K$  is  $\mathcal{M}$ -injective.
- (b) Any  $K \rightarrow A \in \mathcal{M}$  has a left inverse.
- (c) Any  $\mathcal{M}$ -essential  $K \rightarrow A$  is an isomorphism.

1.2 is one of the important *desiderata* for injectivity. The proof is quite « formal » from the following properties of  $\mathcal{M}$  and the  $\mathcal{M}$ -essential morphisms.

1.3 LEMMA. (a) Let  $\mu \in \mathcal{M}$ . Then,  $\mu$  is  $\mathcal{M}$ -essential iff  $\alpha\mu \in \mathcal{M}$  implies  $\alpha \in \mathcal{M}$ .

- (b) If  $\mu \in \mathcal{M}$ , then there is  $\alpha$  with  $\alpha\mu$   $\mathcal{M}$ -essential.
- (c)  $\mathfrak{L}\mathcal{A}$  has pushouts, and for a pushout diagram

$$\begin{array}{ccc}
 \mathbf{G} & \xrightarrow{\mu} & \mathbf{H} \\
 \downarrow & & \downarrow \\
 \mathbf{K} & \xrightarrow{\nu} & \mathbf{P}
 \end{array}$$

$\mu \in \mathcal{M}$  implies  $\nu \in \mathcal{M}$ .

The fact that, for a pushout as in 1.3 (c),  $\mu$  embedding  $\Rightarrow \nu$  embedding, will be derived from the following (essentially equivalent) deep fact:

1.4 LEMMA (Pierce [P<sub>1</sub>]).  $\mathcal{L}\mathcal{A}$  has the « Amalgamation Property »: Given embeddings  $G \xrightarrow{\alpha_i} H_i$  ( $i = 1, 2$ ), there are embeddings  $H_1 \xrightarrow{\tau_1} P$  ( $i = 1, 2$ ) with  $\tau_1\alpha_1 = \tau_2\alpha_2$ .

Before giving the proofs, we make some remarks about the related literature:

[V] shows, in essence, that in archimedean vector lattices, the complete objects are  $\mathcal{M}$ -essentially injective. [A] shows that in archimedean  $f$ -algebras with identity, the complete objects are  $\mathcal{M}$ -essentially injective. [L<sub>1</sub>], [LS], [L<sub>2</sub>] show that in vector lattices, the complete objects are  $\mathcal{M}$ -injective. [AHM] shows that in  $\mathcal{L}\mathcal{A}$ , the complete vector lattices are  $\mathcal{M}$ -injective.

The proofs in [A] and [AHM] use representations of the algebras or groups as algebras or groups of extended real-valued functions, then the Gleason projectivity theorem in topology. The proofs in [V], [L<sub>1</sub>], and [LZ] use the Kantorovich generalization of the Hahn-Banach Theorem. The proof in [L<sub>2</sub>] is a direct argument by Zorn's Lemma (as Kantorovich/Hahn-Banach).

The connection between these various results are discussed in [AHM], including the fact that the result for Arch implies the result for  $\mathcal{L}\mathcal{A}$ ; this point is adumbrated in § 2 below, particularly 2.4.

It will be clear that everything said here is true for vector lattices. The complete situation for  $f$ -rings is unclear to us.

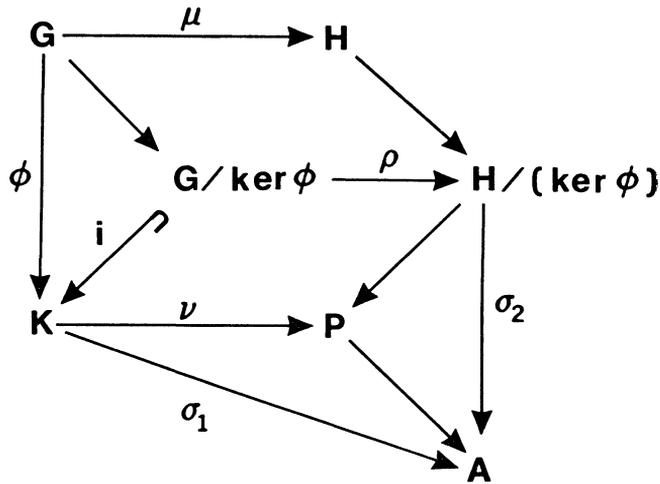
PROOF OF 1.3. (a)  $\Rightarrow$ . Clearly,  $\alpha\mu$  majorizing implies  $\mu$  majorizing. If  $\alpha\mu$  is an embedding, then  $\ker(\alpha) \cap \text{im}(\mu) = (0)$ , and  $\ker[\alpha]$  will be  $(0)$  when  $\mu$  is large.

$\Leftarrow$ . If  $A \xrightarrow{\mu} B$  is not large, then  $I \cap \mu(A) = (0)$  for some ideal  $I \neq (0)$ . Then for the canonical projection  $B \xrightarrow{\alpha} B/I$ , we have  $\alpha \notin \mathcal{M}$ , while  $\alpha\mu \in \mathcal{M}$  whenever  $\mu \in \mathcal{M}$ .

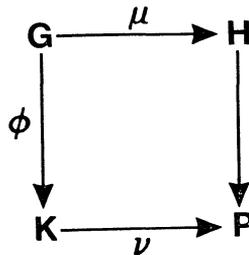
(b) Given  $A \xrightarrow{\mu} B$ , let  $I$  be an ideal of  $B$  maximal for  $I \cap \mu(A) = (0)$ . The projection  $B \xrightarrow{\alpha} B/I$  has  $\alpha\mu$  large, and  $\alpha\mu \in \mathcal{M}$  whenever  $\mu \in \mathcal{M}$ .

(c)  $\mathcal{L}\mathcal{A}$  is a variety (equational class) by [B<sub>4</sub>], and hence has free sums and pushouts, by [J]. We first show, indirectly, that pushouts

preserve embeddings. Consider



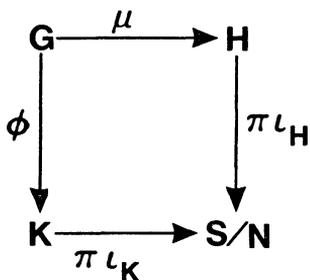
where  $\phi$  and the embedding  $\mu$  are given, then a homomorphism theorem produces the square with  $\rho$  on the bottom, which is clearly a pushout square, with  $\rho$  clearly an embedding. We now push out  $\rho$  and  $i$  to the square with terminus  $P$ . Then, the square



is a pushout, being made from the two interfacing pushout squares in the above diagram. We are to show  $\nu$  is an embedding. To do this, apply 1.4 to the embeddings  $\rho$  and  $i$ : there are *embeddings*  $\sigma_1$  and  $\sigma_2$  with  $\sigma_1 i = \sigma_2 \rho$ . But since  $P$  came from pushing out  $\rho$  and  $i$ , there is  $P \xrightarrow{\sigma} A$  with  $\sigma \nu = \sigma_1$ ; thus  $\nu$  is an embedding.

We now show that for a pushout square as above,  $\mu$  majorizing  $\Rightarrow \nu$  majorizing. This can be done directly: A pushout square as in 1.3 (c)

« is » (see [J]) really



where

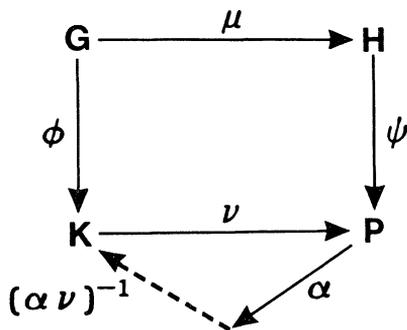


is the free sum,  $N$  is the ideal generated by all  $\iota_H(\mu(g)) - \iota_K(\phi(g))$  ( $g \in G$ ), and  $S \xrightarrow{\pi} S/N$  is the projection. The point is that  $\iota_H(H) \cup \iota_K(K)$  generates  $S$  as an ideal [M]. Thus, with  $\mu$  majorizing, so does  $\iota_H(\mu(G)) \cup \iota_K(K)$ , and so  $\pi(\iota_H(\mu(G))) \cup \pi(\iota_K(K))$  generates  $S/N$  as an ideal, which means that  $\pi \iota_K$  is majorizing.

PROOF OF 1.2. (a)  $\Rightarrow$  (b). Since  $K \xrightarrow{id} K$  lifts over  $\mu$ .

(b)  $\Rightarrow$  (c). Let  $K \xrightarrow{\mu} A$ , be  $\mathcal{M}$ -essential. By (b), there is  $K \xleftarrow{\alpha} A$  with  $\alpha\mu = id_K$ . By 1.3 (a),  $\alpha \in \mathcal{M}$ , so  $\alpha$  is an embedding and  $\mu$  is an isomorphism.

(c)  $\Rightarrow$  (a). Suppose given  $G \xrightarrow{\mu} H \in \mathcal{M}$  and  $G \xrightarrow{\phi} K$ , and consider the diagram



in which: The square is the pushout from 1.3 (c), so that  $v \in \mathcal{M}$ . Then 1.3 (b) provides  $\alpha$  with  $\alpha v$   $\mathcal{M}$ -essential. Assuming (c),  $\alpha v$  is an isomorphism, and  $(\alpha v)^{-1} \alpha \psi$  is the desired lifting of  $\varphi$  over  $\mu$ .

PROOF OF 1.1. We shall use 1.2 (c).

Suppose  $K$  is complete and divisible, and let  $K \xrightarrow{\mu} A$  be  $\mathcal{M}$ -essential. We suppress  $\mu$  and write  $K \subseteq A$ . Since  $K$  is divisible and large,  $K$  is dense. Since  $K$  is archimedean (being complete, by 11.2.2 of [BKW]),  $A$  is archimedean by 1.5 below. We now have  $a = V_A\{k \in K | k \leq a\}$  for each  $a \in A^+$ , because  $K$  is dense in archimedean  $A$  (11.3.6 of [BKW]). Now  $V_K\{k \in K | k \leq a\}$  exists, because  $K$  is complete and majorizes  $A$ . But  $V_A(\ ) = V_K(\ )$  whenever the latter exists, because  $K$  is dense (11.3.5 of [BKW]). Thus  $a^+ \in K$ ; hence  $A^+ \subseteq K$  and  $A = K$ .

Conversely, suppose  $K$  satisfies 1.2 (c). Lemma 1.6 below shows that  $K$  must be archimedean. So let  $K \xrightarrow{\mu} A$  be the embedding of  $K$  into the completion of the divisible hull of  $K$  (see [BKW], pp. 31 and 237). Since  $\mu$  is large and majorizing, it is an isomorphism, hence  $K$  is complete and divisible.

This completes the proof of 1.1. modulo 1.5 and 1.6 below.

1.5 LEMMA [LZ]. Let  $G \rightarrow H$  be  $\mathcal{M}$ -essential: If  $G$  is archimedean, then  $H$  is archimedean.

PROOF.  $A \in \mathcal{L}A$  is archimedean iff  $I_A = (0)$ , where  $I_A = \{a | \exists b \text{ with } n|a| \leq b \ \forall n \in \mathbb{Z}\}$  is the ideal of infinitely small elements. Writing  $G \subseteq H$ , we have here that  $I_A = I_H \cap G$  because  $G$  majorizes,  $I_A = (0)$  because  $G$  is archimedean, whence  $I_H = (0)$  because  $G$  is large.

1.6 LEMMA. If  $G$  is not archimedean, there is  $\mathcal{M}$ -essential  $G \xrightarrow{\mu} H$  which is not onto.

PROOF. In an  $l$ -group, we write  $a \ll b$  to mean that  $n|a| \leq |b| \ \forall n \in \mathbb{Z}^+$ . Then,  $G \in \text{Arch}$  iff  $a \ll b \Rightarrow a = 0$ . We shall prove the following below:

1.7 LEMMA. Let  $G$  be divisible, let  $b \in G$ , and suppose that  $\{s \in G | 0 < s \ll b\} \neq \emptyset$ . Then there is  $E \in \mathcal{L}A$ , an embedding  $G \hookrightarrow E$ , and  $h \in E$  such that (1)  $s \in G$  and  $s \ll b \Rightarrow s \ll h$ ; (2)  $h \ll b$ .

We prove 1.6: Suppose  $G$  is not archimedean. We can assume  $G$  divisible, by replacing  $G$  by its divisible hull. Take  $b$  as in 1.7 and apply 1.7 to produce  $G \hookrightarrow E$  and  $h$ , etc. We can assume that  $G$  majorizes

rizes  $E$ , by replacing  $E$  by the ideal generated by  $G$  (which ideal contains  $h$ , since  $h < b$ ). By 1.3 (b), choose an ideal  $I$  of  $E$  maximal for  $I \cap G = (0)$ , and let  $G \rightarrow H$  be the embedding  $G \hookrightarrow E \rightarrow E/I \cong H$ ; this is majorizing, and *large*. Let  $h' = h + I$ . Since homomorphisms preserve  $\ll$ , we find that  $s \in G$  and  $s \ll b \Rightarrow s \ll h'$ ; and  $h' \ll b$ . Thus  $h' \notin G$  (since  $h' \in G$  would imply that  $h' \ll h'$ , which is impossible).

To prove 1.7, we use an ultraproduct construction. See [BS], but we indicate the basics:

If  $G$  is a set,  $S$  is a set, and  $\mathcal{F}$  is an ultrafilter on  $S$ , the ultrapower  $G^S/\mathcal{F}$  is  $G^S$  modulo  $f \sim g \equiv \{s | f(s) = g(s)\} \in \mathcal{F}$ . Then  $G \rightarrow G^S/\mathcal{F}$  via  $g \mapsto \bar{g} \equiv$  the constant function with value  $g$ . If  $G$  is partially ordered, then so is  $G^S/\mathcal{F}$  via  $[f] \leq [g] \equiv \{s | f(s) \leq g(s)\} \in \mathcal{F}$ . If  $G \in \mathcal{LA}$ , one defines  $+$  in a natural way so that  $G^S/\mathcal{F} \in \mathcal{LA}$ , and  $G$  divisible  $\Rightarrow G^S/\mathcal{F}$  divisible.

1.8 LEMMA (Weinberg [W]). Let  $G$  be a partially ordered set,  $S$  a nonvoid strictly up-directed subset of  $G$ . Then there is an ultrafilter  $\mathcal{F}$  on  $S$  and  $h \in G^S/\mathcal{F}$  such that

- (1)  $\bar{g} < h$  iff  $g <$  some  $s \in S$ ; and
- (2)  $S < g \Rightarrow h < \bar{g}$ .

PROOF. For  $s \in S$ , let  $U_s = \{t | s < t\}$  and let  $\mathcal{F}$  be any ultrafilter with  $\mathcal{F} \supseteq \{U_s | s \in S\}$ . Let  $h$  be the equivalence class of the inclusion function  $S \rightarrow G$ . (1) and (2) can be verified without difficulty.

We prove 1.7: Let  $G$  and  $b$  be as in 1.7, let  $S = \{s \in G | 0 < s \ll b\}$ , and apply 1.8 to produce  $G^S/\mathcal{F} = E$  and  $h \in E$ , etc. Instead of  $\bar{g}$  per 1.8, we just write  $g$ . (1) If  $s \in S$  and  $n \in \mathbb{Z}^+$ , then  $ns < (n+1)s \in S$ , so that  $ns < h$  by 1.8 (1). Thus  $s \ll h$ . (2) If  $n \in \mathbb{Z}^+$ , then  $nS = S < b$ , when  $S < (1/n)b$  ( $G$  is divisible, recall), whence by 1.8 (2),  $h < (1/n)b$ , or  $nh < b$ . Thus,  $h \ll b$ .

This concludes the proof of 1.7, and so 1.6 is proved.

## 2. Behavior of $\mathcal{M}$ -injectivity.

We describe the place of  $\mathcal{M}$ -injectivity in  $\mathcal{LA}$  in Banaschewski's formal framework for relative injectivity given in [B<sub>2</sub>, B<sub>3</sub>]. The reader may not find it totally necessary to have [B<sub>3</sub>] in hand, but this would help.

The setting is a category  $\mathbf{C}$  with a distinguished class  $\mathcal{E}$  of morphisms.  $K \in |\mathbf{C}|$  is  $\mathcal{E}$ -injective if each  $G \rightarrow K$  « lifts » over any  $\mathcal{E}$ -morphism out of  $G$ . An  $\mathcal{E}$ -essential morphism is a  $\mu \in \mathcal{E}$  for which  $\alpha\mu \in \mathcal{E} \Rightarrow \alpha \in \mathcal{E}$ . An  $\mathcal{E}$ -injective hull of  $G \in |\mathbf{C}|$  is an  $\mathcal{E}$ -essential  $G \rightarrow K$  with  $K$   $\mathcal{E}$ -injective.  $\mathcal{E}$ -injectivity is called « properly behaved » if (A) the analogue of 1.2 holds, and (B) each object has an essentially unique  $\mathcal{E}$ -injective hull, and (C) every  $\mathcal{E}$ -injective hull satisfies certain conditions spelled out for  $(\mathcal{L}\mathcal{A}, \mathcal{M})$  in 2.3 below.

$[B_2, B_3]$  present axioms *E1-E6* on  $(\mathbf{C}, \mathcal{E})$  ensuring proper behavior, and, in particular, show that *E1-E4*  $\Rightarrow$  (A). Our 1.3 (b) is *E3* for  $(\mathcal{L}\mathcal{A}, \mathcal{M})$ . *E4* is that  $G \xrightarrow{\mu} H \in \mathcal{E}$  and  $G \xrightarrow{\nu} K$  embed into a square with  $\mu$  on top and with bottom  $\in \mathcal{E}$ ; this follows from existence of push-outs « preserving  $\mathcal{E}$  » as in 1.3 (c). Our proof that 1.3  $\Rightarrow$  1.2 is essentially the proof that *E1-E4*  $\Rightarrow$  (A).

$(\mathcal{L}\mathcal{A}, \mathcal{M})$  doesn't satisfy (B), of course, but 1.1 and 1.2 easily yield

2.1 For  $(\mathcal{L}\mathcal{A}, \mathcal{M})$ :  $\mathcal{M}$ -injective hulls are unique, and  $G$  has an  $\mathcal{M}$ -injective hull (the completion of the divisible hull) iff  $G$  is archimedean.

2.1 has its place in abstract  $\mathcal{E}$ -injectivity, as well: *E5* is this: Each well-ordered direct system in  $\mathcal{E}$  has an upper bound in  $\mathcal{E}$ . This holds for  $(\mathcal{L}\mathcal{A}, \mathcal{M})$  because  $\mathcal{L}\mathcal{A}$  is a variety. And, « *E6* at  $G \in |\mathbf{C}|$  » is that the class of all  $\mathcal{E}$ -essential  $G \rightarrow H$  is *small*. Then, with *E1-E5* assumed, *E6* at  $G$  holds iff  $G$  has a unique  $\mathcal{E}$ -injective hull. ( $\Rightarrow$  is Prop. 3 § 1 of  $[B_2]$  or in Prop. 4 of  $[B_3]$ .  $\Leftarrow$  is easy using (A).) Thus

2.2 For  $(\mathcal{L}\mathcal{A}, \mathcal{M})$ : *E6* holds at  $G$  iff  $G$  is archimedean.

PROOF (Sketch).  $\Leftarrow$ . For  $G$  archimedean and  $G \subseteq H$   $\mathcal{M}$ -essentially, for  $h \in H^+$ ,  $h = VG_h$  for some  $G_h \subseteq G$  as in the proof of 1.1, so that  $|H| \leq 2^{|G|}$ , and the condition follows.  $\Rightarrow$  can be argued from 1.5.

So 2.2 and Banaschewski's results imply 2.1. Looking more closely at *how* yields this novel construction of the completion of the divisible hull of archimedean  $G$ , as a « maximal  $\mathcal{M}$ -essential extension »: let  $G \xrightarrow{\mu_\alpha} H_\alpha$ , or  $(H_\alpha, \mu_\alpha)$ , be a *set* representing all  $\mathcal{M}$ -essential extensions of  $G$  (by *E6* at  $G$ ), made into a partially ordered set  $\mathcal{F}$  and a direct system by  $(H_\alpha, \mu_\alpha) \leq (H_\beta, \mu_\beta)$  if there is  $H_\alpha \xrightarrow{\iota} H_\beta$  (the bonding morphism) with  $\iota\mu_\alpha = \mu_\beta$ . *E5* permits application of a maximality principle to infer a maximal member of  $\mathcal{F}$ .

The following asserts (C) for our circumstance:

2.3 In  $\mathcal{L}\mathcal{A}$ , the following conditions on  $G \xrightarrow{\mu} K \in \mathcal{M}$  are equivalent.

- (a)  $G \xrightarrow{\mu} K$  is an  $\mathcal{M}$ -injective hull of  $G$ .
- (b)  $\mu$  is  $\mathcal{M}$ -essential, and if  $\alpha\mu$  is  $\mathcal{M}$ -essential, then  $\alpha$  is an isomorphism.
- (c)  $K$  is  $\mathcal{M}$ -injective, and if  $\mu = \alpha\beta$  for  $\alpha, \beta \in \mathcal{M}$  and domain  $\alpha$   $\mathcal{M}$ -injective, then  $\alpha$  is an isomorphism.

PROOF. According to Cor. 2, § 1 of [B<sub>2</sub>], 2.3 will hold if we show that (c)  $\Rightarrow$   $G$  has an  $\mathcal{M}$ -injective hull, that is, by 2.1,  $G$  is archimedean. Assume (c). By 1.1,  $K$  is complete, hence archimedean, and so is  $G$ .

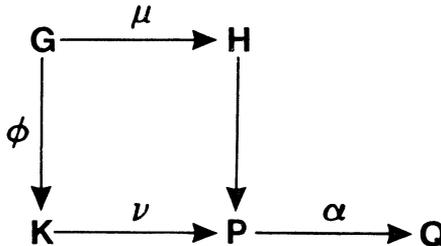
Finally, let Arch be the full subcategory of  $\mathcal{L}\mathcal{A}$  with archimedean objects. The situation in Arch and its relation to  $\mathcal{L}\mathcal{A}$  is a model (as is seen easily now) of these more general observations:

Suppose  $\mathcal{C}$  is a variety with  $\mathcal{E}$  a class of embeddings satisfying E1-E3 and E4': Pushouts preserve  $\mathcal{E}$  (in the sense of 1.3). Here, E5 holds. Let  $\mathcal{S}$  be the full subcategory with objects the  $G$ 's where E6 holds. So  $G \in |\mathcal{S}|$  iff  $G$  has an  $\mathcal{E}$ -injective hull iff  $G$  is a subobject of an  $\mathcal{E}$ -injective object.

2.4 (a) In  $\mathcal{S}$ , E1-E6 hold and  $\mathcal{E}$ -injectivity is properly behaved.

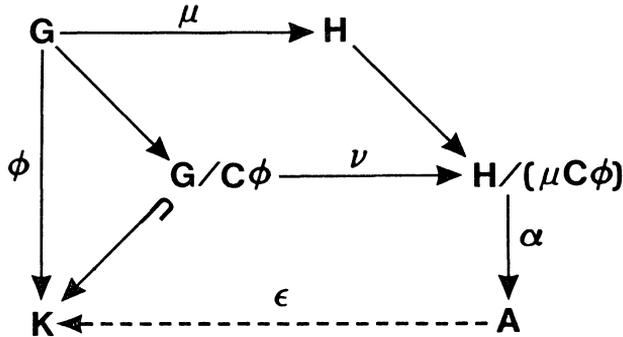
(b) Let  $K \in |\mathcal{C}|$ .  $K$  is  $\mathcal{E}$ -injective in  $\mathcal{C}$  iff  $K$  is  $\mathcal{E}$ -injective in  $\mathcal{S}$ .

PROOF. (a) E4 is the only thing to verify: We construct



from  $\mu \in \mathcal{E}$  and  $\phi$  (both assumed in  $\mathcal{S}$ ) by pushing out to get the  $\mathcal{C}$ -square with  $\nu \in \mathcal{E}$  (by E4'). E3 (cf. 1.3 (b)) provides  $\alpha$  with  $\alpha\nu \in \mathcal{E}$ -essential. This implies that  $Q \in |\mathcal{S}|$ : We have the  $\mathcal{E}$ -injective hull  $K \xrightarrow{\iota} \hat{K}$  and there is  $Q \xrightarrow{\varepsilon} \hat{K}$  with  $\varepsilon(\alpha\nu) = \iota$ . By  $\mathcal{E}$ -essentiality of  $\alpha\nu$ ,  $\varepsilon \in \mathcal{E}$  hence is an embedding.

(b).  $\Rightarrow$  is clear. Conversely, consider



where  $\mu \in \mathcal{E}$  and  $\varphi$  (both in  $\mathcal{C}$ ) are given and  $K$  is  $\mathcal{M}$ -injective in  $\mathcal{S}$ . Let  $C\varphi$  be the congruence for  $\varphi$ , and  $(\mu C\varphi)$  the generated congruence in  $H$ . We create  $\nu$  as shown and  $\nu$  is the bottom of a pushout square; whence  $\nu \in \mathcal{E}$  by  $E4'$ .  $E3$  provides  $\alpha$  with  $\alpha\nu \in \mathcal{E}$ -essential. Since  $G/C\varphi \in |\mathcal{S}|$ ,  $A \in |\mathcal{S}|$  follows as in (a). Then  $\epsilon$  exists, hence also the desired « lift » of  $\varphi$  over  $\mu$ .

### 3. Amalgamations in Arch.

A pair of embeddings  $G \xrightarrow{\sigma_i} H_i$  ( $i = 1, 2$ ) is said to amalgamate if there are embeddings  $H_i \xrightarrow{\tau_i} P$  ( $i = 1, 2$ ) with  $\tau_1\sigma_1 = \tau_2\sigma_2$ . A category  $\mathcal{C}$  is said to have the Amalgamation Property (AP) if, in  $\mathcal{C}$ , each pair of embeddings amalgamates. As noted earlier, Pierce has shown that  $\mathcal{L}A$  has AP [ $P_1$ ], while AP fails in Arch [ $P_2$ ]. Also, there is strong connection between types of injective completeness and versions of AP: On one hand, under simple hypotheses on  $\mathcal{C}$ , relative injective completeness implies AP (see [ $P_1$ ] for a discussion); on the other hand, §2 explains how the version of « AP for  $\mathcal{E}$  » embodied in  $E4$  implies, under further hypotheses,  $\mathcal{E}$ -injective completeness. This motivates us to the following observations about amalgamating in Arch.

**3.1 THEOREM.** In Arch, the embeddings  $G \xrightarrow{\sigma_i} H_i$  ( $i = 1, 2$ ) will amalgamate if one of the following holds:

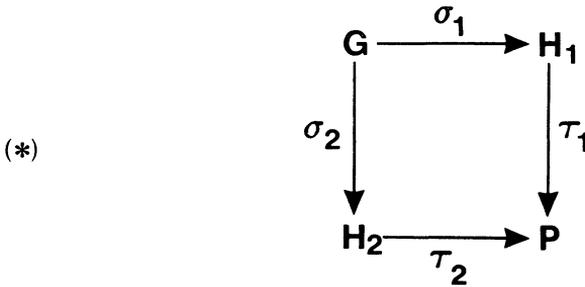
- (a) The  $\sigma_i$  are majorizing.
- (b) The  $\sigma_i$  are large.
- (c)  $\sigma_1$  is majorizing and large.

3.1 (b) is a special case of, but also the core of, Theorem 3 of [P<sub>2</sub>]. In any event, the following yields a proof immediately.

3.2 Conrad [C]). In Arch, each  $G$  has a maximum essential extension  $G \xrightarrow{\varepsilon} G^{\text{ess}}$ : If  $G \xrightarrow{\sigma} H$  is essential, then there is  $H \xrightarrow{\tau} G^{\text{ess}}$  with  $\tau\sigma = \varepsilon$ .

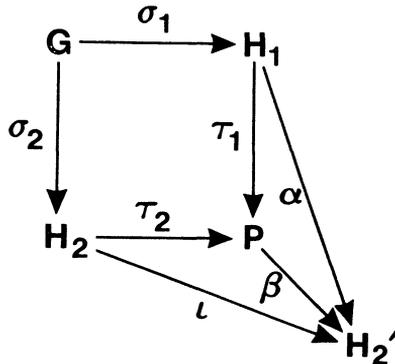
The rest of 3.1 can be proved using either products or pushouts. We choose the latter:

3.3 LEMMA. In a category, suppose



is a pushout square, that  $\sigma_1$  is an embedding, and that  $H_2$  embeds in a  $\sigma_1$ -injective. Then,  $\tau_2$  is an embedding.

PROOF. Consider the diagram



arising from (\*) as follows: First,  $H_2 \xrightarrow{\iota} H_2'$  is the hypothesized embedding into the  $\sigma_1$ -injective  $H_2'$ . Then, there is  $\alpha$  with  $\alpha\sigma_1 = \iota\sigma_2$ . Then, because (\*) is a pushout, there is  $\beta$  with  $\beta\tau_1 = \alpha_1$  and  $\beta\tau_2 = \iota$ . Thus,  $\beta\tau_2$  is an embedding, hence  $\tau_2$  is an embedding.

3.4 COROLLARY. Suppose the category has pushouts. Let  $\mathcal{E}$  be a class of embeddings for which each object embeds into an  $\mathcal{E}$ -injective. Then, the pair of embeddings  $G \xrightarrow{i} H_i$  ( $i = 1, 2$ ) will amalgamate if either of the following holds:

(1) Each  $\sigma_i \in \mathcal{E}$ .

(2)  $\sigma_1 \in \mathcal{E}$  and  $\sigma_1$  is embedding-essential for the category (*i.e.*,  $\varphi\sigma_1$  an embedding  $\Rightarrow \varphi$  embedding).

PROOF. Pushout  $G \xrightarrow{\sigma_i} H_i$  to a diagram like (\*) in 3.2.

If each  $\sigma_i \in \mathcal{E}$ , we apply 3.2 twice to conclude that each  $\tau_i$  is an embedding. This proves (1). If just  $\sigma_1 \in \mathcal{E}$ , then  $\tau_2$  is an embedding by 3.2; so  $\tau_2\sigma_2 = \tau_1\sigma_1$  is an embedding also. Thus, if  $\sigma_1$  essential, then  $\tau_1$  is an embedding. This proves (2).

3.4 and the following prove 3.1 (a) and (c).

3.5 Arch has pushouts.

PROOF. Any variety  $\mathcal{U}$  (like  $\mathcal{L}A$ ) has pushouts [J]. Any  $SP$ -subclass  $\mathcal{R}$  (like Arch) is reflective in  $\mathcal{U}$ , that is, to each  $V \in \mathcal{U}$  corresponds  $rV \in \mathcal{R}$  and a map  $V \xrightarrow{\rho} rV$  with the property: If  $R \in \mathcal{R}$ , and  $\varphi \in \text{Hom}(V, R)$  then there is unique  $\tilde{\varphi} \in \text{Hom}(rV, R)$  with  $\tilde{\varphi}\rho = \varphi$ . Then, if  $\mathcal{U}$  has pushouts and  $\mathcal{R}$  is reflective, then  $\mathcal{R}$  has pushouts (obtained by «reflecting»  $\mathcal{U}$ -pushouts).

Finally, consider, in Arch, the class  $\mathcal{E}$  of majorizing embeddings which are also epimorphisms of Arch (*i.e.*, right cancellable morphisms). With reference to the Banaschewski scheme discussed in § 2, one can verify that  $\mathcal{E}$ -essential =  $\mathcal{E}$ , whence E6 holds, and that the analogue of 1.3 (b) holds (from 3.1 (1), or more directly, from 3.3), whence E3 holds. The other axioms present no trouble, and it follows (from  $[B_2, B_3]$ ; see § 2) that:

3.6 In Arch, majorizing-epi-injectivity is properly behaved.

Let  $\mathcal{R}$  be the full subcategory of Arch whose objects are  $\mathcal{E}$ -injective. Since  $\mathcal{E} \subseteq \text{epics}$ , it follows that  $\mathcal{R}$  is epi-reflective (a reflection morphism  $G \rightarrow rG$  being the embedding into the  $\mathcal{E}$ -injective hull) and evidently, the smallest reflective subcategory for which each reflection morphism is a majorizing epic embedding.

This seems quite interesting, but we have been unable to determine

what the objects of  $\mathcal{R}$  are. The basic problem is that we don't know what the Arch-epimorphisms are, nor even if Arch is co-well-powered. The strongest working conjecture is that an embedding  $G \hookrightarrow H$  is epic iff  $H$  is a subobject of the vector lattice hull of  $G$ . It would then follow that Arch is co-well-powered, that every epic is majorizing so that  $\delta$ -injective = epi-injective, and  $\mathcal{R}$  = vector lattices. The author's opinions on this differ.

The situation for  $\mathcal{L}A$  is that  $G \hookrightarrow H$  is epic iff  $H$  is a subobject of the divisible hull of  $G$  [AC], and hence that epi-injective = divisible. See also [HM].

Bacsich has given an elegant discussion of epi-injectivity for universal theories in [B<sub>1</sub>], which see.

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