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BRUNELLA BRUNO
RICHARD E. PHILLIPS

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On Minimal Conditions Related to Miller-Moreno Type Groups.

BRUNELLA BRUNO (*) - RICHARD E. PHILLIPS (*)

1. Introduction.

In [1] and [2], Belyaev and Sesekin have given a detailed account of locally finite groups G in which G' is infinite while every proper subgroup of G has finite derived group. In [2] such groups are said to be of Miller-Moreno type; these groups are special types of Černikov groups. Details of such groups can be found in [2] and also in our Section 7.2.

In the paper we present what amounts to a three way generalization of the Belyaev-Sesekin results. We denote the class of locally graded groups by $\mathfrak{L}$ ... see § 2 for the relevant definitions. Our principal result is given as

THEOREM 1. *Let $G \in \mathfrak{L}$ and $k \geqslant 1$ be a positive integer. Suppose further that for every properly descending chain of subgroups*

$$G_1 > G_2 > \ldots > G_n > \ldots$$

there is a $t \geqslant 1$ such that the k-th lower central term $\gamma_k(G_t)$ is finite. Then either G is a Černikov group or $\gamma_k(G)$ is finite.

This result provides the lever necessary for proving

(*) Indirizzo degli AA.: B. BRUNO: Istituto di Algebra e Geometria, Università, Via Belzoni 7, 35100 Padova; R. E. PHILLIPS: Departement of Mathematics, Michigan State University, East Lansing, Michigan 48824, USA.

THEOREM 2. *The following conditions on a locally graded group G are equivalent.*

i) *For some $k \geqslant 1$ $\gamma_k(G)$ is infinite and for every proper subgroup H of G $\gamma_k(H)$ is finite.*

ii) *G' is infinite and for every proper subgroup H of G, H' is finite.*

iii) *G is a Černikov group with G' infinite and every proper subgroup of G is either Abelian or finite.*

The generalizations mentioned above are

1) the replacement of the « all proper subgroups » condition implicit in the Miller-Moreno groups by the weaker « minimal condition » on certain subgroups;

2) the replacement of the « derived group » condition by the condition on the k-th lower central term;

3) substituting « locally graded » for « locally finite ».

A portion of the Belyaev-Sesekin results—namely that the locally finite Miller-Moreno groups are Černikov groups—follows directly from Theorem 2. Finer structural properties of such groups do not follow directly, but can be obtained with a little additional work—we will do this in § 7.2.

The *locally graded* condition is present in order that we avoid the Tarski and other such « monsters ». Indeed, the results of Ol'šhanskii [11] and Rips show that our theorems are false without some sort of finiteness condition.

Possible generalizations to the results herein as well as the methods used in our proofs will be discussed in § 2 where more precise terminology is available.

Our Theorem 2 falls within that body of results known as « groups with restricted subgroups ». We refer to the introductions of [13] and [3] or the interesting [6] for general discussions of these types of problems.

Theorem 1 adds to the vast literature on groups satisfying various minimal conditions (see [15; Chapter 3], [10], [12]). Obviously, every $\mathfrak{L}$-group with the minimal condition on subgroups (= min) satisfies the hypotheses of our Theorem 1 and it is not difficult to deduce from Theorem 1 that *every $\mathfrak{L}$-group with min is a Černikov group.* Thus Theorem 1 may be viewed as a generalization of the Šunkov-Kegel-

Wehrfritz theorem for locally finite groups with min [10; p. 172]. We note however that we use the recently confirmed classification of finite simple groups, which gives considerable insight into locally finite simple groups with various minimality conditions. This, in effect, is used to overcome what has been recognized for some time as the principal difficulty in dealing with questions of this type.

2. Notation.

We quickly review some elementary facts to be used in the sequel. If n is a positive integer, $\mathfrak{N}_n$ denotes the class of nilpotent groups of class n or less while $\mathfrak{N}$ denotes the class of nilpotent groups. The terms of the upper central series of a group G are denoted $\zeta_i(G)$ while $R(G)$ is the set of right Engel-elements of G (see [16; Chapter 7] for the relevant definitions). We make frequent use of a result of Baer (see [16; p. 52]) which asserts that in a Noetherian group (= groups with the maximum condition on subgroups),

(2.1) *there is a positive integer n such that*

$$R(G) = \zeta_n(G) .$$

The terms of the lower central series are denoted by $\gamma_i(G)$ (beginning with $\gamma_0(G) = G$). We need both the Schur-Baer properties and related results of P. Hall (see [15; pp. 111-119]).

(2.2) *a) if $G/\zeta_n(G)$ is finite then $\gamma_n(G)$ is finite, and*

b) if $\gamma_n(G)$ is finite then $G/\zeta_{2n}(G)$ is finite.

For any two classes of groups Ω and Λ, $\Omega\Lambda$ is the class of all extensions of Ω-groups by Λ-groups. Thus, with $\mathfrak{F}$ the class of finite groups, $\mathfrak{F}\mathfrak{N}$ is the class of «finite-by-nilpotent» groups.

A group G is *locally graded* if every non-trivial finitely generated subgroup of G has a non-trivial finite image. The class of locally graded groups is very extensive as it contains each of the classes «locally solvable» (or more generally the SN-groups), «locally finite», «residually finite», etc.

Recall that a group G is a Černikov group if G is an «Abelian-by-finite» group with the minimal condition on subgroups. We denote this class by $\mathfrak{C}$ and throughout assume many special properties of these groups (as, for example, in [15; Chapter 3] and [10; 1. E]).

If Σ is any class of groups, $M(\overline{\Sigma})$ is the class of *locally graded* groups G such that G has the minimal condition on non-Σ $(=\overline{\Sigma})$ subgroups; i.e., every properly descending chain

$$G_1 > G_2 > \ldots > G_n > \ldots$$

of subgroups of G has the property that for some k, $j \geqslant k$ implies that $G_j \in \Sigma$. Obviously, $\mathfrak{C} \leqslant M(\overline{\Sigma})$ and if Σ is a subgroup closed class then $\Sigma \leqslant M(\overline{\Sigma})$. Our Theorem 1 may be rephrased as

THEOREM 1. If $k \geqslant 1$, $M(\overline{\mathfrak{F}\mathfrak{N}_k}) = \mathfrak{F}\mathfrak{N}_k \cup \mathfrak{C}$.

It will be convenient to have a special notation for the class of *locally graded* groups G such that $G \notin \Sigma$ while every proper subgroup of G is in Σ; we denote this class by Σ^*. Thus, $\Sigma^* \leqslant M(\overline{\Sigma})$ and Theorem 2 now becomes

THEOREM 2. *The following conditions on a group G are equivalent.*

i) *For some $k \geqslant 1$, $G \in (\mathfrak{F}\mathfrak{N}_k)^*$.*

ii) $G \in (\mathfrak{F}\mathfrak{N}_1)^*$.

iii) $G \in \mathfrak{C}$, $G \notin \mathfrak{F}\mathfrak{N}_1$ *and every proper subgroup of G is either Abelian or finite.*

Possible generalizations of our Theorems could come from changing the class $\mathfrak{F}\mathfrak{N}_k$ to some wider class. For example it may be possible to obtain variants of the Theorems for the class $M(\overline{\mathfrak{S}_d\mathfrak{F}})$; here $\mathfrak{S}_d$ is the class of solvable groups of derived length d. One must keep in mind that there are infinite locally finite simple groups with all proper subgroups in $(\mathfrak{S}_2 \cup \mathfrak{F}) \leqslant \mathfrak{S}_2\mathfrak{F}$ [17] and so these groups would have to be incorporated into any such generalizations. The first author has studied the class $M(\overline{\mathfrak{N}_1\mathfrak{F}})$ in [4]; non-trivial examples of such groups are the p-groups of Heineken and Mohamed [9]. Such complexities can also be expected in the classes $M(\overline{\mathfrak{N}_k\mathfrak{F}})$ and $M(\overline{\mathfrak{N}\mathfrak{F}})$.

The methods used to prove Theorem 1 consist of several steps. Our Section 3 is devoted to developing criteria that insure that certain groups have « enough » large normal subgroups. Such criteria will ultimately be used to study the class $(\mathfrak{F}\mathfrak{N}_K)^*$. In § 4 we show that the groups G in our Theorem 1 are locally finite. The non-existence of simple groups in the class $(\mathfrak{C} \cup \mathfrak{F}\mathfrak{N}_k)^*$ is taken up in § 5. We use, in an essential way and in more than one place, the recent classification of finite simple groups and some recently verified consequences of this for locally finite groups. It may well be possible to prove such a non-simplicity result without using the classification but we have been unable to do so.

The indirect proof of Theorem 1 is taken up in § 6; one passes immediately to a minimal counter example S which is a locally finite group in the class $(\mathfrak{C} \cup \mathfrak{F}\mathfrak{N}_k)^*$. Here the non-simplicity of S is used together with the preliminaries in § 3 to complete the proof of Theorem 1. The much easier proof of Theorem 2 as well as other characterizations of $(\mathfrak{F}\mathfrak{N}_1)^*$-groups (Propositions 3 and 4) is taken up in § 7.1.

In § 7.2 we go on to develop a complete classification of the Miller-Moreno type $\mathfrak{L}$-groups; as noted earlier, this has already been done by Belyaev and Sesekin and we include this section only for completeness.

3. Basic Lemmas.

We here present the notion of an n-decomposable group which is an extension of an idea put forth in [2].

DEFINITION. Let n be a positive integer, $n \geqslant 2$. The group G is n-decomposable (and called a D_n-group) if there are normal subgroups $A_1, \ldots, A_n$ of G such that

(3.1) i) $G = A_1 A_2 \ldots A_n$, *and*

ii) *for any* $i, 1 \leqslant i \leqslant n$, $\langle A_j | j \neq i \rangle < G$

(here, as elsewhere, $<$ means proper subgroup).

The importance of this concept for our purposes is indicated in

LEMMA 1. *Suppose* $n \geqslant 1$ *and that every proper subgroup of* G *is in* $\mathfrak{F}\mathfrak{N}_n$. *If* G *is a* D_{n+2}-*group then* $G \in \mathfrak{F}\mathfrak{N}_n$.

PROOF. Let $A_1, \ldots, A_{n+2}$ be normal subgroups of G that satisfy (3.1). Then

$$\gamma_n(G) = \langle [R_1, \ldots, R_{n+1}] | R_i \in \{A_1, \ldots, A_{n+2}\} \rangle$$

and each commutator $[R_1, \ldots, R_{n+1}]$ lies in the n-th lower central subgroup of a proper subgroup of G; thus $[R_1, \ldots, R_{n+1}]$ is finite. Further, there are only a finite number of commutators $[R_1, \ldots, R_{n+1}]$ and it follows that $\gamma_n(G)$ is the product of a finite number of finite normal subgroups and so is finite. Thus $G \in \mathfrak{F}\mathfrak{N}_n$ and the proof is complete.

The next two lemmas determine situations in which Lemma 1 can be applied.

LEMMA 2. *If G is a non-trivial torsion-free Abelian group, then G satisfies D_n for every $n \geqslant 2$.*

PROOF. Let $n \geqslant 2$ be a positive integer and suppose first that there are primes $q_1, \ldots, q_n$ such that $q_i G < G$. Then the periodic group

$$G/\bigcap_{i=1}^{n} q_i G$$

has n primary components and the lemma follows easily. Thus if p is a prime we must have $pG = G$ with at most $n-1$ exceptions.

Since in an infinite cyclic group Z we have $pZ < Z$ for all p we may assume that G is not free. Let T be a maximal free subgroup of G; then $0 \neq T$ and $H = G/T$ is periodic. If x is a free generator of T and q is a prime for which $qG = G$ then there is a $y \in G$ such that $qy = x$. Since T is free $y \notin T$ so that H has elements of order q. Thus H has an infinite number of primary components and it follows that G satisfies D_n.

The following two facts are, no doubt, well known; since they play an essential role in what follows, we indicate proofs.

(3.2) *Let G be a periodic group;*

a) if G si nilpotent and G/G' has a divisible subgroup of finite index, then G' is finite,

b) if H is a normal divisible Abelian subgroup of G and for some $n \geqslant 1$, $H \leqslant \zeta_n(G)$, then $H \leqslant \zeta_1(G)$. Thus if $G \in (\mathfrak{F}\mathfrak{N} \cap \mathfrak{C})$ and B is the maximal divisible subgroup of G, $B \leqslant \zeta_1(G)$; in particular G is «central-by-finite».

For the proof of $(3.2(a))$ write $G/G' = D \oplus R$ where D is divisible and R reduced. Since the lower central factors of G are images of tensor powers of G/G' [15; pp. 54-57] and $D \otimes C = 0$ for any periodic Abelian group C we see that $\gamma_k(G)/\gamma_{k+1}(G)$ is finite for $k \geqslant 1$; thus, G' is finite.

The proof of $(3.2(b))$ follows easily from the fact that $[H, G, G] =$

$= [H, G]$ (see [15; p. 69]; this also uses a tensor product argument). Since $[H, nG] = 1$ we must have $[H, G] = 1$. The second part of (3.2(b)) follows from the first part together with (2.2(b)).

LEMMA 3. *Let G be a nilpotent group and suppose that for some $n \geqslant 2$, G is not a D_n-group. Then G has a finite normal subgroup V such that G/V is a periodic divisible Abelian group. In particular such groups are periodic and have finite derived groups.*

PROOF. If G/G' is not periodic then G has a non-trivial torsion-free Abelian image. Lemma 2 shows that G is n-decomposable for every $n \geqslant 2$. Thus, we may assume that G/G' is periodic (and so G is periodic [15; p. 55]).

If the reduced part of G/G' is infinite then for every $n \geqslant 2$, G/G' has at least n-direct factors. Thus, for $n \geqslant 2$, $G \in D_n$; we conclude that the reduced part of G/G' is finite and the lemma now follows from (3.2(a)).

4. Reduction to locally finite groups.

In this section we show that the $M(\overline{\mathfrak{F}\mathfrak{N}_K})$-groups are locally finite if they are not in $\mathfrak{F}\mathfrak{N}_K$; for this we require two preparatory lemmas.

LEMMA 4. *Let G be a finitely generated locally graded group with every proper subgroup in $\mathfrak{F}\mathfrak{N}$. Then $G \in \mathfrak{F}\mathfrak{N}$.*

PROOF. We may certainly suppose that G is an infinite finitely generated group. Since G is locally graded G has proper normal subgroups H with G/H finite. Since any such H is a finitely generated $\mathfrak{F}\mathfrak{N}$-group, G is Noetherian. At this point the proof splits into two cases.

CASE 1. *Every finite image of G is cyclic.* Let H be a proper normal subgroup of finite index in G. Since H is finitely generated and in $\mathfrak{F}\mathfrak{N}$ we may suppose that H is finitely generated nilpotent. Thus, H is residually finite [16; p. 129] and so G is residually finite. The assumption that all finite images of G are Abelian now implies G is Abelian, and we now proceed to

CASE 2. *For some normal subgroup H of finite index in G, G/H is not cyclic.* In this case $V_x = H\langle x\rangle$ is a proper subgroup of G for every

$x \in G$. It follows from $(2.2(b))$ that for each $x \in G$ there is a positive integer n_x such that $V_x/\zeta_{n_x}(V_x)$ is finite. Thus $U_x = H \cap \zeta_{n_x}(V_x)$ has finite index in H and if $y \in V_x$ then $[U_x, n_x y] = 1$.

Let $T = \{t_1, \ldots, t_k\}$ be a transversal of H in G and put $U_i = U_{t_i}$. If $\sigma = \max\{n_{t_1}, \ldots, n_{t_k}\}$ and $S = \bigcap_{i=1}^{k} U_i$ then $[S, \sigma x] = 1$ for all $x \in G$. Thus, $S \leqslant R(G)$ and since S has finite index in G, we have $G/R(G)$ finite. We now use (2.1) and deduce that for some $s \geqslant 1$, $G/\zeta_s(G)$ is finite; application of $(2.2(a))$ now completes the proof.

LEMMA 5. *Let $k \geqslant 1$ and suppose $G \in (\mathfrak{F}\mathfrak{N}_k)^*$. If G has a local system of $\mathfrak{F}\mathfrak{N}_k$-subgroups then G is locally finite.*

PROOF. Suppose G satisfies the above hypotheses. Then $\gamma_k(G)$ is locally finite; further, if T is the locally finite radical of G then G/T is a torsion free $\mathfrak{N}_k$-group. If $T < G$ then Lemma 3 gives $G \in D_{k+2}$ and Lemma 1 then implies that $G \in \mathfrak{F}\mathfrak{N}_k$. From this contradiction we have $T = G$, as desired.

PROPOSITION 1. *For $k \geqslant 1$, $M(\overline{\mathfrak{F}\mathfrak{N}_k})$-groups are locally finite if they are not $\mathfrak{F}\mathfrak{N}_k^1$.*

PROOF. We will first prove

(4.1) *$(\mathfrak{F}\mathfrak{N}_k)^*$-groups are locally finite; consequently there are no finitely generated $(\mathfrak{F}\mathfrak{N}_k)^*$-groups.*

For the proof of (4.1) let $G \in (\mathfrak{F}\mathfrak{N}_k)^*$ and suppose H is an infinite finitely generated subgroup of G; by Lemma 4, $H \in \mathfrak{F}\mathfrak{N}$. Thus, H has a non-trivial, nilpotent, torsion-free image and, by Lemma 3, $H \in D_{k+2}$; Lemma 1 now gives $H \in \mathfrak{F}\mathfrak{N}_k$ and (4.1) now follows immediately from Lemma 5.

Now suppose $G \in M(\overline{\mathfrak{F}\mathfrak{N}_k})$ and let H be any finitely generated subgroup of G. If H is infinite, the fact that G is locally graded implies that H has a properly descending chain of subgroups of finite index. Thus, H has an $\mathfrak{F}\mathfrak{N}_k$-subgroup of finite index and so H is Noetherian. Hence, *every finitely generated subgroup of G is Noetherian.*

If G has a finitely generated subgroup U with $U \notin \mathfrak{F}\mathfrak{N}_k$ then U contains a subgroup V with $V \in (\mathfrak{F}\mathfrak{N}_k)^*$; since V is finitely generated we have contradicted (4.1) and we conclude that *every finitely generated subgroup of G is in $\mathfrak{F}\mathfrak{N}_k$*. Thus the set T of all elements of finite order in G is a locally finite normal subgroup of G and G/T is a torsion-free $\mathfrak{N}_k$-group.

Suppose that $T < G$; then G has an infinite cyclic subgroup $\langle x \rangle$ and since the sequence $T\langle x^{2^i} \rangle$, $i = 0, 1, 2, \ldots$ is a properly descending chain of subgroups of G we must have $T \in \mathfrak{F}\mathfrak{N}_k$. But since $G \notin \mathfrak{F}\mathfrak{N}_k$ there are subgroups V of G with $V \in (\mathfrak{F}\mathfrak{N}_k)^*$; by (4.1), $V \leqslant T$ and from the contradiction we have $G = T$.

5. Simple $(\mathfrak{F}\mathfrak{N}_k \cup \mathfrak{C})^*$-groups.

There is now available, thanks to the classification of finite simple groups, enough information regarding locally finite simple groups to establish

PROPOSITION 2. *For $k \geqslant 1$, there are no simple groups in the class $(\mathfrak{F}\mathfrak{N}_k \cup \mathfrak{C})^*$.*

Before proceeding we note that Belyaev has shown in [1] (without using the classification of finite simple groups) that there are no locally finite simple groups in the class $(\mathfrak{F}\mathfrak{N}_1)^*$. Extensions of Belyaev's ideas can be used to show that there are no locally finite simple groups in $(\mathfrak{F}\mathfrak{N}_1 \cup \mathfrak{C})^*$.

There is a related result, essentially more general than Proposition 2, now known and we record this as

PROPOSITION 2′. *Let G be an infinite locally finite simple group with all proper subgroups «solvable-by-finite». Then either $G \simeq PSL(2, F)$ or $G \simeq Sz(F)$ where F is some suitable locally finite field.*

The Proposition 2′ is a consequence of recent work of G. Shute [18] whose results are far too complicated to give in detail here. Before we give a (very brief) sketch of the methods of Shute we note that Proposition 2′ does, in fact, imply Proposition 2. To see this, suppose that G is a simple group in $(\mathfrak{F}\mathfrak{N}_k \cup \mathfrak{C})^*$. Then G is infinite and Proposition 1 implies that G is locally finite. Since the proper subgroups of G are «solvable-by-finite» (from (2.2(b))), Proposition 2′ asserts that G must be one of the two types $PSL(2, F)$ or $Sz(F)$, F some infinite locally finite field. Both of these groups contain subgroups which are neither $\mathfrak{C}$-groups nor in $\mathfrak{F}\mathfrak{N}_k$ ... we refer to [12; p. 59] where these subgroups are explicitly given; Proposition 2 now follows.

The proof of Proposition 2′ will appear in [18]. Here we only indicate the way in which the classification of finite simple groups is used. We make free use of the terminology of [10; Chapter 4] and [5].

Let G be any infinite locally finite simple linear group; i.e., $G \leqslant GL(n, K)$ where K is a locally finite field. From 4.6 of [10] and the classification of finite simple groups there is a Chevalley functor (or type) $\mathfrak{D}$ and a chain G_i of finite simple subgroups of G, each of type $\mathfrak{D}$, such that $G = \bigcup G_i$; here $\mathfrak{D}$ may be of twisted or untwisted type and has fixed rank parameter. The unions of such chains have been analyzed by Shute [18] who shows (along with many other results) that $G = \bigcup G_i$ contains a subgroup V such that $V/\zeta_1(V)$ is isomorphic with either $PSL(2, F)$ or $Sz(F)$, F some infinite locally finite field.

Using these results, Proposition 2′ easily follows provided we show that the group G in Proposition 2′ is linear. Now let G be an infinite locally finite simple group with all proper subgroups « solvable-by-finite ». From [10; p. 114], G is countable and it is easy to prove that G is not « enormous » (see [10; p. 122] for the definition of « enormous ») It now follows from 4.8 of [10] together with the classification that G is linear and this concludes our discussion of Proposition 2.

6. Proof of Theorem 1.

§ 6.1. The following lemma is of fundamental importance for the proofs of Theorems 1 and 2.

LEMMA 6. *Let G be a locally finite group with all proper subgroups either in $\mathfrak{C}$ or $\mathfrak{F}\mathfrak{N}$ and suppose also that G has a proper subgroup of finite index. Then*

a) if G has the minimal condition on subgroups of finite index then either $G \in \mathfrak{C}$ or G is « central-by-finite »;

b) if G does not have the minimal condition on subgroups of finite index then $G \in \mathfrak{F}\mathfrak{N}$.

PROOF OF (*a*). Suppose $G \notin \mathfrak{C}$ and let *U be the unique minimal subgroup of finite index in G*; we may assume that $U < G$. If $U \in \mathfrak{C}$ then $G \in \mathfrak{C}$ also and we have $U \in \mathfrak{F}\mathfrak{N}$. An easy argument shows that U/U' is divisible and (3.2) now implies that U' is finite; there is no loss in assuming that $U' = 1$ and so *U is a divisible Abelian group.* We now prove

(6.1.1) *every proper subgroup of G is in $\mathfrak{F}\mathfrak{N}$.*

For the proof of (6.1.1) suppose that K is a proper subgroup of G with $K \notin \mathfrak{F}\mathfrak{N}$. Then $K \in \mathfrak{C}$; further, the group U_0 generated by the elements of prime order in U is a normal subgroups of G with $U_0 \notin \mathfrak{C}$. Since U_0K is neither $\mathfrak{C}$ nor $\mathfrak{F}\mathfrak{N}$ we must have $U_0K = G$. Then $G/U_0 \simeq \simeq K/K \cap U_0$ is a $\mathfrak{C}$-group; on the other hand G/U_0 contains U/U_0 and since U is divisible and $U \notin \mathfrak{C}$, $U/U_0 \notin \mathfrak{C}$. This contradiction completes the proof of (6.1.1).

Suppose that U is not central in G. Thus there is a C_{p^∞} subgroup D of U with $D \not\leqslant \zeta_1(G)$. Since G is locally finite there is a finite subgroup L of G such that $G = UL$. We now analyze the subgroup $\langle D, L\rangle = = D^L L$; note that since $D^L \leqslant U$ and D^L is generated by a finite number of C_{p^∞}-groups, D^L is a $\mathfrak{C}$-group. Thus, $D^L L$ is in $\mathfrak{C}$ and so $D^L L < G$. From (6.1.1) $D^L L \in \mathfrak{F}\mathfrak{N}$ and $(3.2(b))$ now shows that $D^L L$ is « central-by-finite ». Thus, $[D, L] = 1$ and since $[D, U] = 1$ we have $D \leqslant \zeta_1(G)$. This contradiction completes the proof of part (a).

Proof of (b). Here G does not have the minimal condition on subgroups of finite index and every subgroup of finite index is $\mathfrak{F}\mathfrak{N}$. Let $U < G$ with G/U finite and T be a finite subgroup with $G = UT$. Since G does not have the minimal condition on subgroups of finite index there is a G-subgroup V of U with G/V finite and $VT < G$. Since VT has finite index in G, $VT \notin \mathfrak{C}$ and so for some $k \geqslant 1$, $\gamma_k(VT)$ is finite. Further, since $VT \leqslant N_G(\gamma_k(VT))$, $\gamma_k(VT)$ has only a finite number of conjugates in G. It follows that $(\gamma_k(VT))^G$ is finite. There is also an $s \geqslant 1$ such that $\gamma_s(U)$ is finite; thus $L = \gamma_s(U)(\gamma_k(VT))^G$ is finite and there is no loss in assuming that $L = 1$. We now have

$$U \in \mathfrak{N} \quad \text{and} \quad VT \in \mathfrak{N}.$$

The U-central series

$$V \geqslant [V, U] \geqslant \ldots \geqslant [V, mU] \geqslant \ldots$$

becomes the identity in a finite number of steps. Between each pair $[V, mU]$ and $[V, (m+1)U]$ interpolate a T-central series of finite length. The resulting series is a G-central series of finite length and so there is a t with $V \leqslant \zeta_t(G)$; from $(2.2(a))$ $G \in \mathfrak{F}\mathfrak{N}$ and this completes the proof of (b).

§ 6.2. Proof of Theorem 1. We proceed indirectly by supposing that there is a group $G \in M(\overline{\mathfrak{F}\mathfrak{N}_k})$ and that G is neither a $\mathfrak{C}$-group nor an $\mathfrak{F}\mathfrak{N}_k$-group. Then G has a subgroup V which is minimal with respect to being neither $\mathfrak{F}\mathfrak{N}_k$ nor $\mathfrak{C}$. Thus, every subgroup of V is either in $\mathfrak{F}\mathfrak{N}_k$ or is a $\mathfrak{C}$-group; i.e., V is in the class $(\mathfrak{F}\mathfrak{N}_k \cup \mathfrak{C})^*$. We have already seen that such groups are locally finite (Proposition 1) and have no infinite simple sections (Proposition 2). We proceed to show that the class $(\mathfrak{F}\mathfrak{N}_k \cup \mathfrak{C})^*$ is empty and this will provide a proof of Theorem B.

Let $V \in (\mathfrak{F}\mathfrak{N}_k \cup \mathfrak{C})^*$; we verify

$$V \in \mathfrak{F}\mathfrak{N} . \tag{6.2.1}$$

For the proof of (6.2.1) note first that V can not satisfy the hypotheses of Lemma 6(a). Thus, if V has a proper subgroup of finite index Lemma 6(b) yields (6.2.1). We assume then that V has no proper subgroups of finite index.

Since V has no infinite simple section, V is the union of a chain $\{N_\alpha | \alpha \in I\}$ of proper normal subgroups. Suppose that for some α we have $N_\alpha \in (\mathfrak{C} - \mathfrak{F}\mathfrak{N}_k)$ and let N_α^0 be the maximal divisible Abelian subgroup of N_α. Then from the Corollary of [15; p. 85] we have $V/\mathrm{C}_V(N_\alpha^0)$ finite. Thus $V = C_V(N_\alpha^0)$ and $N_\alpha^0 \leqslant \zeta_1(V)$ and so N_α is « central-by-finite ». Application of $\big(2.2(a)\big)$ gives a contradiction and *we may now assume that for every* $\alpha \in I$, $N_\alpha \in \mathfrak{F}\mathfrak{N}_k$. Since $\gamma_k(N_\alpha)$ is finite, $\mathrm{C}_V\big(\gamma_k(N_\alpha)\big)$ has finite index in V and so for all α, $\gamma_k(N_\alpha) \leqslant \zeta_1(N_\alpha)$ and thus $N_\alpha \in \mathfrak{N}_{k+1}$. It is now clear that $V \in \mathfrak{N}_{k+1}$.

We have established (6.2.1) and it is now easy to complete the proof of Theorem 1. From Lemma 3, $V \in D_{k+2}$ and Lemma 1 now implies that V has a normal subgroup M with $M \in (\mathfrak{C} - \mathfrak{F}\mathfrak{N}_k)$. Thus every subgroup of V above M is in $\mathfrak{C}$ and it follows that V/M is a $\mathfrak{F}\mathfrak{N}$-group with the minimal condition on subgroups. Thus $V/M \in \mathfrak{C}$ [15; p. 68] and since $M \in \mathfrak{C}$ also we have $V \in \mathfrak{C}$; this completes the proof of Theorem 1.

7. Structure of $(\mathfrak{F}\mathfrak{N}_1)^*$-groups.

§ 7.1. Proof of Theorem 2. Some of the implications in Theorem 2 are immediate and we dispense with these first. To prove that (i) and (ii) are equivalent let $G \in (\mathfrak{F}\mathfrak{N}_1)^*$; then from Theorem 1, $G \in \mathfrak{C}$, and $\big(3.2(b)\big)$ now shows that $G \notin \mathfrak{F}\mathfrak{N}_k$. This gives the implication « (ii) implies (i) ».

If $G \in (\mathfrak{F}\mathfrak{N}_k)^*$ again $G \in \mathfrak{C}$ and from $\big(3.2(b)\big)$ the proper subgroups of G are « central-by-finite ». Thus, if H is a proper subgroup of G,

H' is finite and this gives «(i) implies (ii)». We actually have proved the stronger

(7.1.1) *if* $G \in (\mathfrak{F}\mathfrak{N}_k)^*$ *then* $G \in \mathfrak{C}$ *and every proper subgroup of* G *is «central-by-finite».*

Before proceeding to the other implications we require the following (amended) terminology of Hartley [8].

Let p be a prime; denote by $C^n_{p\infty}$ the direct sum of n-copies of a $C_{p\infty}$-group. If $M = C^n_{p\infty}$ is a faithful module for the finite cyclic group $\langle x \rangle$ we say that $\langle x \rangle$ acts divisibly irreducibly on M if for every non-zero divisible subgroup U of M we have $U^{\langle x \rangle} = M$ (see [13] and [14] for different terminology).

Amongst the essential facts for our purposes is Lemma 2.2 of [8] which asserts that M is a divisibly irreducible $\langle x \rangle$-module if and only if M has no decomposition $M = B + C$ where B and $\mathfrak{C}$ are proper, non-zero $\langle x \rangle$-invariant divisible subgroups of M.

These considerations will aid in giving yet other characterizations of $(\mathfrak{F}\mathfrak{N}_k)^* = (\mathfrak{F}\mathfrak{N}_1)^*$-groups.

PROPOSITION 3. *Let* $G \in (\mathfrak{F}\mathfrak{N}_1)^*$. *Then*

i) *for some prime* p *and positive integer* $n \geqslant 1$ *there is a normal subgroup* A *of* G *with* $A \simeq C^n_{p\infty}$ *and an element* y *in* G *of prime power order* q^s *with* $G = A\langle y \rangle$, *and*

ii) $V = \langle y \rangle / C_{\langle y \rangle}(A)$ *has order* q *and* A *is a divisibly irreducible* V*-module.*

PROOF. Suppose that $G \in (\mathfrak{F}\mathfrak{N}_1)^*$, then G is in $\mathfrak{C}$ and has a (unique) maximal divisible subgroup A. From $(2.2(b))$ we have

(7.1.2) *if* $U < G$ *and* $A \leqslant U$ *then* $A \leqslant \zeta_1(U)$.

Let $x \in G$; if $A\langle x \rangle < G$ then (7.1.2) implies that $A \leqslant C_G(x)$. Thus, since $A \nleqslant \zeta_1(G)$ we must have $A\langle x \rangle = G$ for some $x \in G$. Using (7.1.2) again we see that G/A has a unique maximal subgroup and so has prime power order q^K. If $|x| = q^s r$ where $(q, r) = 1$ then $y = x^r$ has order q^s and $G = A\langle y \rangle$. Yet another application of (7.1.2) shows that $y^q \in C_G(A)$.

It remains to show that A is a «divisibly irreducible» V-module. Suppose that $A = B + C$ where B and C are proper, divisible, $\langle y \rangle$-invariant subgroups of A; one shows easily that $B\langle y \rangle$ and $C\langle y \rangle$ are proper subgroups of G and from $(3.2(b))$ we have $A = B + C \leqslant C_G(y)$.

From this contradiction we deduce that A is a p-group and so $A \simeq C^n_{p^\infty}$ for some n. Further A is a divisibly irreducible $\langle y\rangle$-module and so the proof of Proposition 3 is complete.

PROPOSITION 4. *If G satisfies the conditions* (i) *and* (ii) *of Proposition* 3, *then $G \in \mathfrak{C}$, $G \notin \mathfrak{F}\mathfrak{N}_1$ and every proper subgroup of G is Abelian or finite.*

PROOF. Suppose G satisfies the stated conditions; G is obviously in $\mathfrak{C}$—if G' were finite then $(3.2(b))$ implies that G is « central-by-finite », contrary to the divisible irreducibility of A.

Let B be an infinite proper subgroup of G. We may certainly suppose that $B \nleqslant A$ and so B contains elements of the form ay^j where $a \in A$ and $y^j \neq 1$. Suppose that $y^j \notin C_G(A)$; since V has prime order q, $\langle y\rangle = \langle y^j\rangle$. Denote by B^0 the maximal (necessarily infinite) divisible subgroup of B; then

$$B^0 = (B^0)^{\langle ay^j\rangle} = (B^0)^{\langle y^j\rangle} = (B^0)^{\langle y\rangle} = A\,,$$

the last equality forced by the « divisible irreducibility » of A. Thus, $A \leqslant B$ and now $\langle y\rangle = \langle y^j\rangle \leqslant B$ which gives $B = G$. From this contradiction we may assume that $ay^j \in B$ implies $y^j \in C_G(A)$. Thus $B \leqslant C_G(A)$, and (7.1.2) implies that $C_G(A)$ is « central-by-cyclic » and so Abelian. Thus, B is Abelian and Proposition 4 follows.

Propositions 3 and 4 give another characterization of $(\mathfrak{F}\mathfrak{N}_1)^*$-groups and the equivalence of parts (ii) and (iii) of Theorem 2 now follow easily.

§ 7.2. Using the results of Hartley [8] it is possible to give presentations of the groups satisfying the conditions (i) and (ii) of Proposition 3 and thus give presentations for the $(\mathfrak{F}\mathfrak{N}_1)^*$-groups. We assume, as in Proposition 3, that

$$A \simeq C^n_{p^\infty}\,, \qquad |y| = q^s \qquad \text{and} \qquad G = A\langle y\rangle \in (\mathfrak{F}\mathfrak{N}_1)^*\,.$$

Since the divisibly irreducible modules for V are known [8; Theorem 3.4] we need only describe the possible extensions of A by $\langle y\rangle/(\langle y\rangle \cap A)$. Let θ be a homomorphism from $\langle y\rangle$ into $\mathrm{Aut}\,(A)$ with $\mathrm{Ker}\,\theta = \langle y^q\rangle$ and A a divisibly irreducible module for y^θ. If $p \neq q$, then G is a split extension of A by $\langle y\rangle$ and this gives us the first type;

(α) $G = A \lhd_\theta \langle y \rangle$; $p \neq q$, $y^{q^s} = 1$ (*here as elsewhere*, $\lhd_\theta$ *is the semidirect product defined by* A, y *and* θ).

If $p = q$, then one possibility is again the split extension.

(β) $G = A \lhd_\theta \langle y \rangle$; $y^{p^s} = 1$.

If $A \cap \langle y \rangle \neq 1$ let r be minimal such that $y^{p^r} \in A$. Then $y^{p^r} \in \zeta_1(G) \cap \cap A$. Since G' is infinite, the divisible irreducibility of A implies $A \leqslant G'$. Thus, $y^{p^r} \in G' = \{[a, y] | a \in A\}$ and so there is an $a \in A$ such that $y^{p^r} = = [a, y]$. Since y^{p^r} is central in G, $(y^{p^r})^p = [a, y]^p = [a, y^p] = 1$ and thus $y^{p^{r+1}} = 1$. Hence $\langle y \rangle \cap A$ is a central subgroup of G of order p.

From Proposition 5.9 of [8] we see that $|C_A(y)| = p$. Thus our final type is the semidirect product with amalgamation (c.f. [7; p. 29])

(γ) $C = (A \lhd_\theta \langle y \rangle)_{C_A(y) = \langle y^{p^r} \rangle}$; $r \geqslant 1$, $y^{p^{r+1}} = 1$.

It is immediate that each of the three types (α), (β), (γ) satisfy the conditions of Proposition 3 and therefore are $(\mathfrak{F}\mathfrak{N}_1)^*$-groups.

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