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Two-Dimensional Constitutive Equations.

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SUMMARY - Constitutive equations appropriate to the two-dimensional deformation of isotropic materials with memory are developed from first principles. These are, of course, considerably simpler than the corresponding three-dimensional equations, from which they could also be derived.

1. Introduction.

In a series of three papers Green and Rivlin [1, 2] and Green, Rivlin and Spencer [3] discussed the formulation of constitutive equations appropriate to the continuum mechanics of materials with memory. They took as their starting point the constitutive assumption that the Cauchy stress $\boldsymbol{\sigma}$ in an element of the material at an instant of time, t say, depends on the history of the deformation gradient matrix $\boldsymbol{g}(\tau)$ at times up to and including time t. In mathematical terms, the Cauchy stress was assumed to be a *functional* of the history of the deformation gradient matrix.

It was shown, from a consideration of the effect on the stress of superposing on the assumed deformation a rigid time-dependent rotation, that the dependence of the stress on the deformation gradient matrix must take the form

$$\boldsymbol{\sigma} = \boldsymbol{g}(t)\mathscr{F}[\boldsymbol{C}(\tau)]\boldsymbol{g}^{\dagger}(t)\,, \tag{1.1}$$

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where $\boldsymbol{C}(\tau)$ is the Cauchy strain at time τ, $\mathscr{F}$ is a symmetric matrix functional and the dagger denotes the transpose.

It was then shown how further restrictions resulting from symmetry, and particularly isotropy, of the material could be introduced into the constitutive equation for the stress by solving an appropriate invariant-theoretical problem. In [1, 2, 3] this was discussed in the case when $\mathscr{F}$ may be expressed as the sum of a number of multiple integrals of the argument function. In [4] it was shown how a canonical form can be obtained for $\mathscr{F}$ in the case when the material is isotropic. Analogous results in the more general case when $\mathscr{F}$ is an arbitrary functional of $\boldsymbol{C}(\tau)$ were obtained by Wineman and Pipkin [5].

In the present paper we discuss the analogous problem in the case when the deformation is two-dimensional. The canonical forms for the constitutive equation which are obtained are much simpler than those for three-dimensional deformations. While the results obtained here could, of course, be obtained by specialization of the more general three-dimensional ones, they are derived here independently. This enables us to avoid the need to appeal to the rather complicated three-dimensional representation theorems for invariants and to use the much simpler two-dimensional theorems.

In this paper we make the assumption that the deformation gradients—and hence the Cauchy strain—are piece-wise continuous, having at most a countably infinite number of salti, and have bounded variation. This is a less general class of functions than that to which the representation theorems of Wineman and Pipkin [5] apply. However, it is sufficiently general for the purposes for which one would wish to use it. It has the merit that the appropriate representation theorems are derived in a somewhat less abstract manner and in a somewhat more explicit form than would be the case if the more general class of functions employed in [5] were used. The same advantage could, of course, be obtained in the three-dimensional case.

We make the constitutive assumption that the stress is a continuous functional of the deformation gradient history for piece-wise continuous histories of the type mentioned above. However, it may well be that the material with which we are concerned does not admit the production in it of discontinuities in the deformation gradients. If this is the case, we embed the space of physically allowable deformation gradient histories in the larger space considered here and correspondingly enlarge the space of functionals so that they depend continuously on the argument functions throughout the whole of this space. That this can, in fact, be done follows from the Hahn-Banach theorem (see, for example, [6]). The resulting constitutive equation is, of course, then applied only in the case when the deformation gradient history is a physically possible one.

2. Restrictions on the constitutive equation.

In this section we discuss some restrictions which may be imposed on the two-dimensional constitutive equations for plane strain of a material with memory. We assume that the two-dimensional Cauchy stress at time t, denoted $\boldsymbol{\sigma} = \boldsymbol{\sigma}(t)$, depends on the history of the two-dimensional deformation gradients. Let σ_{ij} $(i, j = 1, 2)$ be the components of $\boldsymbol{\sigma}$ in a two-dimensional rectangular cartesian coordinate system x.

Suppose that a particle, which initially at time $\tau = 0$ has vector position $\boldsymbol{X}$, with components X_A $(A = 1, 2)$ in the system x, moves to vector position $\boldsymbol{x}(\tau)$, with components $x_i(\tau)$, at time τ. Then our basic constitutive assumption takes the form

$$\boldsymbol{\sigma} = \mathscr{F}[\boldsymbol{g}(\tau)]\,, \qquad -\infty < \tau \leqslant t\,, \tag{2.1}$$

where $\boldsymbol{g}(\tau)$ is the 2×2 deformation gradient matrix defined by

$$\boldsymbol{g}(\tau) = \|g_{iA}(\tau)\| = \|x_{i,A}(\tau)\|\,, \tag{2.2}$$

$,A$ denotes the operator $\partial/\partial X_A$ and $\mathscr{F}$ is a 2×2 matrix functional.

We now superpose on the assumed deformation $\boldsymbol{X} \to \boldsymbol{x}(\tau)$ a time-dependent rigid rotation about an axis normal to the plane of the strain, so that the resulting deformation is $\boldsymbol{X} \to \boldsymbol{a}(\tau)\,\boldsymbol{x}(\tau)$, where $\boldsymbol{a}(\tau)$ is a proper orthogonal 2×2 matrix. Then the corresponding stress at time t is $\boldsymbol{a}(t)\boldsymbol{\sigma}\boldsymbol{a}^\dagger(t)$, where the dagger denotes the transpose. It follows that $\boldsymbol{\sigma}$ must be expressible in the form

$$\boldsymbol{\sigma} = \boldsymbol{g}\mathscr{F}[\boldsymbol{C}(\tau)]\boldsymbol{g}^\dagger\,, \tag{2.3}$$

where $\boldsymbol{g} = \boldsymbol{g}(t)$ and $\boldsymbol{C}(\tau)$, the Cauchy strain at time τ, is defined by

$$\boldsymbol{C}(\tau) = \boldsymbol{g}^\dagger(\tau)\boldsymbol{g}(\tau)\,. \tag{2.4}$$

If the material is isotropic in the plane of the deformation, the functional $\mathscr{F}$ in (2.3), which is in general different from that in (2.1), must satisfy the relation

$$\mathscr{F}[\overline{\boldsymbol{C}}(\tau)] = \boldsymbol{S}\mathscr{F}[\boldsymbol{C}(\tau)]\boldsymbol{S}^{\dagger}\,, \tag{2.5}$$

where

$$\overline{\boldsymbol{C}}(\tau) = \boldsymbol{S}\boldsymbol{C}(\tau)\boldsymbol{S}^{\dagger}\,, \tag{2.6}$$

and $\boldsymbol{S}$ is an arbitrary orthogonal 2×2 matrix. We note that both $\mathscr{F}$ and $\boldsymbol{C}(\tau)$ are symmetric matrices. Accordingly, we may interpret the relation (2.5) as stating that $\mathscr{F}$ is a symmetric second-order tensor-valued functional of the symmetric second-order tensor function $\boldsymbol{C}(\tau)$ which is form-invariant under the 2-dimensional full orthogonal group $\boldsymbol{S}$.

In the next section we discuss how the restrictions imposed on the form of $\mathscr{F}$, by the requirement that it satisfy the relation (2.5), can be made explicit.

3. Canonical form.

The relation (2.5) can easily be converted into a scalar relation by employing an auxiliary symmetric 2×2 matrix $\boldsymbol{\psi}$. We define $\overline{\boldsymbol{\psi}}$ by

$$\overline{\boldsymbol{\psi}} = \boldsymbol{S}\boldsymbol{\psi}\boldsymbol{S}^{\dagger}\,. \tag{3.1}$$

Then, from (2.5) we obtain

$$\operatorname{tr}\{\overline{\boldsymbol{\psi}}\mathscr{F}[\overline{\boldsymbol{C}}(\tau)]\} = \operatorname{tr}\{\boldsymbol{\psi}\mathscr{F}[\boldsymbol{C}(\tau)]\} = \Phi[\boldsymbol{\psi}, \boldsymbol{C}(\tau)] \quad \text{say}\,, \tag{3.2}$$

and note that

$$\mathscr{F}[\boldsymbol{C}(\tau)] = \frac{\partial\Phi}{\partial\boldsymbol{\psi}}\,. \tag{3.3}$$

The relations (3.2) state that Φ is a scalar-valued functional of the second-order symmetric tensor function $\boldsymbol{C}(\tau)$ and a linear scalar function of the second-order symmetric tensor $\boldsymbol{\psi}$, invariant under the full orthogonal group. We now show how these facts enable us to obtain a canonical expression for Φ and hence, from (3.3), for $\mathscr{F}$.

We make the assumption that $\boldsymbol{C}(\tau) = \boldsymbol{\delta}$ (the 2×2 unit matrix) for $\tau \leqslant t_0$ say. Then the support of the argument function $\boldsymbol{C}(\tau)$ in the functional $\mathscr{F}$ may be taken as $[t_0, t]$. We also suppose that $\boldsymbol{C}(\tau)$ is piece-wise continuous, having at most a countably infinite number of salti, and is of bounded variation (1). We define

$$(3.4)\qquad \begin{aligned} \operatorname{norm} \boldsymbol{C}(\tau) &= \sup \{\operatorname{tr} [\boldsymbol{C}(\tau)]^2\}^{\frac{1}{2}}, \\ \operatorname{norm} \mathscr{F} &= \{\operatorname{tr} (\mathscr{F}^2)\}^{\frac{1}{2}}, \end{aligned}$$

and suppose that $\mathscr{F}$ is a continuous functional of $\boldsymbol{C}(\tau)$.

We now divide the interval $[t_0, t]$ into μ sub-intervals $[\tau_{\alpha-1}, \tau_\alpha)$ $(\alpha = 1, \dots, \mu)$, where $\tau_0 = t_0$, $\tau_\mu = t$ and $\tau_{\alpha-1} < \tau_\alpha$. We construct a symmetric second-order tensor-valued function $\boldsymbol{C}_\mu(\tau)$ such that

$$(3.5)\qquad \begin{aligned} \boldsymbol{C}_\mu(\tau) &= \boldsymbol{C}(\tau_{\alpha-1}), \qquad \tau_{\alpha-1} \leqslant \tau < \tau_\alpha \\ \boldsymbol{C}_\mu(t) &= \boldsymbol{C}(t). \end{aligned}$$

The times τ_α are so chosen that if a saltus exists in any one of the components of $\boldsymbol{C}(\tau)$ it occurs at one of these times.

Since $\mathscr{F}[\boldsymbol{C}(\tau)]$ is assumed to be a continuous functional of its argument functions, $\mathscr{F}[\boldsymbol{C}_\mu(\tau)] \to \mathscr{F}[\boldsymbol{C}(\tau)]$ as $\mu \to \infty$ and $\sup (\tau_\alpha - \tau_{\alpha-1}) \to 0$.

Correspondingly, Φ is a continuous scalar functional of $\boldsymbol{C}(\tau)$ and $\boldsymbol{\psi}$, where

$$(3.6)\qquad \operatorname{norm} \boldsymbol{\psi} = \{\operatorname{tr} (\boldsymbol{\psi}^2)\}^{\frac{1}{2}}$$

and $\Phi[\boldsymbol{\psi}, \boldsymbol{C}_\mu(\tau)] \to \Phi[\boldsymbol{\psi}, \boldsymbol{C}(\tau)]$ as $\mu \to \infty$ and $\sup (\tau_\alpha - \tau_{\alpha-1}) \to 0$. Since the relation (3.2) is valid with $\boldsymbol{C}(\tau)$ replaced by $\boldsymbol{C}_\mu(\tau)$, it follows that $\Phi[\boldsymbol{\psi}, \boldsymbol{C}_\mu(\tau)]$ is a scalar function of the $p + 2$ second-order symmetric tensors $\boldsymbol{C}(\tau_\alpha)$ $(\alpha = 0, 1, \dots, \mu)$ and $\boldsymbol{\psi}$, invariant under the full orthogonal group. It is, of course, linear in $\boldsymbol{\psi}$. It follows that Φ must be expressible as a function of the elements of an irreducible integrity basis for the $\mu + 2$ argument tensors. Let $I_1, \dots, I_\nu$ be the elements in such a basis which are independent of $\boldsymbol{\psi}$ and let $K_1, \dots, K_\lambda$ be those elements which are linear in $\boldsymbol{\psi}$. Then,

$$(3.7)\qquad \Phi[\boldsymbol{\psi}, \boldsymbol{C}_\mu(\tau)] = \sum_{\gamma=1}^{\lambda} \Phi_\gamma(I_1, \dots, I_\nu) K_\gamma .$$

(1) Each of the components of $\boldsymbol{C}(\tau)$ is assumed to have these properties.

It follows that

$$(3.8)\qquad \mathscr{F}[C_\mu(\tau)] = \frac{\partial \Phi[\boldsymbol{\psi}, C_\mu)\tau)]}{\partial \boldsymbol{\psi}} = \sum_{\gamma=1}^{\lambda} \Phi_\gamma(I_1, \ldots, I_\nu)\frac{\partial K_\gamma}{\partial \boldsymbol{\psi}}$$

$I_1, \ldots, I_\nu$ may be taken [7] as the set of invariants

$$(3.9)\qquad \operatorname{tr} \boldsymbol{C}(\tau_\alpha)\,, \quad \operatorname{tr}[\boldsymbol{C}(\tau_\alpha)\boldsymbol{C}(\tau_\beta)] \quad (\alpha, \beta = 0, 1, \ldots, \mu)\,.$$

Again, $K_1, \ldots, K_\lambda$ may be taken as the set of invariants

$$(3.10)\qquad \operatorname{tr} \boldsymbol{\psi}\,, \quad \operatorname{tr} \boldsymbol{\psi}\boldsymbol{C}(\tau_\alpha) \quad (\alpha = 0, 1, \ldots, \mu)\,.$$

In then follows from (3.8) that

$$(3.11)\qquad \mathscr{F}[\boldsymbol{C}_\mu(\tau)] = \sum_{\beta=1}^{\mu} \Phi_\beta(I_1, \ldots, I_\nu)\boldsymbol{C}(\tau_\beta) + \Phi_0(I_1, \ldots, I_\nu)\boldsymbol{\delta}\,,$$

where $\boldsymbol{\delta}$ is the 2×2 unit matrix.

We now let $\mu \to \infty$ and sup $(\tau_\alpha - \tau_{\alpha-1}) \to 0$. The function $\mathscr{F}[\boldsymbol{C}_\mu(\tau)]$ then becomes the functional $\mathscr{F}[\boldsymbol{C}(\tau)]$ $\big(\tau = [t_0, t]\big)$ and equation (3.11) becomes

$$(3.12)\qquad \mathscr{F}[\boldsymbol{C}(\tau)] = \mathscr{G}[J_1(\xi_1), J_2(\xi_1, \xi_2), \boldsymbol{C}(\tau)] + \mathscr{H}[J_1(\xi), J_2(\xi_1, \xi_2)]\boldsymbol{\delta}\,,$$

where

$$(3.13)\qquad J_1(\xi) = \operatorname{tr} \boldsymbol{C}(\xi)\,, \quad J_2(\xi_1, \xi_2) = \operatorname{tr}[\boldsymbol{C}(\xi_1)\boldsymbol{C}(\xi_2)]\,.$$

$\mathscr{G}$ and $\mathscr{H}$ are tensor-valued and scalar-valued functionals respectively of the indicated argument functions, $\mathscr{G}$ being linear in $\boldsymbol{C}(\tau)$. The range of ξ, ξ_1, ξ_2 and τ is $[t_0, t]$.

4. Integral representation.

We now consider the set of functions $\boldsymbol{C}(\tau)$ which consists of one such function and all the functions which can be obtained from it by time-independent orthogonal transformations. Such a set of functions is called an *orbit* of any $\boldsymbol{C}(\tau)$ in the set. It is evident from their definitions in (3.13) that $J_1(\xi)$ and $J_2(\xi_1, \xi_2)$ are the same for all functions $\boldsymbol{C}(\tau)$ in a single orbit. Accordingly, for each orbit, $\mathscr{G}$ is a linear functional of $\boldsymbol{C}(\tau)$ and $\mathscr{H}$ is a constant. Using the version of Riesz's theorem presented in [8] it follows that, on any specified orbit, $\mathscr{G}$ can be expressed in the form

$$\mathscr{G} = \int_{t_0}^{t} f(t, \tau)\, d\boldsymbol{C}(\tau)\,. \tag{4.1}$$

The kernel $f(t, \tau)$ in (4.1) depends, of course, on the particular choice of orbit, i.e. on the values of the functions $J_1(\xi)$ and $J_2(\xi_1, \xi_2)$. It is accordingly a scalar functional of these functions.

From (4.1) and (3.12) we obtain

$$\mathscr{F}[\boldsymbol{C}(\tau)] = \int_{t_0}^{t} f[t, \tau, J_1(\xi), J_2(\xi_1, \xi_2)]\, d\boldsymbol{C}(\tau) + \mathscr{H}[J_1(\xi), J_2(\xi_1, \xi_2)]\boldsymbol{\delta}\,. \tag{4.2}$$

By substituting this expression in (2.3) we obtain the constitutive equation for $\boldsymbol{\sigma}$. If $\boldsymbol{\sigma} = 0$ when the material undergoes no deformation, so that $\boldsymbol{C}(\tau) = \boldsymbol{g} = \boldsymbol{\delta}$ and $\operatorname{tr} \boldsymbol{C}(\xi) = \operatorname{tr}[\boldsymbol{C}(\xi_1)\boldsymbol{C}(\xi_2)] = 2$, then

$$\mathscr{H}[2, 2] = 0\,. \tag{4.3}$$

If the material is incompressible,

$$\boldsymbol{\sigma} = \boldsymbol{g}\mathscr{F}[\boldsymbol{C}(\tau)]\boldsymbol{g}^{\dagger} - p\boldsymbol{\delta}\,, \tag{4.4}$$

where p is an arbitrary hydrostatic pressure and $\mathscr{F}$ is given by an expression of the form (4.2).

If the material is of the hereditary type, we may replace $f(t, \tau)$ in (4.1) by $f(t-\tau)$ and correspondingly in (4.2) the dependence of f on t and τ is through $t-\tau$. Also, in (4.2) the functional dependence of f and $\mathscr{H}$ on $J_1(\xi)$ and $J_2(\xi_1, \xi_2)$ is of the hereditary type.

It is convenient for our purposes to write

$$\boldsymbol{E}(\tau) = \boldsymbol{C}(\tau) - \boldsymbol{\delta}\,, \tag{4.5}$$

and we note that if $\boldsymbol{g}(\tau) = \boldsymbol{\delta}$, i.e. for zero deformation history, $\boldsymbol{E}(\tau) = 0$.

With (4.5) we may rewrite (4.2) in the form

$$\mathscr{F}[\boldsymbol{C}(\tau)] = \int_{t_0}^{t} f[t, \tau, \bar{J}_1(\xi), \bar{J}_2(\xi_1, \xi_2)]\, d\boldsymbol{E}(\tau) + \mathscr{H}[\bar{J}_1(\xi), \bar{J}_2(\xi_1, \xi_2)]\boldsymbol{\delta}\,, \tag{4.6}$$

where

$$\bar{J}_1(\xi) = \operatorname{tr} \boldsymbol{E}(\xi) \quad \text{and} \quad \bar{J}_2(\xi_1, \xi_2) = \operatorname{tr}\, [\boldsymbol{E}(\xi_1)\boldsymbol{E}(\xi_2)]\,. \tag{4.7}$$

5. Plane stress.

We consider a thin plane sheet of initial thickness h. We suppose that the material of the sheet is either isotropic or has transverse isotropy about an axis normal to it. We adopt a two-dimensional rectangular cartesian reference system x in the mid-plane of the sheet. We suppose that the sheet is deformed by forces applied to its edges so that its mid-plane remains plane. We define the two-dimensional Cauchy stress $\boldsymbol{\sigma} = \|\sigma_{ji}\|$ in the following manner. σ_{1i} is the resultant force acting on an element of cross-sectional area which is normal to the 1-axis in the deformed state and has unit length parallel to the 2-axis in that state. σ_{2i} is analogously defined.

In the deformation a particle in the mid-plane which is initially in vector position $\boldsymbol{X}$ moves to vector position $\boldsymbol{x}(\tau)$ at time τ. The deformation gradient matrix $\boldsymbol{g}(\tau)$ is then defined in terms of these vectors by equations (2.2). As in the case of plane strain, we make the constitutive assumption that the stress $\boldsymbol{\sigma}$ at time t is a functional of the deformation gradient history and so is given by an expression of the form (2.1). Then, by an argument similar to that in §§ 2-4, we arrive at expressions for $\boldsymbol{\sigma}$ identical in form with those given by (2.3) with (4.2) or (4.6). We note that these forms will be valid whether or not the material is incompressible.

6. Small deformations.

Let $\boldsymbol{u}(\tau)$ be the displacement vector at time τ. Then,

$$\boldsymbol{u}(\tau) = \boldsymbol{x}(\tau) - \boldsymbol{X}\,. \tag{6.1}$$

We define the displacement gradient matrix $\boldsymbol{\gamma}(\tau)$ by

$$\boldsymbol{\gamma}(\tau) = \boldsymbol{g}(\tau) - \boldsymbol{\delta}\,, \tag{6.2}$$

and its norm by

$$\operatorname{norm} \boldsymbol{\gamma}(\tau) = \sup \{\operatorname{tr}[\boldsymbol{\gamma}(\tau)\boldsymbol{\gamma}^{\dagger}(\tau)]\}^{\frac{1}{2}}\,. \tag{6.3}$$

We now suppose that norm $\boldsymbol{\gamma}(\tau) \ll 1$. This implies that the extensions undergone by linear elements of the material and the rotations undergone by volume elements are small. We can then approximate $\boldsymbol{E}(\tau)$ by

$$\boldsymbol{E}(\tau) \approx 2\boldsymbol{e}(\tau) = \boldsymbol{\gamma}(\tau) + \boldsymbol{\gamma}^{\dagger}(\tau)\,. \tag{6.4}$$

$\boldsymbol{e}(\tau)$ is, of course, the usual two-dimensional infinitesimal strain matrix.

With this approximation we may replace the factors $\boldsymbol{g}$ and $\boldsymbol{g}^{\dagger}$ in (4.6) by $\boldsymbol{\delta}$ and $d\boldsymbol{E}(\tau)$ by $2\,d\boldsymbol{e}(\tau)$ to obtain

$$\boldsymbol{\sigma} = \int_{t_0}^{t} 2f[t, \tau, \bar{J}_1(\xi), \bar{J}_2(\xi_1, \xi_2)]\, d\boldsymbol{e}(\tau) + \mathscr{H}[\bar{J}_1(\xi), \bar{J}_2(\xi_1, \xi_2)]\boldsymbol{\delta}\,. \tag{6.5}$$

We could, of course, also introduce the approximation (6.4) into the expressions (4.7) for $\bar{J}_1$ and $\bar{J}_2$. However, we will not do this for the moment.

We now suppose that u_1 and u_2 depend on time in the same manner, i.e.

$$\boldsymbol{u}(\tau) = \hat{\boldsymbol{u}}\varphi(\tau)\,, \tag{6.6}$$

where $\hat{\boldsymbol{u}}$ is a time-independent vector and $\varphi(\tau)$ is a function of bounded variation which has at most a countably infinite number of salti. From (6.4) it follows that

$$\boldsymbol{e}(\tau) = \hat{\boldsymbol{e}}\varphi(\tau)\,, \tag{6.7}$$

where

$$\hat{\boldsymbol{e}} = \tfrac{1}{2}[\nabla\hat{\boldsymbol{u}} + (\nabla\hat{\boldsymbol{u}})^{\dagger}]\,. \tag{6.8}$$

Then, (6.5) becomes

$$\sigma = \hat{e}\int_{t_0}^{t} 2f[t, \tau, \bar{J}_1(\xi), \bar{J}_2(\xi_1, \xi_2)]\, d\varphi(\tau) + \mathscr{H}[\bar{J}_1(\xi), \bar{J}_2(\xi_1, \xi_2)]\boldsymbol{\delta}\,. \tag{6.9}$$

From (6.9) we readily obtain

$$\begin{aligned} \sigma_{11} - \sigma_{22} &= (\hat{e}_{11} - \hat{e}_{22})\int_{t_0}^{t} 2f[t, \tau, \bar{J}_1(\xi), \bar{J}_2(\xi_1, \xi_2)]\, d\varphi(\tau)\,, \\ \sigma_2^2 &= \hat{e}_{12}\int_{t_0}^{t} 2f[t, \tau, \bar{J}_1(\xi), \bar{J}_2(\xi_1, \xi_2)]\, d\varphi(\tau)\,. \end{aligned} \tag{6.10}$$

From (6.10) it follows that

$$\frac{\sigma_{11} - \sigma_{22}}{\sigma_{12}} = \frac{\hat{e}_{11} - \hat{e}_{22}}{\hat{e}_{12}}\,. \tag{6.11}$$

If we introduce the approximation norm $\gamma(\tau) \ll 1$ into the expressions (4.7), we obtain

$$\bar{J}_1(\xi) = 2 \operatorname{tr} \boldsymbol{e}(\xi)\,, \qquad \bar{J}_2(\xi_1, \xi_2) = 4 \operatorname{tr} \boldsymbol{e}(\xi_1)\boldsymbol{e}(\xi_2)\,. \tag{6.12}$$

7. Polynomial functionals.

We now assume that in (2.3), the tensor-valued functional $\mathscr{F}$ is a polynomial functional of $\boldsymbol{C}(\tau)$. It is evident that it may also be regarded as a polynomial functional of $\boldsymbol{E}(\tau)$ defined by (4.5). Then,

$$\sigma = \boldsymbol{g}\mathscr{F}[\boldsymbol{E}(\tau)]\boldsymbol{g}^{\dagger} \qquad (t_0 \leqslant \tau \leqslant t)\,. \tag{7.1}$$

We assume that $\boldsymbol{C}(\tau)$, and hence $\boldsymbol{E}(\tau)$, is of bounded variation and has, at most, a countably infinite number of salti.

Using a result analogous to that proven in [8] for the case of three-dimensional constitutive equations, we can express $\mathscr{F}$ as the sum of a number of multiple integrals thus:

$$\mathscr{F}[\boldsymbol{E}(\tau)] = \sum_N \int_{t_0}^{t} \dots \int_{t_0}^{t} \boldsymbol{\Phi}_N , \tag{7.2}$$

where $\boldsymbol{\Phi}_N$ is the matrix defined by

$$\boldsymbol{\Phi}_N = \| f_{iji_1j_1 \dots i_N j_N}(t, \tau_1, \dots, \tau_N)\, dE_{i_1 j_1}(\tau_1) \dots dE_{i_N j_N}(\tau_N) \| . \tag{7.3}$$

The isotropy condition (2.5) yields the result that $\boldsymbol{\Phi}_N$ is an isotropic multilinear form in the tensors $d\boldsymbol{E}(\tau_P)$ $(P = 1, \dots, N)$.

Let $\boldsymbol{\psi}$ be an arbitrary symmetric tensor. We define the scalar $\boldsymbol{\Phi}_N$ by

$$\Phi_N = \operatorname{tr} \boldsymbol{\psi} \boldsymbol{\Phi}_N . \tag{7.4}$$

Then,

$$\boldsymbol{\Phi}_N = \frac{\partial \Phi_N}{\partial \boldsymbol{\psi}} . \tag{7.5}$$

Φ_N is a multilinear isotropic invariant of the $N + 1$ symmetric tensors $\boldsymbol{\psi}$, $d\boldsymbol{E}(\tau_P)$ $(P = 1, \dots, N)$. It can then be expressed as a linear combination of products of the elements of an isotropic integrity basis for the $N + 1$ tensors. Each of these products is linear in each of the tensors and has a coefficient which is a function of $t, \tau_1, \dots, \tau_N$.

The elements of an isotropic integrity basis for the $N + 1$ tensors $\boldsymbol{\psi}$ and $d\boldsymbol{E}(\tau_P)$ $(P = 1, \dots, N)$ which are linear in $\boldsymbol{\psi}$ are

$$\operatorname{tr} \boldsymbol{\psi} , \quad \operatorname{tr} \boldsymbol{\psi}\, d\boldsymbol{E}(\tau_P) . \tag{7.6}$$

Those which are independent of $\boldsymbol{\psi}$ and linear in their argument tensors are

$$\begin{aligned} &\operatorname{tr} d\boldsymbol{E}(\tau_P) = L_P , \quad \text{say} , \\ &\operatorname{tr} d\boldsymbol{E}(\tau_P)\, d\boldsymbol{E}(\tau_Q) = M_{PQ} , \quad \text{say} , \quad (P \neq Q) . \end{aligned} \tag{7.7}$$

Thus, Φ_N may be expressed in the form

$$\Phi_N = \alpha \operatorname{tr} \boldsymbol{\psi} + \sum_{P=}^{N} \alpha_P \operatorname{tr} [\boldsymbol{\psi}\, dE(\tau_P)] . \tag{7.8}$$

α and α_P are linear combinations of products of invariants chosen from (7.7) with coefficients which are functions of $t, \tau_1, \ldots, \tau_N$. Each of the products in the expression for α is multilinear in $d\boldsymbol{E}(\tau_P)$ $(j = 1, \ldots, P)$ and each of the products in the expression for α_P is multilinear in $d\boldsymbol{E}(\tau_Q)$ $(Q = 1, \ldots, N;\ Q \neq P)$. From (7.5) and (7.8) we have

$$\boldsymbol{\Phi}_N = \alpha\boldsymbol{\delta} + \sum_{P=1}^{N} \alpha_P\, d\boldsymbol{E}(\tau_P)\,. \tag{7.9}$$

Introducing (7.9) into (7.2) it is easily seen that $\mathscr{F}$ can be expressed in the form

$$\mathscr{F}[\boldsymbol{E}(\tau)] = \sum_N \mathscr{F}_N[\boldsymbol{E}(\tau)]\,, \tag{7.10}$$

where

$$\mathscr{F}_N[\boldsymbol{E}(\tau)] = \boldsymbol{\delta}\int_{t_0}^{t}\ldots\int_{t_0}^{t}\beta_N + \int_{t_0}^{t}\ldots\int_{t_0}^{t}\gamma_N\, d\boldsymbol{E}(\tau)\,. \tag{7.11}$$

β_N is a linear combination of products of the form $L_1L_2\ldots L_P M_{P+1,P+2}\ldots$ $\ldots M_{N-1,N}$ with coefficients which are functions of $t, \tau_1, \ldots, \tau_N$ and γ_N is a linear combination of products of the form $L_1L_2\ldots L_P M_{P+1,P+2}\ldots$ $\ldots M_{N-2,N-1}$ with coefficients which are functions of $t, \tau, \tau_1, \ldots, \tau_{N-1}$. Thus,

$$\begin{aligned}
\mathscr{F}_1 &= \boldsymbol{\delta}\int_{t_0}^{t} h_{10}(t, \tau)\,\mathrm{tr}\, d\boldsymbol{E}(\tau) + \int_{t_0}^{t} g_{00}(t, \tau)\, d\boldsymbol{E}(\tau)\,,\\
\mathscr{F}_2 &= \boldsymbol{\delta}\int_{t_0}^{t}\int_{t_0}^{t}\{h_{20}(t, \tau_1, \tau_2)\,\mathrm{tr}\, d\boldsymbol{E}(\tau_1)\,\mathrm{tr}\, d\boldsymbol{E}(\tau_2)\\
&\qquad + h_{01}(t, \tau_1, \tau_2)\,\mathrm{tr}\,[d\boldsymbol{E}(\tau_1)\, d\boldsymbol{E}(\tau_2)]\}\\
&\qquad + \int_{t_0}^{t}\int_{t_0}^{t} g_{10}(t, \tau, \tau_1)[\mathrm{tr}\, \mathrm{d}\boldsymbol{E}(\tau_1)]\, d\boldsymbol{E}(\tau)\,,\\
\mathscr{F}_3 &= \boldsymbol{\delta}\int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t}\{h_{30}(t, \tau_1, \tau_2, \tau_3)\,\mathrm{tr}\, d\boldsymbol{E}(\tau_1)\,\mathrm{tr}\, d\boldsymbol{E}(\tau_2)\,\mathrm{tr}\, d\boldsymbol{E}(\tau_3)\\
&\qquad + h_{11}(t, \tau_1, \tau_2, \tau_3)\,\mathrm{tr}\, d\boldsymbol{E}(\tau_1)\,\mathrm{tr}\,[d\boldsymbol{E}(\tau_2)\, d\boldsymbol{E}(\tau_3)]\}\\
&\qquad + \int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t}\{g_{20}(t, \tau, \tau_1, \tau_2)\,\mathrm{tr}\, d\boldsymbol{E}(\tau_1)\,\mathrm{tr}\, d\boldsymbol{E}(\tau_2)\\
&\qquad + g_{01}(t, \tau, \tau_1, \tau_2)\,\mathrm{tr}\,[d\boldsymbol{E}(\tau_1)\, d\boldsymbol{E}(\tau_2)]\}\, d\boldsymbol{E}(\tau)\,.
\end{aligned} \tag{7.12}$$

The connection between these results and that given in (4.6) can easily be established by partial integration. For example, with the assumption that h_{10} is differentiable with respect to τ, we have from $(7.12)_1$,

$$\mathscr{F}_1 = \boldsymbol{\delta}\left\{h_{10}(t, t)\,\mathrm{tr}\,\boldsymbol{E}(t) - \int_{t_0}^{t} \frac{\partial h_{10}}{\partial \tau}\,\mathrm{tr}\,\boldsymbol{E}(\tau)\,d\tau\right\} - \int_{t_0}^{t} g_{00}(t-\tau)\,d\boldsymbol{E}(\tau)\,. \tag{7.13}$$

Similar partial integrations of the expressions for $\mathscr{F}_2, \mathscr{F}_3$, etc. yield with (7.10) an expression for $\mathscr{F}$ of the form (4.6) in which f and $\mathscr{H}$ are polynomial functionals of $\bar{J}_1(\xi)$ and $\bar{J}_2(\xi_1, \xi_2)$.

If $\boldsymbol{E}(\tau)$ is sufficiently small, the expressions, $\mathscr{F}_1, \mathscr{F}_1 + \mathscr{F}_2, \ldots$ provide a hierarchy of approximations to $\mathscr{F}$. If the displacement gradients $\boldsymbol{\gamma}(\tau)$ defined in (6.2) are such that norm $\boldsymbol{\gamma}(\tau) \ll 1$, then we may further approximate $\boldsymbol{E}(\tau)$ by $2\boldsymbol{e}(\tau)$, defined in (6.4), and we may replace $\boldsymbol{g}$ by $\boldsymbol{\delta}$ in the corresponding expressions for $\boldsymbol{\sigma}$. In the case when the material is incompressible, we can then absorb the terms involving $\boldsymbol{\delta}$ in (7.12) in the arbitrary hydrostatic pressure term. Then the successive approximates to $\boldsymbol{\sigma}$ are

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 - p\boldsymbol{\delta}\,, \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 - p\boldsymbol{\delta}\,, \qquad \ldots \tag{7.14}$$

where

$$\begin{aligned} \boldsymbol{\sigma}_1 &= 2\int_{t_0}^{t} g_{00}(t, \tau)\,d\boldsymbol{e}(\tau)\,, \\ \boldsymbol{\sigma}_2 &= 4\int_{t_0}^{t}\int_{t_0}^{t} g_{10}(t, \tau, \tau_1)[\mathrm{tr}\,d\boldsymbol{e}(\tau_1)]\,d\boldsymbol{e}(\tau)\,, \\ \boldsymbol{\sigma}_3 &= 8\int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t}\{g_{20}(t, \tau, \tau_1, \tau_2)\,\mathrm{tr}\,d\boldsymbol{e}(\tau_1)\,\mathrm{tr}\,d\boldsymbol{e}(\tau_2) \\ &\qquad + g_{01}(t, \tau, \tau_1, \tau_2)\,\mathrm{tr}\,[d\boldsymbol{e}(\tau_1)\,d\boldsymbol{e}(\tau_2)]\}\,d\boldsymbol{e}(\tau)\,. \end{aligned} \tag{7.15}$$

8. Simple fluids.

For plane flow of an incompressible simple fluid, the initial constitutive assumption (2.1) is replaced by

$$\boldsymbol{\sigma} = \mathscr{F}[\boldsymbol{g}_t(\tau)] - p\boldsymbol{\delta}\,, \tag{8.1}$$

where $\boldsymbol{g}_t(\tau)$ is the two-dimensional deformation gradient measured with respect to the current configuration at time t. Then,

(8.2) $$\boldsymbol{g}_t(\tau) = \|\partial x_i(\tau)/\partial x_j(t)\| .$$

Correspondingly, the constitutive relation (4.4) is replaced by

(8.3) $$\boldsymbol{\sigma} = \mathscr{F}[\boldsymbol{C}_t(\tau)] - p\boldsymbol{\delta} ,$$

where

(8.4) $$\boldsymbol{C}_t(\tau) = \boldsymbol{g}_t^{\dagger}(\tau)\boldsymbol{g}_t(\tau) .$$

Alternatively we may write

(8.5) $$\boldsymbol{\sigma} = \mathscr{F}[\boldsymbol{E}_t(\tau)] - p\boldsymbol{\delta} ,$$

where (cf. (4.5))

(8.6) $$\boldsymbol{E}_t(\tau) = \boldsymbol{C}_t(\tau) - \boldsymbol{\delta} .$$

Isotropy of the fluid then yields a restriction on the form of $\mathscr{F}$ given by (4.6) and (4.7) with $\boldsymbol{E}(\tau)$ repelaced by $\boldsymbol{E}_t(\tau)$.

If in (8.5) $\mathscr{F}$ is a polynomial functional of $\boldsymbol{E}_t(\tau)$, then $\mathscr{F}$ may be expressed in the form (7.10), where $\mathscr{F}_N$ is given by (7.11), with $d\boldsymbol{E}(\tau)$ replaced by $d\boldsymbol{E}_t(\tau)$ both in (7.11) and in the expressions (7.7) for $\boldsymbol{L}_P$ and $\boldsymbol{M}_{PQ}$. With this replacement we obtain from (7.10), (7.11) and (8.5)

(8.7) $$\boldsymbol{\sigma} = \sum_N \int_{t_0}^{t} \cdots \int_{t_0}^{t} \gamma_N \, d\boldsymbol{E}_t(\tau) - p\boldsymbol{\delta} .$$

Here the term in (7.11) involving $\boldsymbol{\delta}$ has been absorbed into the arbitrary hydrostatic pressure.

If $\boldsymbol{E}_t(\tau)$ is differentiable, then $d\boldsymbol{E}_t(\tau)$ may be replaced in (8.7) and in the expressions for γ_N by $\boldsymbol{E}_t'(\tau)\, d\tau$, where $\boldsymbol{E}_t'(\tau) = d\boldsymbol{E}_t(\tau)/d\tau$. (We note that $\boldsymbol{E}_t(\tau)$ is necessarily differentiable if the fluid does not exhibit instantaneous elasticity). Then, (8.7) may be rewritten as

(8.8) $$\boldsymbol{\sigma} = \sum_N \int_{t_0}^{t} \cdots \int_{t_0}^{t} \gamma_N \boldsymbol{E}_t'(\tau)\, d\tau - p\delta .$$

If $\boldsymbol{E}_t'(\tau)$ is sufficiently small, a hierarchy of slow flow approximations may be easily read off from this equation. This assumption is, of course, equivalent to the assumption that the velocity gradients are small.

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