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Reflective Subcategories and Dense Subcategories.

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Introduction.

In [M], S. Mardešić defined the notion of a dense subcategory $\mathcal{K} \subset \mathcal{C}$, generalizing the situation one has in the Shape Theory of topological spaces, where $\mathcal{K} = \mathrm{HCW}$ (= the homotopy category of CW-complexes) and $\mathcal{C} = \mathrm{HTOP}$ (= the homotopy category of topological spaces). In [G], E. Giuli observed that « dense subcategories » are a generalization of « reflective subcategories » and characterized (epi-) dense subcategories of TOP.

In this paper we prove that the concepts of density and reflectivity are symmetric with respect to the passage to pro-categories; this means that, if $\mathcal{K} \subset \mathcal{C}$, then $\mathcal{K}$ is dense in $\mathcal{C}$ if and only if pro-$\mathcal{K}$ is reflective in pro-$\mathcal{C}$.

In order to do this we establish two necessary and sufficient conditions for $\mathcal{K}$ being dense in $\mathcal{C}$. In the last section we discuss relations between epi-density and epi-reflectivity.

1. Pro-categories and pro-representable functors.

Let $\mathcal{C}$ be a category; an inverse system $\boldsymbol{X} = (X_i, p_{ij}, I)$ in $\mathcal{C}$, is a family of $\mathcal{C}$-objects $\{X_i : i \in I\}$, indexed on a directed set I and equipped with $\mathcal{C}$-morphisms (bonding morphisms) $p_{ij}\colon X_j \to X_i$, $\forall i \leqslant j$

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in I, such that $p_{ii} = 1_{X_i}$ and $p_{ij} \cdot p_{jt} = p_{it}$, for any $i \leqslant j \leqslant t$ in I.

The inverse systems in $\mathcal{C}$ are the objects of the category pro-$\mathcal{C}$, whose morphisms, from $\boldsymbol{X}$ to $\boldsymbol{Y} = (Y_a, q_{ab}, A)$, are given by the formula (see [AM; App.] and [Gr; § 2]):

$$[\boldsymbol{X}, \boldsymbol{Y}] = \varprojlim_i \varinjlim_a [X_i, Y_a] \,. \tag{1.1}$$

The above definition of (pro-$\mathcal{C}$)-morphisms may be explicitated as follows (see [M; § 1] or [MS; Ch. I, § 1]).

A map of system (f, f_a): $\boldsymbol{X} \to \boldsymbol{Y}$ consists of a function $f\colon A \to I$ and of a collection of $\mathcal{C}$-morphisms $f_a\colon X_{f(a)} \to Y_a$, $a \in A$, such that for $a \leqslant a'$ there is an $i \geqslant f(a), f(a')$ such that $f_a \cdot p_{f(a)i} = q_{aa'} \cdot f_{a'} \cdot p_{f(a')i}$. Two maps of systems (f, f_a), (f', f'_a): $\boldsymbol{X} \to \boldsymbol{Y}$ are considered equivalent, provided for each $a \in A$ there is an $i \geqslant f(a), f'(a)$ such that $f_a \cdot p_{f(a)i} = f'_a \cdot p_{f'(a)i}$.

A (pro-$\mathcal{C}$)-morphism $\boldsymbol{f}$: $\boldsymbol{X} \to \boldsymbol{Y}$ is an equivalence class of maps of systems.

Let us note that $\mathcal{C}$ is (equivalent to) the full subcategory of pro-$\mathcal{C}$, whose objects are rudimentary inverse systems $\boldsymbol{X} = (X)$, indexed on a one-point set.

Every inverse system $\boldsymbol{X} = (X_i, p_{ij}, I)$ in $\mathcal{C}$ induces a direct system $([X_i, -], p^*_{ij}, I)$ of covariant functors from $\mathcal{C}$ to the category SET of sets, (cfr. [MS; Ch. I, Remark 5]). Then we can form the colimit of this direct system in the functor category $\mathrm{SET}^{\mathcal{C}}$:

$$h^{\boldsymbol{X}} = \varinjlim_i \big([X_i, -], p^*_{ij}, I\big) \,. \tag{1.2}$$

Definition 1.3. A covariant functor F: $\mathcal{C} \to \mathrm{SET}$ is said to be pro-representable on $\mathcal{C}$, by means of an $\boldsymbol{X} \in$ pro-$\mathcal{C}$, if there exists a natural isomorphism $F \cong h^{\boldsymbol{X}}$.

It is clear that any representable functor $[X, -]$ is pro-representable by means of the rudimentary system $\boldsymbol{X} = (X)$.

It is also clear that if $h^{\boldsymbol{X}}$ and $h^{\boldsymbol{Y}}$ are two pro-representations of F, then $\boldsymbol{X}$ and $\boldsymbol{Y}$ are isomorphic (pro-$\mathcal{C}$)-objects (cfr. [Gr; § 2]).

Proposition 1.4. *The correspondence $\boldsymbol{X} \mapsto h^{\boldsymbol{X}}$ establishes a contravariant isomorphism between pro-$\mathcal{C}$ and the full subcategory of $\mathrm{SET}^{\mathcal{C}}$ of all pro-representable functors.*

PROOF. It must to be proved that, if $\boldsymbol{X} = (X_i, p_{ij}, I)$ and $\boldsymbol{Y} = (Y_a, q_{ab}, A)$ are inverse systems in $\mathcal{C}$, then there is a bijection NAT $(h^{\boldsymbol{X}}, h^{\boldsymbol{Y}}) \cong [\boldsymbol{Y}, \boldsymbol{X}]$. One has:

$$
\begin{aligned}
& NAT(h^{\boldsymbol{X}}, h^{\boldsymbol{Y}}) = \quad \text{(by (1.2))} \\
&= NAT\Big(\varinjlim_i [X_i, -], \varinjlim_a [Y_a, -]\Big) \cong \quad \text{(by [}P\text{; Th. 2, p. 90])} \\
&\cong \varprojlim_i NAT([X_i, -], \varinjlim_a [Y_a, -]) \cong \quad \text{(by Yoneda lemma)} \\
&\cong \varprojlim_i \varinjlim_a [Y_a, X_i] = \quad \text{(by (1.1))} \\
&= [\boldsymbol{Y}, \boldsymbol{X}] .
\end{aligned}
$$

COROLLARY 1.5. Let $(\boldsymbol{X}^\lambda)_{\lambda \in \Lambda}$ be an inverse system in pro-$\mathcal{C}$. Then one has $\boldsymbol{X} = \varprojlim_\lambda \boldsymbol{X}^\lambda$ in pro-$\mathcal{C}$ if and only if $h^{\boldsymbol{X}} = \varinjlim_\lambda h^{\boldsymbol{X}^\lambda}$ in the category of all pro-representable functors.

PROOF. Recall from [AM; Prop. 4.4, App.] that, for any category $\mathcal{C}$, pro-$\mathcal{C}$ is closed under the formation of limits of inverse systems.

2. Dense subcategories and reflective subcategories.

All subcategories are assumed to be full.

Recall from [M; § 2, Def. 1] the following definition.

DEFINITION 2.1. Let $\mathcal{K} \subset \mathcal{C}$ and let X be a $\mathcal{C}$-object. A $\mathcal{K}$-expansion of X is an inverse system $\boldsymbol{K} = (K_i, p_{ij}, I)$ in $\mathcal{K}$, together with a (pro-$\mathcal{C}$)-morphism $\boldsymbol{p} = (p_i)\colon X \to \boldsymbol{K}$, such that:

(a) $\forall H \in \mathcal{K}$, $\forall f\colon X \to H$ in $\mathcal{C}$, there is a $\mathcal{K}$-morphism $f_i\colon K_i \to H$ such that $f_i \cdot p_i = f$.

(b) If $f_i, g_i\colon K_i \to H$ are $\mathcal{K}$-morphisms with $f_i \cdot p_i = g_i \cdot p_i$, then there is a $j \geqslant i$ in I, such that $f_i \cdot p_{ij} = g_i \cdot p_{ij}$.

$\mathcal{K}$ is dense in $\mathcal{C}$ provided every $\mathcal{C}$-object X admits a $\mathcal{K}$-expansion.

PROPOSITION 2.2. Let $\mathcal{K}$ be a subcategory of $\mathcal{C}$ and $J\colon \mathcal{K} \hookrightarrow \mathcal{C}$ be the inclusion functor. $\mathcal{K}$ is dense in $\mathcal{C}$ if and only if, for every

$\mathcal{C}$-object X, the covariant functor $[X, J(\ )]$: $\mathcal{K} \to \mathrm{SET}$ is pro-representable on $\mathcal{K}$.

PROOF. Let $\boldsymbol{p} = (p_i)$: $X \to \boldsymbol{K} = (K_i, p_{ij}, I)$ be a $\mathcal{K}$-expansion of $X \in \mathcal{C}$. Each $\mathcal{C}$-morphism p_i: $X \to K_i$, $i \in I$, induces a natural transformation p_i^*: $[K_i, -] \to [X, J(\ )]$ such that, if $i \leqslant j$ in I, then $p_j^* \cdot p_{ij}^* = p_i^*$. Therefore we obtain a natural transformation $\boldsymbol{p}^*$: $h^{\boldsymbol{K}} = \varinjlim_i [K_i, -] \to$ $\to [X, J(\ )]$.

It has been pointed out in [MS; Ch. I, Remark 5] that conditions (a) and (b) above are equivalent to the requirement that $\boldsymbol{p}^*$ be a natural isomorphism.

Conversely, let ψ: $\varinjlim_i [K_i, -] \to [X, J(\ )]$ be given and, for each $i \in I$, let $\psi(1_{K_i}) = p_i$: $X \to K_i$. Then the morphisms $\{p_i$: $X \to K_i$: $i \in I\}$ so determined constitute a (pro-$\mathcal{C}$)-morphism $\boldsymbol{p}$: $X \to \boldsymbol{K}$, and it turns out that $\psi = \boldsymbol{p}^*$; hence $\boldsymbol{p}$ is a $\mathcal{K}$-expansion for $X \in \mathcal{C}$.

(2.3) Recall now ([HS]) that, if $\mathcal{K} \subset \mathcal{C}$, then, in order that $\mathcal{K}$ be reflective in $\mathcal{C}$, the following conditions are equivalent:

(r_1) $\forall X \in \mathcal{C}$, $[X, J(\ )]$: $\mathcal{K} \to \mathrm{SET}$ is representable on $\mathcal{K}$.

(r_2) the inclusion functor J: $\mathcal{K} \hookrightarrow \mathcal{C}$ has a left adjoint.

Now, it is clear, from Proposition 2.2 and condition (r_1) above, that the concept of pro-representability is the right generalization of that of representability, when passing from reflective subcategories to dense subcategories.

In the next theorem we state a condition, similar to (r_2), in order that a subcategory $\mathcal{K}$ of $\mathcal{C}$ be dense in $\mathcal{C}$.

If J: $\mathcal{K} \hookrightarrow \mathcal{C}$ is an inclusion functor, let us denote by J^*: pro-$\mathcal{K} \to$ $\to$ pro-$\mathcal{C}$, the corresponding inclusion of the pro-categories.

Since $\mathcal{K} \subset$ pro-$\mathcal{C}$, then $J^*_{|\mathcal{K}} = J$.

THEOREM 2.4. Let J: $\mathcal{K} \hookrightarrow \mathcal{C}$. $\mathcal{K}$ is dense in $\mathcal{C}$ if and only if J^*: pro-$\mathcal{K} \to$ pro-$\mathcal{C}$ has a left adjoint.

PROOF. Let Λ': pro-$\mathcal{C} \to$ pro-$\mathcal{K}$ be left adjoint to J^*. If $X \in \mathcal{C}$ and $\Lambda'(X) = \boldsymbol{K} = (K_i, p_{ij}, I)$, then, for each $H \in \mathcal{K}$, there is a bijection

$$[X, J(H)] \cong [\boldsymbol{K}, H] = \varinjlim_i [K_i, H] = h^{\boldsymbol{K}}(H)\,,$$

therefore a natural isomorphism $[X, J(\ )] \cong h^{\boldsymbol{K}}$. In view of Proposition 2.2, $\boldsymbol{K}$ is a $\mathcal{K}$-expansion of X.

Conversely, suppose $\mathcal{K}$ is dense in $\mathcal{C}$. Any $\mathcal{C}$-object X admits a $\mathcal{K}$-expansion $\boldsymbol{p}\colon X \to \boldsymbol{K}$. This gives a correspondence $X \mapsto \Lambda'(X) = \boldsymbol{K}$, from $\mathcal{C}$ to pro-$\mathcal{K}$, which is functorial since, if $\boldsymbol{q}\colon Y \to \boldsymbol{H}$ is a $\mathcal{K}$-expansion of $Y \in \mathcal{C}$, and if $f\colon X \to Y$ is a $\mathcal{C}$-morphism, then there is a unique (pro-$\mathcal{K}$)-morphism $\Lambda'(f)\colon \boldsymbol{K} \to \boldsymbol{H}$, which makes the following diagram commutative (cfr. [MS; Ch. I, § 3]):

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "f"'] & \boldsymbol{K} \arrow[d, "\Lambda'(f)"] \\ Y \arrow[r, "q"] & \boldsymbol{H}. \end{tikzcd}$$

Now, let $\boldsymbol{X} = (X_i, p_{ij}, I) \in$ pro-$\mathcal{C}$; applying Λ' to each X_i, we obtain an inverse system in pro-$\mathcal{K}$, $(\Lambda'(X_i), \Lambda'(p_{ij}), I)$. By [AM; Prop. 4.4, App.], there exists in pro-$\mathcal{K}$ the limit

$$\Lambda(\boldsymbol{X}) = \varprojlim_i \left(\Lambda'(X_i), \Lambda'(p_{ij}), I\right).$$

This formula extends the functor $\Lambda'\colon \mathcal{C} \to$ pro-$\mathcal{K}$ to a functor Λ: pro-$\mathcal{C} \to$ pro-$\mathcal{K}$. It remains to show that Λ is left adjoint to J^*. Since for each $i \in I$ there is natural isomorphism

$$[X_i, J(\;)] \cong [\Lambda'(X_i), -] = h^{\Lambda'(X_i)},$$

then, taking the colimit on I and applying (1.1) and Cor. 1.5, it follows that

$$[\boldsymbol{X}, J(\;)] \cong [\Lambda(\boldsymbol{X}), -] = h^{\Lambda(\boldsymbol{X})}.$$

Given now an $\boldsymbol{L} = (L_a, q_{ab}, A) \in$ pro-$\mathcal{K}$, from above we get bijections

$$[\boldsymbol{X}, J(L_a)] \cong [\Lambda(\boldsymbol{X}), L_a], \qquad \forall a \in A .$$

This time, taking the limit on A, it follows at once from (1.1)

$$[\boldsymbol{X}, J^*(\boldsymbol{L})] \cong [\Lambda(\boldsymbol{X}), \boldsymbol{L}],$$

and we have finished.

Corollary 2.5. Let $\mathcal{K} \subset \mathcal{C}$. $\mathcal{K}$ is dense in $\mathcal{C}$ if and only if pro-$\mathcal{K}$ is reflective in pro-$\mathcal{C}$.

This follows immediately from the equivalence of conditions (r_1) and (r_2) in (2.3).

(2.6) Now we want to explicitate the construction of the reflection $\boldsymbol{\lambda}_{\boldsymbol{X}}\colon \boldsymbol{X} \to \Lambda(\boldsymbol{X})$, for a given $\boldsymbol{X} = (X_j, p_{jj'}, J) \in$ pro-$\mathcal{C}$.

For each $j \in J$, let $\boldsymbol{\lambda}^j\colon X_j \to \boldsymbol{K}^j = (K_i^j, q_{ii'}^j, I_j)$ be a $\mathcal{K}$-expansion of X_j. Since for any $p_{jj'}\colon X_{j'} \to X_j$, there is a unique $\boldsymbol{q}^{jj'}\colon \boldsymbol{K}^{j'} \to \boldsymbol{K}^j$ such that $\boldsymbol{q}^{jj'}\cdot\boldsymbol{\lambda}^{j'} = \boldsymbol{\lambda}^j\cdot p_{jj'}$ ([MS; Ch. I, § 3]), then we obtain an inverse system in pro-$\mathcal{K}$, $(\boldsymbol{K}^j, \boldsymbol{q}^{jj'}, J)$, whose limit $\Lambda(\boldsymbol{X})$, according to [AM; Prop. 4.4, App.], is obtained in the following way:

let $F = \{(j, i)\colon j \in J,\ i \in I_j\}$, and put on it the relation

$$(j, i) \leqslant (j', i') \Leftrightarrow [j \leqslant j' \text{ in } J \text{ and } q_{ii'}^{jj'}\colon K_{i'}^{j'} \to K_i^j \text{ is a } \mathcal{K}\text{-morphism constituing the bonding morphism } \boldsymbol{q}^{jj'}]\,.$$

Then F becomes a directed set and one easily verifies that $\Lambda(\boldsymbol{X}) = (K_i^j, q_{ii'}^{jj'}, F)$. Finally, $\boldsymbol{\lambda}_{\boldsymbol{X}}\colon \boldsymbol{X} \to \Lambda(\boldsymbol{X})$ is such that $(\boldsymbol{\lambda}_{\boldsymbol{X}})_{(j,i)} = \lambda_i^j\colon X_j \to K_i^j$.

Remark 2.7. Suppose $\mathcal{K}$ is reflective in $\mathcal{C}$, then (cfr. [G; Prop. 1.1]) it is trivially dense in $\mathcal{C}$; so pro-$\mathcal{K}$ is reflective in pro-$\mathcal{C}$. If $X \in \mathcal{C}$ has a reflection $r\colon X \to rX$, $rX \in \mathcal{K}$, then the rudimentary system $\boldsymbol{X} = (X)$ admits the reflection $\boldsymbol{r} = (r)\colon \boldsymbol{X} \to \boldsymbol{rX} = (rX)$. Moreover, given $\boldsymbol{X} = (X_i, p_{ij}, I)$ in pro-$\mathcal{C}$, then one has $\Lambda(\boldsymbol{X}) = (rX_i, rp_{ij}, I)$, while the reflection morphism $\boldsymbol{r}\colon \boldsymbol{X} \to \Lambda(\boldsymbol{X})$ is the level morphism given by $\boldsymbol{r} = \{r_i\colon X_i \to rX_i,\ \forall i \in I\}$.

3. EPI-reflections and EPI-densities.

Definition 3.1. Let $\boldsymbol{f} = (f_a)\colon X \to \boldsymbol{Y} = (Y_a, q_{ab}, A)$ be a (pro-$\mathcal{C}$)-morphism. We call $\boldsymbol{f}$ a strong (pro-$\mathcal{C}$)-epimorphism if for each $a \in A$, there is a $b \geqslant a$ such that $f_b\colon X \to Y_b$ is a $\mathcal{C}$-epimorphism.

According to [M; § 1, Lemma 1], if $\boldsymbol{f}$ is a strong (pro-$\mathcal{C}$)-epimorphism, then there exists a $\boldsymbol{Y}' \simeq \boldsymbol{Y}$ in pro-$\mathcal{C}$ and a (pro-$\mathcal{C}$)-morphism $\boldsymbol{f}' = (f_a')\colon X \to \boldsymbol{Y}'$, such that each f_a' is a $\mathcal{C}$-epimorphism, and $\boldsymbol{f}' = \boldsymbol{f}$.

The definition of strong (pro-$\mathcal{C}$)-epimorphism extends easily to a (pro-$\mathcal{C}$)-morphism $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$.

It is clear that a strong (pro-$\mathcal{C}$)-epimorphism is a (pro-$\mathcal{C}$)-epimorphism.

PROPOSITION 3.2. Let $\boldsymbol{f} = (f_j)\colon X \to \boldsymbol{Y} = (Y_j, q_{jj'}, J)$ be a (pro-$\mathcal{C}$)-epimorphism. If all bonding morphisms $q_{jj'}\colon Y_{j'} \to Y_j$ of $\boldsymbol{Y}$ are $\mathcal{C}$-epimorphisms, then $\boldsymbol{f}$ is a strong (pro-$\mathcal{C}$)-epimorphism.

PROOF. Let $j \in J$ and let $h, g\colon Y_j \to Z$, $Z \in \mathcal{C}$, be $\mathcal{C}$-morphisms such that $h \cdot f_j = g \cdot f_j$. Then, since $\boldsymbol{h} = (h)$ and $\boldsymbol{g} = (g)$ are (pro-$\mathcal{C}$)-morphisms from $\boldsymbol{Y}$ to Z such that $\boldsymbol{h} \cdot \boldsymbol{f} = \boldsymbol{g} \cdot \boldsymbol{f}$, it follows that $\boldsymbol{h} = \boldsymbol{g}$ in pro-$\mathcal{C}$. This last equality means ([M; § 1]) that there is a $j' \geqslant j$ such that $h \cdot q_{jj'} = g \cdot q_{jj'}$, so, by the assumption that $q_{jj'}$ is an epimorphism, it follows $h = g$.

DEFINITION 3.3. Let $\mathcal{K}$ be dense in $\mathcal{C}$. $\mathcal{K}$ is epi-dense in $\mathcal{C}$ if every $\mathcal{C}$-object X admits a $\mathcal{K}$-expansion $\boldsymbol{p}\colon X \to \boldsymbol{K}$, which is a strong (pro-$\mathcal{C}$)-epimorphism.

PROPOSITION 3.4. If $\mathcal{K}$ is epi-dense in $\mathcal{C}$, then pro-$\mathcal{K}$ is epi-reflective in pro-$\mathcal{C}$. Every reflection morphism is a strong (pro-$\mathcal{C}$)-epimorphism. If pro-$\mathcal{K}$ is (strong epi)- reflective in pro-$\mathcal{C}$, then $\mathcal{K}$ is epi-dense in $\mathcal{C}$.

PROOF. Let $\boldsymbol{Y} = (Y_j, q_{jj'}, J) \in$ pro-$\mathcal{C}$ and let $\boldsymbol{\lambda}_{\boldsymbol{Y}}\colon \boldsymbol{Y} \to \Lambda(\boldsymbol{Y})$ be its reflection, as in (2.6). Recall that $\boldsymbol{\lambda}_{\boldsymbol{Y}} = (\lambda_i^j)_{(j,i)\in F}$; since we may assume, without any restriction, that each λ_i^j is a $\mathcal{C}$-epimorphism, it follows that $\boldsymbol{\lambda}_{\boldsymbol{Y}}\colon \boldsymbol{Y} \to \Lambda(\boldsymbol{Y})$ is a strong (pro-$\mathcal{C}$)-epimorphism. The proof of the second part is immediate.

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