Luciano Stramaccia

Reflective subcategories and dense subcategories

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Reflective Subcategories and Dense Subcategories.

LUCIANO STRAMACCIA (*)

Introduction.

In [M], S. Mardešić defined the notion of a dense subcategory $\mathcal{K} \subset C$, generalizing the situation one has in the Shape Theory of topological spaces, where $\mathcal{K} = \text{HCW}$ (the homotopy category of CW-complexes) and $C = \text{HTOP}$ (the homotopy category of topological spaces). In [G], E. Giuli observed that « dense subcategories » are a generalization of « reflective subcategories » and characterized (epi-) dense subcategories of TOP.

In this paper we prove that the concepts of density and reflectivity are symmetric with respect to the passage to pro-categories; this means that, if $\mathcal{K} \subset C$, then $\mathcal{K}$ is dense in $C$ if and only if pro-$\mathcal{K}$ is reflective in pro-$C$.

In order to do this we establish two necessary and sufficient conditions for $\mathcal{K}$ being dense in $C$. In the last section we discuss relations between epi-density and epi-reflectivity.

1. Pro-categories and pro-representable functors.

Let $C$ be a category; an inverse system $X = (X_i, p_{ij}, I)$ in $C$, is a family of $C$-objects $\{X_i : i \in I\}$, indexed on a directed set $I$ and equipped with $C$-morphisms (bonding morphisms) $p_{ij} : X_j \to X_i, \forall i < j$.

(*) Indirizzo dell'A.: Istituto di Geometria, Università di Perugia, Via Vanvitelli n. 1, 06100 Perugia.
in $I$, such that $p_{ii} = 1_{X_i}$ and $p_{ij} \cdot p_{jt} = p_{it}$, for any $i < j < t$ in $I$.

The inverse systems in $C$ are the objects of the category pro-$C$, whose morphisms, from $X$ to $Y = (Y_a, q_{ab}, A)$, are given by the formula (see [AM; App.] and [Gr; § 2]):

$$[X, Y] = \lim_{\leftarrow i} \lim_{\rightarrow a} [X_i, Y_a].$$

The above definition of (pro-$C$)-morphisms may be explicitated as follows (see [M; § 1] or [MS; Ch. I, § 1]).

A map of system $(f, f_a): X \to Y$ consists of a function $f: A \to I$ and of a collection of $C$-morphisms $f_a: X_{f(a)} \to Y_a$, $a \in A$, such that for $a < a'$ there is an $i > f(a), f(a')$ such that $f_a \cdot p_{f(a)i} = q_{aa'} \cdot f_{a'} \cdot p_{f(a')i}$. Two maps of systems $(f, f_a), (f', f'_a): X \to Y$ are considered equivalent, provided for each $a \in A$ there is an $i > f(a), f'(a)$ such that $f_a \cdot p_{f(a)i} = f'_a \cdot p_{f'(a)i}$.

A (pro-$C$)-morphism $f: X \to Y$ is an equivalence class of maps of systems.

Let us note that $C$ is (equivalent to) the full subcategory of pro-$C$, whose objects are rudimentary inverse systems $X = (X)$, indexed on a one-point set.

Every inverse system $X = (X_i, p_{ii}, I)$ in $C$ induces a direct system $([X_i, -], p_{ij}^*, I)$ of covariant functors from $C$ to the category SET of sets, (cfr. [MS; Ch. I, Remark 5]). Then we can form the colimit of this direct system in the functor category $\text{SET}^C$:

$$h^X = \lim_{\rightarrow i} ([X_i, -], p_{ij}^*, I).$$

**Definition 1.3.** A covariant functor $F: C \to \text{SET}$ is said to be pro-representable on $C$, by means of an $X \in \text{pro-}C$, if there exists a natural isomorphism $F \cong h^X$.

It is clear that any representable functor $[X, -]$ is pro-representable by means of the rudimentary system $X = (X)$.

It is also clear that if $h^X$ and $h^Y$ are two pro-representations of $F$, then $X$ and $Y$ are isomorphic (pro-$C$)-objects (cfr. [Gr; § 2]).

**Proposition 1.4.** The correspondence $X \mapsto h^X$ establishes a contravariant isomorphism between pro-$C$ and the full subcategory of $\text{SET}^C$ of all pro-representable functors.
PROOF. It must to be proved that, if \( X = (X_i, p_{ii}, I) \) and \( Y = (Y_a, q_{ab}, A) \) are inverse systems in \( C \), then there is a bijection 
\[ \text{NAT}(h^X, h^Y) \cong [Y, X]. \]
One has:

\[
\text{NAT}(h^X, h^Y) = \quad \text{(by (1.2))}
\]
\[
= \text{NAT} \left( \lim_{\longrightarrow} [X_i, -], \lim_{\longrightarrow} [Y_a, -] \right) \cong \quad \text{(by [P; Th. 2, p. 90])}
\]
\[
\cong \lim_{\longleftarrow} \text{NAT}([X_i, -], \lim_{\longrightarrow} [Y_a, -]) \cong \quad \text{(by Yoneda lemma)}
\]
\[
\cong \lim_{\longleftarrow} \lim_{\longrightarrow} [Y_a, X_i] = \quad \text{(by (1.1))}
\]
\[
= [Y, X].
\]

**Corollary 1.5.** Let \( (X^i)_{i \in A} \) be an inverse system in \( \text{pro}-C \). Then one has \( X = \lim_{\longleftarrow} X^i \) in \( \text{pro}-C \) if and only if \( h^X = \lim_{\longleftarrow} h^X^i \) in the category of all \( \text{pro-representable functors} \).

**Proof.** Recall from [AM; Prop. 4.4, App.] that, for any category \( C \), \( \text{pro}-C \) is closed under the formation of limits of inverse systems.

### 2. Dense subcategories and reflective subcategories.

All subcategories are assumed to be full.

Recall from [M; § 2, Def. 1] the following definition.

**Definition 2.1.** Let \( \mathcal{K} \subseteq C \) and let \( X \) be a \( C \)-object. A \( \mathcal{K} \)-expansion of \( X \) is an inverse system \( K = (K_i, p_{ii}, I) \) in \( \mathcal{K} \), together with a (\( \text{pro}-C \))-morphism \( p = (p_i): X \to K \), such that:

\begin{itemize}
  \item [(a)] \( \forall H \in \mathcal{K}, \forall f: X \to H \) in \( C \), there is a \( \mathcal{K} \)-morphism \( f_i: K_i \to H \) such that \( f_i \cdot p_i = f \).
  \item [(b)] If \( f_i, g_i: K_i \to H \) are \( \mathcal{K} \)-morphisms with \( f_i \cdot p_i = g_i \cdot p_i \), then there is a \( j \geq i \) in \( I \), such that \( f_i \cdot p_{jj} = g_i \cdot p_{jj} \).
\end{itemize}

\( \mathcal{K} \) is dense in \( C \) provided every \( C \)-object \( X \) admits a \( \mathcal{K} \)-expansion.

**Proposition 2.2.** Let \( \mathcal{K} \) be a subcategory of \( C \) and \( J: \mathcal{K} \hookrightarrow C \) be the inclusion functor. \( \mathcal{K} \) is dense in \( C \) if and only if, for every
C-object $X$, the covariant functor $[X, J(\cdot)]: \mathcal{K} \to \text{SET}$ is pro-representable on $\mathcal{K}$.

**Proof.** Let $p = (p_i): X \to K = (K_i, p_{ii}, I)$ be a $\mathcal{K}$-expansion of $X \in \mathcal{C}$. Each $\mathcal{C}$-morphism $p_i: X \to K_i$, $i \in I$, induces a natural transformation $p_i^*: [K_i, -] \to [X, J(\cdot)]$ such that, if $i < j$ in $I$, then $p_j^* \circ p_i^* = p_i^*$. Therefore we obtain a natural transformation $p^*: h^K = \lim_{\rightarrow} [K_i, -] \to [X, J(\cdot)]$.

It has been pointed out in [MS; Ch. I, Remark 5] that conditions (a) and (b) above are equivalent to the requirement that $p^*$ be a natural isomorphism.

Conversely, let $\psi: \lim_{\rightarrow} [K_i, -] \to [X, J(\cdot)]$ be given and, for each $i \in I$, let $\psi(1_{K_i}) = p_i: X \to K_i$. Then the morphisms $\{p_i: X \to K_i: i \in I\}$ so determined constitute a (pro-$\mathcal{C}$)-morphism $p: X \to K$, and it turns out that $\psi = p^*$; hence $p$ is a $\mathcal{K}$-expansion for $X \in \mathcal{C}$.

(2.3) Recall now ([HS]) that, if $\mathcal{K} \subset \mathcal{C}$, then, in order that $\mathcal{K}$ be reflective in $\mathcal{C}$, the following conditions are equivalent:

$(r_1) \forall X \in \mathcal{C}, [X, J(\cdot)]: \mathcal{K} \to \text{SET}$ is representable on $\mathcal{K}$.

$(r_2)$ the inclusion functor $J: \mathcal{K} \hookrightarrow \mathcal{C}$ has a left adjoint.

Now, it is clear, from Proposition 2.2 and condition $(r_1)$ above, that the concept of pro-representability is the right generalization of that of representability, when passing from reflective subcategories to dense subcategories.

In the next theorem we state a condition, similar to $(r_2)$, in order that a subcategory $\mathcal{K}$ of $\mathcal{C}$ be dense in $\mathcal{C}$.

If $J: \mathcal{K} \hookrightarrow \mathcal{C}$ is an inclusion functor, let us denote by $J^*: \text{pro-}\mathcal{K} \to \text{pro-}\mathcal{C}$ the corresponding inclusion of the pro-categories.

Since $\mathcal{K} \subset \text{pro-}\mathcal{C}$, then $J^*_\mathcal{K} = J$.

**Theorem 2.4.** Let $J: \mathcal{K} \hookrightarrow \mathcal{C}$. $\mathcal{K}$ is dense in $\mathcal{C}$ if and only if $J^*: \text{pro-}\mathcal{K} \to \text{pro-}\mathcal{C}$ has a left adjoint.

**Proof.** Let $A': \text{pro-}\mathcal{C} \to \text{pro-}\mathcal{K}$ be left adjoint to $J^*$. If $X \in \mathcal{C}$ and $A'(X) = K = (K_i, p_{ii}, I)$, then, for each $H \in \mathcal{K}$, there is a bijection

$$[X, J(H)] \cong [K, H] = \lim_{\rightarrow} [K_i, H] = h^K(H),$$

therefore a natural isomorphism $[X, J(\cdot)] \cong h^K$. In view of Proposition 2.2, $K$ is a $\mathcal{K}$-expansion of $X$. 
Conversely, suppose \( \mathcal{K} \) is dense in \( \mathcal{C} \). Any \( \mathcal{C} \)-object \( X \) admits a \( \mathcal{K} \)-expansion \( p : X \to K \). This gives a correspondence \( X \mapsto A'(X) = K \), from \( \mathcal{C} \) to \( \text{pro-}\mathcal{K} \), which is functorial since, if \( q : Y \to H \) is a \( \mathcal{K} \)-expansion of \( Y \in \mathcal{C} \), and if \( f : X \to Y \) is a \( \mathcal{C} \)-morphism, then there is a unique (pro-\( \mathcal{K} \))-morphism \( A'(f) : K \to H \), which makes the following diagram commutative (cfr. [MS; Ch. I, § 3]):

\[
\begin{array}{ccc}
X & \xrightarrow{p} & K \\
\downarrow{f} & & \downarrow{A'(f)} \\
Y & \xrightarrow{q} & H
\end{array}
\]

Now, let \( X = (X_i, p_{ii}, I) \in \text{pro-}\mathcal{C} \); applying \( A' \) to each \( X_i \), we obtain an inverse system in \( \text{pro-}\mathcal{K} \), \( (A'(X_i), A'(p_{ii}), I) \). By [AM; Prop. 4.4, App.], there exists in \( \text{pro-}\mathcal{K} \) the limit

\[
A(X) = \lim_{\leftarrow i} (A'(X_i), A'(p_{ii}), I).
\]

This formula extends the functor \( A' : \mathcal{C} \to \text{pro-}\mathcal{K} \) to a functor \( A : \text{pro-}\mathcal{C} \to \text{pro-}\mathcal{K} \). It remains to show that \( A \) is left adjoint to \( J^* \). Since for each \( i \in I \) there is natural isomorphism

\[
[X_i, J(\_)] \cong [A'(X_i), \_] = \mathbb{h}^{A'(X_i)},
\]

then, taking the colimit on \( I \) and applying (1.1) and Cor. 1.5, it follows that

\[
[X, J(\_)] \cong [A(X), \_] = \mathbb{h}^{A(X)}.
\]

Given now an \( L = (L_a, q_{ab}, A) \in \text{pro-}\mathcal{K} \), from above we get bijections

\[
[X, J(L_a)] \cong [A(X), L_a], \quad \forall a \in A.
\]

This time, taking the limit on \( A \), it follows at once from (1.1)

\[
[X, J^*(L)] \cong [A(X), L],
\]

and we have finished.
COROLLARY 2.5. Let $\mathcal{K} \subset C$. $\mathcal{K}$ is dense in $C$ if and only if pro-$\mathcal{K}$ is reflective in pro-$C$.

This follows immediately from the equivalence of conditions $(r_1)$ and $(r_2)$ in (2.3).

(2.6) Now we want to explicitate the construction of the reflection

$$\lambda_X: X \to \Lambda(X),$$

for a given $X = (X_i, p_{ij}, J) \in \text{pro-}C$.

For each $j \in J$, let $\lambda^j: X_j \to K_j = (K_j, q_{ij}, I_j)$ be a $\mathcal{K}$-expansion of $X_j$. Since for any $p_{ij}: X_j \to X_i$, there is a unique $q^{ij}: K_i \to K_j$ such that $q^{ij} \cdot \lambda^j = \lambda^j \cdot p_{ij}$ ([MS; Ch. I, § 3]), then we obtain an inverse system in pro-$\mathcal{K}$, $(K_i, q^{ij}, J)$, whose limit $\Lambda(X)$, according to [AM; Prop. 4.4, App.], is obtained in the following way:

let $F = \{(j, i): j \in J, i \in I_j\}$, and put on it the relation

$$(j, i) < (j', i') \iff [j < j' \text{ in } J \text{ and } q^{ij'}: K_i \to K'_i \text{ is a } \mathcal{K}\text{-morphism consti-}$$

$$tuting the bonding morphism } q^{ij'}].$$

Then $F$ becomes a directed set and one easily verifies that

$$\Lambda(X) = (K'_i, q^{ij'}, F).$$

Finally, $\lambda_X: X \to \Lambda(X)$ is such that $(\lambda_X)_{(j, i)} = \lambda^j_i: X_j \to K'_i$.

REMARK 2.7. Suppose $\mathcal{K}$ is reflective in $C$, then (cfr. [G; Prop. 1.1]) it is trivially dense in $C$; so pro-$\mathcal{K}$ is reflective in pro-$C$. If $X \in C$ has a reflection $r: X \to rX$, $rX \in \mathcal{K}$, then the rudimentary system $X = (X)$ admits the reflection $r = (r): X \to rX = (rX)$. Moreover, given $X = (X_i, p_{ij}, I)$ in pro-$C$, then one has $\Lambda(X) = (rX_i, rp_{ij}, I)$, while the reflection morphism $r: X \to \Lambda(X)$ is the level morphism given by $r = \{r_i: X_i \to rX_i, \forall i \in I\}$.

3. EPI-reflections and EPI-densities.

DEFINITION 3.1. Let $f = (f_a): X \to Y = (Y_a, q_{ab}, A)$ be a (pro-$C$)-morphism. We call $f$ a strong (pro-$C$)-epimorphism if for each $a \in A$, there is a $b > a$ such that $f_b: X \to Y_b$ is a $C$-epimorphism.

According to [M; § 1, Lemma 1], if $f$ is a strong (pro-$C$)-epimorphism, then there exists a $Y' \cong Y$ in pro-$C$ and a (pro-$C$)-morphism $f' = (f'_a): X \to Y'$, such that each $f'_a$ is a $C$-epimorphism, and $f' = f$. 
The definition of strong (pro-C)-epimorphism extends easily to a (pro-C)-morphism \( f: X \to Y \).

It is clear that a strong (pro-C)-epimorphism is a (pro-C)-epimorphism.

**Proposition 3.2.** Let \( f = (f_j): X \to Y = (Y_j, q_{i,j'}, J) \) be a (pro-C)-epimorphism. If all bonding morphisms \( q_{i,j'}: Y_{i'} \to Y_j \) of \( Y \) are C-epimorphisms, then \( f \) is a strong (pro-C)-epimorphism.

**Proof.** Let \( j \in J \) and let \( h, g: Y_j \to Z, Z \in C \), be C-morphisms such that \( h \cdot f = g \cdot f_j \). Then, since \( h = (h) \) and \( g = (g) \) are (pro-C)-morphisms from \( Y \) to \( Z \) such that \( h \cdot f = g \cdot f \), it follows that \( h = g \) in pro-C. This last equality means ([M; § 1]) that there is a \( j' > j \) such that \( h \cdot q_{i,j'} = g \cdot q_{i,j'} \), so, by the assumption that \( q_{i,j'} \) is an epimorphism, it follows \( h = g \).

**Definition 3.3.** Let \( K \) be dense in \( C \). \( K \) is epi-dense in \( C \) if every C-object \( X \) admits a \( K \)-expansion \( p: X \to K \), which is a strong (pro-C)-epimorphism.

**Proposition 3.4.** If \( K \) is epi-dense in \( C \), then pro-\( K \) is epi-reflective in pro-C. Every reflection morphism is a strong (pro-C)-epimorphism. If pro-\( K \) is (strong epi)-reflective in pro-C, then \( K \) is epi-dense in \( C \).

**Proof.** Let \( Y = (Y_j, q_{i,j'}, J) \in \text{pro-} C \) and let \( \lambda_Y: Y \to A(Y) \) be its reflection, as in (2.6). Recall that \( \lambda_Y = (\lambda^j_i)_{i,j \in J} \); since we may assume, without any restriction, that each \( \lambda^j_i \) is a C-epimorphism, it follows that \( \lambda_Y: Y \to A(Y) \) is a strong (pro-C)-epimorphism. The proof of the second part is immediate.

**References**


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