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An Existence Result for a Linear Abstract Stochastic Equation in Hilbert Spaces.

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1. Introduction.

We consider the problem

$$(P) \qquad \begin{cases} du(t) = A(t)\,u(t)\,dt + \sum_{i=1}^{N} B^{(i)}(t)\,dW_i(t) \\ u(0) = u_0 \end{cases}$$

where $W(t) = \big(W_1(t), \dots, W_N(t)\big)$ is a N-dimensional Wiener process in a probability space $(\Omega, \mathcal{E}, P)$ and $\mathcal{A} = \{A(t), t \in [0, T]\}$, $\mathcal{B}^{(i)} = \{B^{(i)}(t), t \in [0, T]\}$ $(i = 1, \dots, N)$ are linear (generally unbounded) operators in a Hilbert space H.

This equation naturally arises in the filtering theory ([1], [7], [8]) and has been studied by variational methods by ([6], [8]) and, using the semigroup theory, by ([3] and [4]).

In some previous papers ([4]) we consider the case where $N = 1$ and $B(t) = F$ (i.e. independent on t) is the infinitesimal generator of a strongly continuous group; under these hypotheses (and some additional else) we are able to solve (P) reducing it to a deterministic problem. In this paper, using a method like in [2], we prove some existence and uniqueness result for (P). We assume, roughly speaking,

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that

$$\Gamma(t) = A(t) + \sum_{i=1}^{N} B^{(i)*}(t) B^{(i)}(t) \leqslant kI \qquad (k \in \mathbb{R})$$

regarding $A(t)$ and $B(t)$ as linear operators in the space H and in an other Hilbert space Y contained in all the domains of $A(t)$.

These hypotheses can be verified also if $\Gamma(t)$ degenerates as we show in Example 1. In Example 2 we give an application to a hyperbolic case.

2. Notations and hypotheses.

Let $\mathcal{A} = \{A(t),\ t \in [0, T]\}$ and $\mathcal{B} = (\mathcal{B}^{(1)}, \dots, \mathcal{B}^{(N)})$ with $\mathcal{B}^{(i)} = \{B^{(i)}(t), t \in [0, T]\}$ families of linear operators in the Hilbert space H.

We say that $(\mathcal{A}, \mathcal{B})$ fulfils the hypotesis $\Gamma(H, \omega)$ if

(1)
- *a*) the resolvent set $\varrho(A(t))$ of $A(t)$ contains $\Sigma_\omega = \{\lambda \in \mathbb{C}: \operatorname{Re} \lambda > \omega\}$, $t \in [0, T]$ and $(\lambda - A(\cdot))^{-1}$ is strongly measurable, $\forall \lambda \in \Sigma_\omega$,
- *b*) for every $t \in [0, T]$ we have $D(A(t)) \subset \bigcap_{i=1}^{N} D(B^{(i)}(t))$,
- *c*) for any $x \in D(A(t))$ and $t \in [0, T]$ we have $2(A(t)x, x)_H + \sum_{i=1}^{N} |B^{(i)}(t)x|_H^2 \leqslant 2\omega |x|_H^2$.

It is well-known that if (1) holds then it is

$$|(n - A(t))^{-1}|_{\mathcal{L}(H)} \leqslant \frac{1}{n - \omega} \qquad \forall n > \omega. \tag{2}$$

Let us define

$$J_n(t) = n(n - A(t))^{-1}, \quad A_n(t) = A(t) J_n(t),$$

$$B_n^{(i)}(t) = F^{(i)}(t) J_n(t), \qquad B(t) = (B^{(1)}(t), B^{(2)}(t), \dots, B^{(N)}(t)).$$

The proof of the following lemma is straightforward.

LEMMA 1. *Assume that* $(\mathcal{A}, \mathcal{B})$ *fulfils* $\Gamma(H, \omega)$ *and moreover*

(3) $B^{(i)}(t)$ *is closed in* H *and there exist* $k_1, k_2 > 0$ *such that* $|B^{(i)}(t)\cdot x|_H^2 \leqslant k_1 |A(t)x|_H^2 + k_2 |x|_H^2$

then if $n > 2\omega$ *and* $x \in D(A(t))$ *we have:*

$$|J_n(t)|_H \leqslant 2 , \tag{4}$$

$$|A_n(t)x|_H^2 \leqslant 4|A(t)x|_H^2 , \tag{5}$$

$$|B_n^{(i)}(t)x|_H^2 \leqslant 4\big(k_1|A(t)x|_H^2 + k_2|x|_H^2\big) . \tag{6}$$

Consider now the approximating problem:

$$(P_n) \qquad \begin{cases} du_n(t) = A_n(t)\,u_n(t)\,dt + \sum_{i=1}^{N} B_n^{(i)}(t)\,u_n(t)\,dW_i(t) \\ u_n(0) = 0 \,. \end{cases}$$

LEMMA 2. *Assume that* $(\mathcal{A}, \mathcal{B})$ *fulfils* $\Gamma(H, \omega)$ *and that* $u_0 \in L^2(\Omega, \mathcal{F}_0^W P; H)$ (1). Then the problem (P_n) has a unique solution and the following estimate holds

$$E\big(|u_n(t)|_H^2\big) \leqslant \exp[4\omega t]\, E\big(|u_0|_H^2\big) \qquad \forall n > 2\omega . \tag{7}$$

PROOF. The existence and uniqueness follow from general existence theorems (see for example [1]). Moreover by the Itô formula it follows

$$|u_n(t)|_H^2 = |u_0|_H^2 + \int_0^t \Big[2\big(A_n(s)\,u_n(s), u_n(s)\big)_H + \sum_{i=1}^{N} |B_n^{(i)}(s)\,u_n(s)|_H^2\Big] ds + 2\sum_{i=1}^{N} \int_0^t \big(B_n^{(i)}(s)\,u_n(s), u_n(s)\big)_H dW_i(s) . \tag{8}$$

(1) $\mathcal{F}_t^W$ is the σ-algebra generated by $\{W(s),\ 0 \leqslant s \leqslant t\}$. $L^2(\Omega, \mathcal{F}_t^W, P; H)$ is the space of F_t^W-measurable functions $X\colon \Omega \to H$ that are square-integrable with respect to P, endowed with the usual norm.

On the other hand we have, if $n > 2\omega$

$$(9)\qquad 2(A_n(t)x, x)_H + \sum_{i=1}^{N} |B_n^{(i)}(t)x|_H^2 = 2(A(t)J_n(t)x, x)_H + \\ + \sum_{i=1}^{N} |B^{(i)}(t)J_n(t)x|_H^2 = 2(A(t)J_n(t)x, J_n(t)x)_H + \\ + \sum_{i=1}^{N} |B^{(i)}(t)J_n(t)x|_H^2 - \frac{2}{n}|A_n(t)x|_H^2 \leqslant 2\omega|J_n(t)x|_H^2 \leqslant 4\omega|x|_H^2 .$$

Using this estimate, we have from (8) that

$$E(|u_n(t)|_H^2) \leqslant E(|u_0|_H^2) + 4\omega\int_0^t E(|u_n(s)|_H^2)\, ds$$

and the thesis follows by the Gronwall's lemma.

3. Existence.

THEOREM 1. *Assume that* $(\mathcal{A}, \mathcal{B})$ *fulfils* $\Gamma(H, \omega)$ *and* (3). *Assume moreover that*:

$$(10)\qquad \begin{cases} a) \text{ there exists a Hilbert space } Y \text{ with} \\ \qquad Y \subset \bigcap_{t\in[0,T]} D(A(t)) \\ \text{and } \eta \in \mathbb{R} \text{ such that } (\mathcal{A}^Y, \mathcal{B}^Y)\ (^2) \text{ fulfils } \Gamma(Y, \eta)\,, \\ b) \text{ there exists } M > 0 \text{ such that} \\ \qquad |A(t)x|_H \leqslant M|y|_Y \qquad \forall x \in Y\,. \end{cases}$$

Then if $u_0 \in \mathcal{M}(\Omega, \mathcal{F}_0^W, P; Y)$ (3) *there exists a unique (classical) solu-*

(2) If L is a linear operator in H we put

$$\begin{cases} D(L^Y) = \{x \in D(L) \subset Y : Lx \in Y\} \\ L^Y y = Lx \qquad \forall x \in D(L^Y) \end{cases}$$

Moreover by $\mathcal{A}^Y$ we denote the family $\{A^Y(t)\}$.

(3) $\mathcal{M}(\Omega, \mathcal{E}, P; Y)$ is the space of $\mathcal{E}$-measurable function $X: \Omega \to Y$.

tion u to the equation:

$$(11)\qquad u(t) = u_0 + \int_0^t A(s)\,u(s)\,ds + \sum_{i=1}^{N} \int_0^t B^{(i)}(s)\,u(s)\,dW_i(s)\,.$$

PROOF. One needs only to consider the case where $u_0 \in L^2(\Omega, \mathcal{F}_0^W, P; Y)$: the general case is proved by standard arguments (cfr. [5]).

1° STEP (estimates). From Lemmas 1 and 2 we get

$$(12)\qquad E\big(|u_n(t)|_H^2\big) \leqslant E\big(|u_0|_H^2\big) \exp[4\omega T] = c_1\,,$$

$$(13)\qquad E\big(|u_n(t)|_Y^2\big) \leqslant E\big(|u_0|_Y^2\big) \exp[4\eta T] = c_2\,,$$

Besides, using (10 *b*), we have

$$(14)\qquad E\big(|A_n(t)u_n(t)|_H^2\big) \leqslant 4E\big(|A(t)u_n(t)|_H^2\big) \leqslant \leqslant 4M^2 E\big(|u_n(t)|_Y^2\big) \leqslant 4M^2 c_2 = c_3\,,$$

$$(15)\qquad E(B_n^{(i)}|(t)u_n(t)|_H^2) \leqslant 4E(k_1|A(t)u_n(t)|_H^2 + k_2|u_n(t)|)_H^2 \leqslant \leqslant 4k_1 M c_2 + 4k_2 c_1 = c_4\,.$$

2° STEP. It is

$$(16)\qquad E\big(|u_n(t) - u_m(t)|_H^2\big) \xrightarrow{n,m\to\infty} 0 \quad \text{uniformly in } t \in [0, T]\,.$$

By Itô formula we have:

$$(17)\qquad d|u_n(t) - u_m(t)|_H^2 = \Big\{2\big(A_n(t)u_n(t) - A_m(t)u_m(t),\, u_n(t) - u_m(t)\big)_H + + \sum_{i=1}^{N} |B_n^{(i)}(t)u_n(t) - B_m^{(i)}(t)u_m(t)|_H^2\Big\}\,\mathrm{d}t + + 2\sum_{i=1}^{N} \big(B_n^{(i)}(t)u_n(t) - B_m^{(i)}(t)u_m(t),\, u_n(t) - u_m(t)\big)_H\,\mathrm{d}W_i(t)\,.$$

On the other hand it is

$$2\big(A_n(t)\,u_n(t)-A_m(t)\,u_m(t),u_n(t)-u_m(t)\big)_H=$$
$$=2\big(A(t)\,J_n(t)\,u_n(t)-A(t)\,J_m(t)\,u_m(t),\;u_n(t)-u_m(t)\big)_H=$$
$$=2\big(A(t)\,J_n(t)\,u_n(t)-A(t)\,J_m(t)\,u_m(t),\;J_n(t)\,u_n(t)-J_m(t)\,u_m(t)\big)_H+$$
$$+\,2A_n(t)\,u_n(t)-A_m(t)\,u_m(t),\frac{1}{m}\,A_m(t)\,u_m(t)-\frac{1}{n}\,A_n(t)\,u_n(t)\big)_H\,.$$

Hence by estimate (1) c) we have:

$$2\big(A_n(t)\,u_n(t)-A_m(t)\,u_m(t),\;u_n(t)-u_m(t)\big)_H+$$
$$+\sum_{i=1}^{N}|B_n^{(1)}(t)\,u_n(t)-B_m^{(1)}(t)\,u_m(t)|=$$
$$=2\big(A(t)\big(J_n(t)\,u_n(t)-J_m(t)\,u_m(t)\big),\;J_n(t)\,u_n(t)-J_m(t)\,u_m(t)\big)_H+$$
$$+\sum_{i=1}^{N}|B^{(i)}(t)\big(J_n(t)\,u_n(t)-J_m(t)\,u_m(t)\big)|_H^2+$$
$$+\,2\left(A_n(t)\,u_n(t)-A_m(t)\,u_m(t),\frac{1}{m}\,A_m(t)\,u_m(t)-\frac{1}{n}\,A_n(t)\,u_n(t)\right)_H\leqslant$$
$$\leqslant 2\omega|J_n(t)\,u_n(t)-J_m(t)\,u_m(t)|_H^2+$$
$$+\,4\left(\frac{1}{n}+\frac{1}{m}\right)\big(|A_n(t)u_n(t)|_H^2+|A_m(t)\,u_m(t)|_H^2\big)\leqslant$$
$$\leqslant 2\omega|u_n(t)-u_m(t)|_H^2+4\left(\frac{1}{n}+\frac{1}{m}\right)(1+2\omega)\big(|A_n(t)\,u_n(t)|_H^2+|A_m(t)\,u_m(t)|_H^2\big)\,.$$

Using (12), (13), (14), (15), (17) we get

$$\text{(18)}\qquad E\big(|u_n(t)-u_m(t)|_H^2\big)\leqslant 2\omega\int_0^t E\big(|u_n(s)-u_m(s)|_H^2\big)\,ds+$$
$$+\,4(1+2\omega)\left(\frac{1}{n}+\frac{1}{m}\right)\int_0^t\Big(E\big(|A_n(s)\,u_n(s)|_H^2\big)+E\big(|A_m(s)\,u_m(s)|_H^2\big)\Big)\,ds\leqslant$$
$$\leqslant c_5(1+2\omega)\left(\frac{1}{n}+\frac{1}{m}\right)+2\omega\int_0^t E\big(|u_n(s)-u_m(s)|_H^2\big)\,ds\,.$$

Then by Gronwall lemma we obtain (16).

3° STEP. Put $u(t) = \lim_{n\to\infty} u_n(t)$, here the limit is in $L^2(\Omega, \mathcal{F}_t^W, P, H)$ uniformly in $t \in [0, T]$.

Now owing to (14) it is $\forall \tau \in [0, T]$

$$(19)\qquad \begin{cases} u(t) \in D_{A(t)} \quad \text{a.e. in } [0, \tau], \text{ w.p. 1} \\ A_n(t)\, u_n(t) \to A(t)\, u(t) \quad \text{weakly in} \quad L^2([0, \tau], L^2(\Omega, \mathcal{F}_\tau^W, P, H)) \end{cases}$$

In fact the operator:

$$\mathcal{A} \equiv \begin{cases} D_{\mathcal{A}} = \Big\{u \in L^2([0, \tau]; L^2(\Omega, \mathcal{F}_\tau^W, P, H)) : u(t) \in D_{A(t)} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{w.p. 1 a.e. in } [0, \tau] \\ A(t)\, u(t) \in L^2([0, \tau]; L^2(\Omega, \mathcal{F}_\tau^W, P, M))\Big\} \\ (\mathcal{A}u)(t) = A(t)\, u(t) \end{cases}$$

is the infinitesimal generator of a linear contraction semigroup in $L^2([0, \tau]; L^2(\Omega, \mathcal{F}_\tau^W, P, H))$.

As far as the sequence $B_n^{(i)}(t) u_n(t)$ is concerned we remark that

$$(20)\qquad B_n^{(i)}(t) u_n(t) = B^{(i)}(t) A^{-1}(t) A_n(t) u_n(t)$$

and

$$(21)\qquad |B^{(i)}(t) A^{-1}(t)|_{\mathcal{L}(H)} \leqslant \text{const}\,.$$

then also $\forall \tau \in [0, T]$

$$B_n^{(i)}(t) u_n(t) \to B(t) u(t) \qquad \text{weakly in } L^2([0, \tau]; L^2(\Omega, \mathcal{F}_\tau^W, P, H))\,.$$

Finally $\forall \tau \in [0, T]$

$$(22)\qquad \int_0^t A_n(s) u_n(s)\, ds \to \int_0^t A(s) u(s)\, ds \qquad \text{weakly in } L^2([0, \tau]; L^2(\Omega, \mathcal{F}_\tau^W, P, H))\,,$$

$$(23)\qquad \int_0^t B_n^{(i)}(s) u_n(s)\, dW_s \to \int_0^t B^{(i)}(s) u(s)\, dW_s \qquad \text{weakly in } L^2([0, \tau]; L^2(\Omega, \mathcal{F}_\tau^W, P, H))\,.$$

This latter follows from the fact that the mapping

$$g(t) \to \int_0^t g(s)\, dW_s$$

is continuous from $L^2([0, \tau]; L^2(\Omega, \mathcal{F}_\tau^W, P, H))$ to itself.
(22) and (23) imply (11).

4° Step (uniqueness). Let v another solution to (P), put $z = u - v$; then by Itô formula we have

$$E(|z(t)|_H^2) = E\int_0^t \Big(2\big(A(t)z(t), z(t)\big)_H + \sum_{i=1}^N |B^{(i)}(t)z(t)|_H^2\Big) dt \leqslant 2\omega E\int_0^t |z(s)|_H^2\, ds$$

and the thesis follows from Gronwall's lemma.

Remark. Actually the solution $u(t)$ is a Markov process with respect to the family $\{\mathcal{F}_t^W, t \in [0, T]\}$; in fact

$$u(t) = \lim_{n\to\infty} u_s(t) \qquad \text{in } L^2(\Omega, \mathcal{E}, P; H)$$

and u_n is the solution to a linear problem with bounded coefficients (cfr. for example [5]).

Examples.

Example 1. Consider the problem

$$\text{(19)} \qquad \begin{cases} du = \frac{1}{2}(u_{xx} + u_{yy})\, dt + u_y\, dW_1(t) + hu\, dW_2(t) \\ u(0) = u_0 \end{cases}$$

We put

$$H = L^2(\mathbb{R}^2)\,, \qquad Y = H^2(\mathbb{R}^2)\,,$$

$$Au = \tfrac{1}{2}(u_{xx} + u_{yy})\,, \qquad B^{(1)}u = u_y\,, \qquad B^{(2)}u = hu\,,$$

where for simplicity we assume $h \in C_0^\infty(\mathbb{R}^2)$. We have

$$(A + \tfrac{1}{2}B^{(1)}B^{(1)} + \tfrac{1}{2}B^{(2)}B^{(2)})u = \tfrac{1}{2}(u_{xx} + h^2 u) . \tag{20}$$

It is easy to verify the hypotheses of Theorem 1.

EXAMPLE 2. The previous theory can be extended to complex Hilbert spaces, changing only condition (1) c) as it follows:

$$(1')\ c) \qquad 2\,\mathrm{Re}\,\big(A(t)x, x\big)_H + \sum |B^{(i)}(t)x|_H^2 \leqslant 2\omega |x|_H^2 \qquad x \in D\big(A(t)\big) .$$

Then we can give the following example:

$$A(t) = \begin{cases} D\big(A(t)\big) = H^2(\mathbb{R}) \\ A(t)\,u\big)(x) = \Big(\big(\tfrac{1}{2} + ia(t, x)\big)\,u_x\Big)_x \end{cases}$$

$$B = \begin{cases} D(B) = H^1(\mathbb{R}) \\ Bu(x) = u_x \end{cases}$$

with $a(\cdot, \cdot) \in C_b(0, T \times \mathbb{R})$ and $a(t, x) \geqslant \varepsilon > 0$.

In such a case (1′) c) is fulfilled with $\omega = 0$ and all the hypotheses of Theorem 1 with $Y = H^2(\mathbb{R})$ are fulfilled.

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