RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

JINDŘICH BEČVÁŘ

A generalization of a theorem of F. Richman and C. P. Walker

Rendiconti del Seminario Matematico della Università di Padova, tome 66 (1982), p. 43-55

<http://www.numdam.org/item?id=RSMUP_1982__66__43_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 66 (1982)

A generalization of a Theorem of F. Richman and C. P. Walker.

JINDŘICH BEČVÁŘ (*)

All groups in this paper are abelian, concerning the terminology and notation we refer to [1]. If G is a group then G_t and G_p are the torsion part of G and the p-component of G_t respectively. Let α be an ordinal. Since $(p^\alpha G)_p = p^\alpha(G_p)$, we shall write only $p^\alpha G_p$. Further, it is natural to use the symbol $p^\alpha G[p]$. The cardinality of α will be denoted by $|\alpha|$. Let G be a group and p a prime. A subgroup H of G is said to be p-isotype (isotype) in G if $p^\alpha H = H \cap p^\alpha G$ for each ordinal α (for each ordinal α and each prime p). If $K \subset N$ are subgroups of a group G then every subgroup A of G, which is maximal with respect to the property $A \cap N = K$, is said to be $N - K$-high in G (following F. V. Krivonos [3]).

Let G be a p-group. If A is a neat subgroup of G^1 then there is a pure subgroup P of G such that $P \cap G^1 = A$ (R. W. Mitchell [4], resp. A. R. Mitchell-R. W. Mitchell [5]). The question which subgroups of G^1 are the intersections of G^1 with a pure subgroup of G was settled by F. Richman and C. P. Walker [6]. Let G be an arbitrary group. If K is a subgroup of $p^\beta G$ and P a K-high subgroup of G then $p^\alpha P = P \cap p^\alpha G$ for each ordinal $\alpha \leqslant \beta + 1$; every $p^\beta G$-high subgroup of G is p-isotype in G (J. M. Irwin - E. A. Walker [2]).

Let G be a group and A a subgroup of $p^\beta G$. The purpose of this paper is to give the necessary and sufficient conditions to the following two statements:

(*) Indirizzo dell'A.: Matematicko-fyzikálnì fakulta, Sokolovská 83, 186 00 Praha 8 (Czechoslovakia).

a) For each $p^\beta G - A$-high subgroup P of G, $p^\beta P = P \cap p^\alpha G$ for each ordinal $\alpha \leqslant \beta + 1$ (theorem 1).

b) There is a $p^\beta G - A$-high subgroup P of G such that $p^\alpha P = P \cap p^\alpha G$ for each ordinal $\alpha \leqslant \beta$ (theorem 2).

Note that theorem 1 generalizes the recalled results from [4] and [2], theorem 2 generalizes the main result from [6]. Moreover, theorems 1 and 2 together give an interesting look at these « purification » problems.

1. THEOREM 1. *Let G be a group, p a prime, β an ordinal and A a subgroup of $p^\beta G$. The following are equivalent:*

(i) *A is p-neat in $p^\beta G$.*

(ii) *If P is a $p^\beta G - A$-high subgroup of G then $p^\alpha P = P \cap p^\alpha G$ for each ordinal $\alpha \leqslant \beta + 1$.*

(iii) *There is a subgroup P of G such that $A = P \cap p^\beta G$ and $p^{\beta+1} P = P \cap p^{\beta+1} G$.*

PROOF. Suppose (i). Let P be a $p^\beta G - A$-high subgroup of G. We prove that $p^\alpha P = P \cap p^\alpha G$ for each ordinal $\alpha \leqslant \beta + 1$. It is sufficient to show that if this equality holds for α ($\alpha \leqslant \beta$) then it holds for $\alpha + 1$. Let $x \in P \cap p^{\alpha+1} G$, i.e. $x = pg$, where $g \in p^\alpha G$; by assumption, $x \in P \cap p^\alpha G = p^\alpha P$. If $g \in P$ then $g \in P \cap p^\alpha G = p^\alpha P$ and $x = pg \in p^{\alpha+1} P$. If $g \notin P$ then there are an element $b \in P$ and an integer r such that $b + rg \in p^\beta G \setminus A$. Obviously $b \in P \cap p^\alpha G = p^\alpha P$ and $(p, r) = 1$. Now, $pb + prg \in p^\beta G \cap P = A$ and further $pb + prg \in p^{\beta+1} G \cap A = pA$. Hence there is an element $a \in A$ such that $pb + prg = pa$; obviously $a \in P \cap p^\alpha G = p^\alpha P$. Since $(p, r) = 1$, there are integers u, v such that $1 = up + vr$. Consequently $g = upg + vrg = ux + vrg$ and $x = pg = upx + vprg = upx + vpa - vpb = p(ux + va - vb)$, where $ux + va - vb \in p^\alpha P$. Hence $x \in p^{\alpha+1} P$.

Suppose (iii). Obviously $p^\beta P \subset P \cap p^\beta G = A$ and hence $p^{\beta+1} P \subset pA$. Now,

$$pA \subset A \cap p^{\beta+1} G = P \cap p^\beta G \cap p^{\beta+1} G = p^{\beta+1} P \subset pA .$$

COROLLARY 1 (J. M. Irwin - E. A. Walker [2]). *Let G be a group, p a prime, β an ordinal and K a subgroup of $p^\beta G$. If P is a K-high subgroup of G then $p^\alpha P = P \cap p^\alpha G$ for each ordinal $\alpha \leqslant \beta + 1$.*

PROOF. If P is a K-high subgroup of G then $A = P \cap p^{\beta}G$ is a K-high subgroup of $p^{\beta}G$ and hence A is neat in $p^{\beta}G$. Since P is $p^{\beta}G - A$-high in G, the desired result follows from theorem 1.

The following corollaries 2, 3, 4 and proposition 1 generalize some results from [2].

COROLLARY 2. Let G be a group, p a prime, $\beta, \gamma > 0$ ordinals and A a subgroup of $p^{\beta}G$. Let P be a $p^{\beta}G - A$-high subgroup of G.

(i) If $p^{\alpha}A = A \cap p^{\alpha}(p^{\beta}G)$ for each ordinal $\alpha \leqslant \gamma$ then $p^{\alpha}P = P \cap p^{\alpha}G$ for each ordinal $\alpha \leqslant \beta + \gamma$.

(ii) If A is p-isotype in $p^{\beta}G$ then P is p-isotype in G.

(iii) If $A \cap p^{\beta+1}G = 0$ then P is p-isotype in G.

(iv) If A is p-neat in $p^{\beta}G$ and $A \cap p^{\beta+2}G = 0$ then P is p-isotype in G.

PROPOSITION 1. Let G be a group, $A \subset N$ subgroups of G and q a prime. If A is q-neat in N and $N_q = 0$ then each $N - A$-high subgroup of G is q-isotype in G.

PROOF. Let P be a $N - A$-high subgroup of G. Suppose $g \in G \setminus P$ and $qg \in P$. There are an element $b \in P$ and an integer r such that $rg + b \in N \setminus A$; obviously $(r, q) = 1$. Now, $qrg + qb \in P \cap N = A$, there is an element $a \in A$ such that $qrg + qb = qa$ and hence $rg = a - b$. There are integers u, v such that $uq + vr = 1$ and consequently $g = uqg + vrg = uqg + va - vb \in P$, a contradiction. Hence $(G/P)_q = 0$ and P is q-isotype in G (see lemma 103.1 [1]).

COROLLARY 3. Let G be a group, p a prime, β an ordinal and A a subgroup of $p^{\beta}G$. If A is neat in $p^{\beta}G$, $A \cap p^{\beta+2}G = 0$ and $(p^{\beta}G)_q = 0$ for each prime $q \neq p$ then each $p^{\beta}G - A$-high subgroup of G is isotype in G.

COROLLARY 4. Let H be a subgroup of a group G, p be a prime and β an ordinal. Suppose that for each nonzero element $h \in H$, $h_p(h) \leqslant \beta + 1$. Then

(i) H is contained in a p-isotype subgroup A of G such that for each $a \in A \setminus H$, $h_p(a) \leqslant \beta$.

(ii) If $H \cap p^{\beta}G$ is p-neat in $p^{\beta}G$ then H is contained in a p-isotype subgroup B of G such that for each $a \in B \setminus H$, $h_p(a) < \beta$.

(iii) If $H \cap p^\beta G$ is neat in $p^\beta G$ and $(p^\beta G)_q = 0$ for each prime $q \neq p$ then H is contained in an isotype subgroup B of G such that for each $a \in B \setminus H$, $h_p(a) < \beta$.

2. DEFINITION. Let G be a group, p a prime and β a nonzero ordinal. The minimum of cardinals

$$r(p^\alpha G[p]/p^\beta G[p]) , \qquad \alpha < \beta ,$$

will be called the (p, β)-rank of G and denoted by $r_{(p,\beta)}(G)$.

PROPOSITION 2. Let G be a group, p a prime and β a nonzero ordinal. If H is a $p^\beta G$-high subgroup of G then

$$r_{(p,\beta)}(G) = \min_{\alpha<\beta} r(p^\alpha H_p) .$$

PROOF. Let H be a $p^\beta G$-high subgroup of G and α an ordinal, $\alpha < \beta$. Since the subgroup $p^\alpha G \cap H$ is $p^\beta G$-high in $p^\alpha G$ and $p^\alpha G \cap H = p^\alpha H$ (see corollary 1), $p^\alpha G[p] = p^\beta G[p] \oplus p^\alpha H[p]$. Consequently

$$r(p^\alpha H_p) = r(p^\alpha H[p]) = r(p^\alpha G[p]/p^\beta G[p]) .$$

REMARK 1. Let G be a group and H a $p^\beta G$-high subgroup of G. If $\beta = \delta + 1$ then

$$r_{(p,\beta)}(G) = r(p^\delta G[p]/p^\beta G[p]) = f_\delta(G)$$

is the δ-th Ulm-Kaplansky invariant of G_p and

$$r_{(p,\beta)}(G) = r(p^\delta H_p) = r(p^\delta H[p])$$

by proposition 2. If β is a limit ordinal and $r_{(p,\beta)}(G) > 0$ then for each ordinal $\alpha < \beta$ there is a natural number k such that $p^\alpha H[p] \neq p^{\alpha+k} H[p]$. For, otherwise $p^\alpha H_p$ is divisible, $p^\alpha H_p \subset p^\beta G \cap H = 0$ and proposition 2 implies a contradiction. Hence $r_{(p,\beta)}(G)$ is infinite. Let γ be an ordinal such that $r_{(p,\beta)}(G) = r(p^\gamma H[p])$ and let ε be the least ordinal such that β is cofinal with ε (i.e. $\varepsilon = \mathrm{cof}(\beta)$); hence ε is a cardinal.

According to the precedent consideration,

$$r_{(p,\beta)}(G) = r(p^\gamma H[p]) = |p^\gamma H[p]| \geqslant |\{\alpha;\ \gamma \leqslant \alpha < \beta\}| \geqslant \varepsilon .$$

REMARK 2. If G is a p-group and H a G^1-high subgroup of G then by proposition 2,

$$r_{(p,\omega)}(G) = \min_{n<\omega} r(p^n H)$$

is the final rank of H.

LEMMA 1. *Let G be a group, p a prime, β a nonzero ordinal and A a subgroup of $p^\beta G$. If there is a subgroup P of G such that $P \cap p^\beta G = p^\beta P = A$ then*

$$r(p^{\beta+1}G \cap A/pA) \leqslant r_{(p,\beta)}(G) .$$

PROOF. Let $\{a_i + pA;\ i \in I\}$ be a basis of $p^{\beta+1}G \cap A/pA$ and H be a $p^\beta G$-high subgroup of G. Suppose $|I| > r(p^\alpha H_p)$ for some $\alpha < \beta$. For each index $i \in I$ there are elements $g_i \in p^\beta G$ and $x_i \in p^\alpha P$ such that $a_i = pg_i = px_i$. Since

$$x_i - g_i \in G[p] = p^\beta G[p] \oplus H[p] ,$$

$x_i - g_i = g_i' + h_i$, where $g_i' \in p^\beta G[p]$ and $h_i \in H[p]$ for each $i \in I$. Further

$$h_i = x_i - g_i - g_i' \in H_p \cap p^\alpha G = p^\alpha H_p$$

by corollary 1. Since the set $\{h_i;\ i \in I\}$ is linearly dependent, there are a finite subset J of I and integers r_i, $i \in J$, such that

$$\sum_{i\in J} r_i h_i = 0$$

and $r_i h_i \neq 0$ for each $i \in J$. Hence

$$\sum_{i\in J} r_i x_i \in p^\beta G \cap P = A ,$$

$$\sum_{i\in J} r_i a_i \in pA ,$$

and

$$\sum_{i\in J} r_i(a_i + pA) = pA \ .$$

Since the set $\{a_i + pA;\ i \in I\}$ is linearly independent, $r_i a_i \in pA$ and hence $p|r_i$ for each index $i \in J$, a contradiction. Hence $|I| \leqslant r(p^\alpha H_p)$ for every ordinal $\alpha < \beta$ and proposition 2 implies the desired result.

LEMMA 2. Let G be a group, p a prime, β a nonzero ordinal and B a subgroup of $p^\beta G$. If B is a direct sum of cyclic groups and $r(B) \leqslant r_{(p,\beta)}(G)$ then there is a subgroup X of G such that

(i) $X \cap p^\beta G = B$,

(ii) $\forall b \in B,\ \forall \alpha < \beta,\ \exists x \in X \cap p^\alpha G,\ b = px.$

PROOF. Suppose $B \neq 0$, otherwise the assertion is trivial. Let H be a $p^\beta G$-high subgroup of G.

Case 1. $\beta = \delta + 1$.

By remark 1, $r_{(p,\beta)}(G) = r(p^\delta H[p])$. Write

$$B = \bigoplus_{\gamma<\sigma} \langle a_\gamma \rangle \qquad \text{and} \qquad p^\delta H[p] = \bigoplus_{\gamma<\sigma'} \langle h_\gamma \rangle \ ,$$

where σ and σ' are ordinals such that $\sigma \leqslant \sigma'$. For each ordinal $\gamma < \sigma$ let $H_\gamma = \bigoplus_{\gamma'<\gamma} \langle h_{\gamma'} \rangle$.

By transfinite induction we shall define an ascending chain of subgroups Z_γ, $\gamma \leqslant \sigma$, with the following properties:

$$(1,\gamma) \qquad Z_\gamma \cap p^\beta G = B \ ,$$

$$(2,\gamma) \qquad Z_\gamma \subset p^\delta G \ ,$$

$$(3,\gamma) \qquad \forall \gamma' < \gamma \ , \quad \exists c \in Z_\gamma \ , \quad a_{\gamma'} = pc \ ,$$

$$(4,\gamma) \qquad (p^\beta G + Z_\gamma) \cap H[p] \subset H_\gamma \ .$$

a) Put $Z_0 = B$; the subgroup Z_0 has obviously all the properties (1,0)-(4,0).

b) Suppose that Z_γ has been defined $(0 \leqslant \gamma < \sigma)$ and define $Z_{\gamma+1}$. If there is no element $c \in (p^\beta G + Z_\gamma)$ with $a_\gamma = pc$ then let $c_\gamma \in p^\delta G$

be an arbitrary element such that $a_\gamma = pc_\gamma$. If $c \in (p^\beta G + Z_\gamma) \subset p^\sigma G$ is an element such that $a_\gamma = pc$ then let $c_\gamma = c + h_\gamma$; $c_\gamma \notin p^\beta G + Z_\gamma$ by (4,γ). Define $Z_{\gamma+1} = \langle Z_\gamma, c_\gamma \rangle$ and verify the properties (1,γ+1)-(4,γ+1). Let $z + rc_\gamma \in p^\beta G$, where $z \in Z_\gamma$ and r is an integer. If $p|r$ then $z + rc_\gamma \in Z_\gamma \cap p^\beta G = B$. If $(p, r) = 1$ then it is easy to see that $c_\gamma \in p^\beta G + Z_\gamma$, which contradicts with the definition of the element c_γ. Hence (1,γ+1) holds; (2,γ+1) and (3,γ+1) are trivial. Let $g + z + rc_\gamma = h \in H[p]$, where $g \in p^\beta G$, $z \in Z_\gamma$ and r is an integer. If $p|r$ then $h \in (p^\beta G + Z_\gamma) \cap H[p] \subset H_\gamma \subset H_{\gamma+1}$ by (4,γ). If $(p, r) = 1$ then there are integers u, v such that $1 = pu + rv$. Since

$$ra_\gamma = prc_\gamma = p(h - g - z) = p(-g - z)\,,$$

$$a_\gamma = pua_\gamma + rva_\gamma = p(ua_\gamma - vg - vz)\,,$$

where $ua_\gamma - vg - vz \in p^\beta G + Z_\gamma$, and according to the definition of c_γ, $c_\gamma = c + h_\gamma$, where $c \in p^\beta G + Z_\gamma$ and $a_\gamma = pc$. Hence

$$g + z + rc + rh_\gamma = h\,,$$

$$h - rh_\gamma = g + z + rc \in (p^\beta G + Z_\gamma) \cap H[p] \subset H_\gamma$$

and consequently $h \in H_{\gamma+1}$, i.e. (4,γ+1) holds.

c) Suppose that γ is a limit ordinal. If for each ordinal $\gamma' < \gamma$ a subgroup $Z_{\gamma'}$ with the properties (1,γ')-(4,γ') has been defined then the subgroup $Z_\gamma = \bigcup_{\gamma'<\gamma} Z_{\gamma'}$ has obviously the properties (1,γ)-(4,γ).

Finally put $X = Z_\sigma$; the properties (i), (ii) arise from (1,σ)-(3,σ).

Case 2. β is a limit ordinal.

Let $\varepsilon = \operatorname{cof}(\beta)$. Suppose that β is a limit of an ascending sequence of ordinals α_δ, $\delta < \varepsilon$. Let $B = \bigoplus_{\gamma<\sigma} \langle a_\gamma \rangle$, where $a_\gamma \neq 0$ for each $\gamma < \sigma$ and σ is a cardinal. By assumption, $r_{(p,\beta)}(G) \geqslant r(B) = \sigma$.

Case 2.1. $\sigma > \varepsilon$.

We use the transfinite induction to define an ascending chain of subgroups Z_γ, $\gamma \leqslant \sigma$, with the following properties:

(5,γ) $$Z_\gamma \cap p^\beta G = B\,,$$

(6,γ) $$\forall \gamma' < \gamma\,, \quad \forall \alpha < \beta\,, \quad \exists c \in Z_\gamma \cap p^\alpha G\,, \quad a_{\gamma'} = pc\,,$$

(7,γ) $$r\big((p^\beta G + Z_\gamma) \cap H[p]\big) \leqslant |\gamma| \cdot \varepsilon\,.$$

a) Put $Z_0 = B$.

b) Suppose that Z_γ has been defined $(0 \leqslant \gamma < \sigma)$ and define $Z_{\gamma+1}$. By transfinite induction we shall construct an ascending chain of subgroups Y_δ, $\delta \leqslant \varepsilon$, with the following properties:

$$(8,\delta) \qquad Y_\delta \cap p^\beta G = B\,,$$

$$(9,\delta) \qquad \forall \delta' < \delta\,, \qquad \exists c \in Y_\delta \cap p^{\alpha_{\delta'}} G\,, \qquad a_\gamma = pc\,,$$

$$(10,\delta) \qquad r\big((p^\beta G + Y_\delta) \cap H[p]\big) \leqslant |\gamma| \cdot \varepsilon + |\delta|\,.$$

b$_1$) Put $Y_0 = Z_\gamma$.

b$_2$) Suppose that Y_δ has been defined $(0 \leqslant \delta < \varepsilon)$ and define $Y_{\delta+1}$. If there is no element $c \in (p^\beta G + Y_\delta) \cap p^{\alpha_\delta} G$ with $a_\gamma = pc$ then let $c_\delta \in p^{\alpha_\delta} G$ be an arbitrary element such that $a_\gamma = pc_\delta$. If $c \in (p^\beta G + Y_\delta) \cap p^{\alpha_\delta} G$ is an element such that $a_\gamma = pc$ then let $c_\delta = c + h$, where $h \in p^{\alpha_\delta} H[p] \setminus (p^\beta G + Y_\delta)$; such element h exists, since

$$r(p^{\alpha_\delta} H[p]) \geqslant r_{(p,\beta)}(G) \geqslant \sigma > |\gamma| \cdot \varepsilon + |\delta| \geqslant r\big((p^\beta G + Y_\delta) \cap H[p]\big)\,.$$

Define $Y_{\delta+1} = \langle Y_\delta, c_\delta \rangle$ and verify the properties $(8,\delta+1)$-$(10,\delta+1)$. Let $y + rc_\delta \in p^\beta G$, where $y \in Y_\delta$ and r is an integer. If $p | r$ then $y + rc_\delta \in Y_\delta \cap p^\beta G = B$. If $(p, r) = 1$ then it is easy to see that $c_\delta \in p^\beta G + Y_\delta$ which contradicts with the choice of the element c_δ. Hence $(8,\delta+1)$ holds; $(9,\delta+1)$ is trivial. Further write

$$(p^\beta G + Y_{\delta+1}) \cap H[p] = (p^\beta G + Y_\delta) \cap H[p] \oplus R$$

and suppose $R = \langle h_1 \rangle \oplus R'$, $h_1 \neq 0$ and $h_2 \in R'$. Write

$$h_1 = g_1 + y_1 + r_1 c_\delta\,, \qquad h_2 = g_2 + y_2 + r_2 c_\delta\,,$$

where $g_1, g_2 \in p^\beta G$, $y_1, y_2 \in Y_\delta$ and r_1, r_2 are integers. Obviously $(p, r_1) = 1$, there are integers u, v such that

$$r_2 c_\delta = upr_2 c_\delta + vr_1 r_2 c_\delta = ur_2 a_\gamma + vr_2(h_1 - g_1 - y_1)$$

and hence

$$h_2 = g_2 + y_2 + ur_2 a_\gamma + vr_2 h_1 - vr_2 g_1 - vr_2 y_1 \in$$
$$\in (p^\beta G + Y_\delta) \cap H[p] \oplus \langle h_1 \rangle .$$

Thus $r(R) \leqslant 1$ and $(10,\delta+1)$ holds.

b_3) Let δ be a limit ordinal. If for each ordinal $\delta' < \delta$ a subgroup $Y_{\delta'}$ with the properties $(8,\delta')$-$(10,\delta')$ has been defined then it is not difficult to see that the subgroup $Y_\delta = \bigcup_{\delta'<\delta} Y_{\delta'}$ has the properties $(8,\delta)$-$(10,\delta)$.

Now, the subgroup $Z_{\gamma+1} = Y_\varepsilon$ has the properties $(5,\gamma+1)$-$(7,\gamma+1)$.

c) Let γ be a limit ordinal. If for each ordinal $\gamma' < \gamma$ a subgroup $Z_{\gamma'}$ with the properties $(5,\gamma')$-$(7,\gamma')$ has been defined then the subgroup $Z_\gamma = \bigcup_{\gamma'<\gamma} Z_{\gamma'}$ has the properties $(5,\gamma)$-$(7,\gamma)$.

Finally put $X = Z_\sigma$; the properties (i) and (ii) arise from $(5,\sigma)$ and $(6,\sigma)$.

Case 2.2. $\sigma \leqslant \varepsilon$.

By transfinite induction we shall construct an ascending chain of subgroups Y_δ, $\delta \leqslant \varepsilon$, with the following properties:

$(11,\delta)$ $$Y_\delta \cap p^\beta G = B ,$$

$(12,\delta)$ $$\forall \gamma < \min(\delta, \sigma) , \qquad \forall \delta' < \delta , \qquad \exists c \in Y_\delta \cap p^{\alpha_{\delta'}} G , \qquad a_\gamma = pc ,$$

$(13,\delta)$ $$r((p^\beta G + Y_\delta) \cap H[p]) \leqslant \sum_{\delta' \leqslant \delta} |\delta'| .$$

a) Put $Y_0 = B$.

b) Suppose that Y_δ has been defined $(0 \leqslant \delta < \varepsilon)$ and define $Y_{\delta+1}$. We use the transfinite induction to define an ascending chain of subgroups Z_γ, $\gamma \leqslant \min(\delta+1, \sigma)$, with the following properties:

$(14,\gamma)$ $$Z_\gamma \cap p^\beta G = B ,$$

$(15,\gamma)$ $$\forall \gamma' < \gamma , \qquad \exists c \in Z_\gamma \cap p^{\alpha_\delta} G , \qquad a_{\gamma'} = pc ,$$

$(16,\gamma)$ $$r((p^\beta G + Z_\gamma) \cap H[p]) \leqslant \sum_{\delta' \leqslant \delta} |\delta'| + |\gamma| .$$

b_1) Put $Z_0 = Y_\delta$.

b_2) Suppose that Z_γ has been defined $(0 \leqslant \gamma < \min(\delta+1, \sigma))$ and define $Z_{\gamma+1}$. If there is no element $c \in (p^\beta G + Z_\gamma) \cap p^{\alpha_\delta} G$ with $a_\gamma = pc$ then let $c_\gamma \in p^{\alpha_\delta} G$ be an arbitrary element such that $a_\gamma = pc_\gamma$. If $c \in (p^\beta G + Z_\gamma) \cap p^{\alpha_\delta} G$ is an element such that $a_\gamma = pc$ then let $c_\gamma = c + h$, where $h \in p^{\alpha_\delta} H[p] \setminus (p^\beta G + Z_\gamma)$. Such element h exists, since by remark 1,

$$r(p^{\alpha_\delta} H[p]) \geqslant r_{(p,\beta)}(G) \geqslant \varepsilon > \sum_{\delta' \leqslant \delta} |\delta'| + |\gamma| \geqslant r((p^\beta G + Z_\gamma) \cap H[p]) .$$

Now, define $Z_{\gamma+1} = \langle Z_\gamma, c_\gamma \rangle$; as in the case 2.1, it is not difficult to verify the properties (14,γ+1)-(16,γ+1).

b_3) Let γ be a limit ordinal. If for each ordinal $\gamma' < \gamma$ a subgroup $Z_{\gamma'}$ with the properties (14,γ')-(16,γ') has been defined then the subgroup $Z_\gamma = \bigcup_{\gamma' < \gamma} Z_{\gamma'}$ has the properties (14,γ)-(16,γ).

The subgroup $Y_{\delta+1} = Z_{\min(\delta+1,\sigma)}$ has the properties (11,δ+1)-(13,δ+1).

c) Let δ be a limit ordinal. If for each ordinal $\delta' < \delta$ a subgroup $Y_{\delta'}$ with the properties (11,δ')-(13,δ') has been defined then the subgroup $Y_\delta = \bigcup_{\delta' < \delta} Y_{\delta'}$ has the properties (11,δ)-(13,δ).

Finally put $X = Y_\varepsilon$; the properties (i), (ii), arise from (11,ε) and (12,ε).

LEMMA 3. Let G be a group, p a prime, β a nonzero ordinal and B a subgroup of $p^\beta G$. If B is a direct sum of infinite and p-primary cyclic groups and

$$r(p^{\beta+1} G \cap B/pB) \leqslant r_{(p,\beta)}(G)$$

then there is a subgroup Y of G such that

(i) $Y \cap p^\beta G = B$,

(ii) $\forall b \in B,\ \forall \alpha < \beta,\ \exists y \in Y \cap p^\alpha G,\ b = py$.

PROOF. Write $B/pB = p^{\beta+1} G \cap B/pB \oplus B_2/pB$. Let $\{b_i + pB;\ i \in I\}$ be a basis of the group $p^{\beta+1} G \cap B/pB$; put $B_1 = \langle b_i;\ i \in I \rangle$. Obviously $B_1 + pB = p^{\beta+1} G \cap B$ and hence $B = B_1 + B_2$. Since B is a direct sum of infinite and p-primary cyclic groups,

$$r(B_1) \leqslant |I| \leqslant r_{(p,\beta)}(G);$$

by lemma 2, there is a subgroup X of G such that

$$(17) \qquad X \cap p^\beta G = B_1 ,$$

$$(18) \qquad \forall b \in B_1 , \quad \forall \alpha < \beta , \quad \exists x \in X \cap p^\alpha G , \quad b = px .$$

Write $B_2 = \bigoplus_{\gamma<\sigma} \langle a_\gamma \rangle$, where σ is an ordinal.

By transfinite induction we shall define an ascending chain of subgroups Y_α, $\alpha \leqslant \beta$, with the following properties:

$$(19,\alpha) \qquad X \subset Y_\alpha ,$$

$$(20,\alpha) \qquad Y_\alpha \cap p^\beta G = B ,$$

$$(21,\alpha) \qquad \forall \gamma < \sigma , \quad \forall \alpha' < \alpha , \quad \exists c \in Y_\alpha \cap p^{\alpha'} G , \quad a_\gamma = pc ,$$

a) Put $Y_0 = B_2 + X$; the subgroup Y_0 has the properties (19,0)-(21,0).

b) Suppose that Y_α has been defined $(0 \leqslant \alpha < \beta)$ and define $Y_{\alpha+1}$. By transfinite induction we shall construct an ascending chain of subgroups Z_γ, $\gamma \leqslant \sigma$, with the following properties:

$$(22,\gamma) \qquad Y_\alpha \subset Z_\gamma ,$$

$$(23,\gamma) \qquad Z_\gamma \cap p^\beta G = B ,$$

$$(24,\gamma) \qquad \forall \gamma' < \gamma , \quad \exists c \in Z_\gamma \cap p^\alpha G , \quad a_{\gamma'} = pc .$$

b$_1$) Put $Z_0 = Y_\alpha$.

b$_2$) Suppose that Z_γ has been defined $(0 \leqslant \gamma < \sigma)$ and define $Z_{\gamma+1}$. If there is an element $c \in Z_\gamma \cap p^\alpha G$ such that $a_\gamma = pc$ then define $c_\gamma = c$. If there is no element $c \in (p^\beta G + Z_\gamma) \cap p^\alpha G$ with $a_\gamma = pc$ then let $c_\gamma \in p^\alpha G$ be an arbitrary element such that $a_\gamma = pc_\gamma$. Finally, if there is an element $c \in (p^\beta G + Z_\gamma) \cap p^\alpha G$ with $a_\gamma = pc$ then there is an element $c' \in Z_\gamma \cap p^\alpha G$ with $a_\gamma = pc'$. For, write

$c = v + z$, where $v \in p^\beta G$ and $z \in Z_\gamma$; obviously $z \in p^\alpha G$. Hence $a_\gamma = pv + pz$, i.e. $pz \in p^\beta G \cap Z_\gamma = B$ by $(23,\gamma)$ and further $pv \in p^{\beta+1} G \cap \cap B = B_1 + pB$. There are elements $b_1 \in B_1$ and $b \in B$ such that $pv = b_1 + pb$. By (18), we can write $b_1 = px$, where $x \in X \cap p^\alpha G$, and hence $a_\gamma = p(x + b + z)$, where $c' = x + b + z \in Z_\gamma \cap p^\alpha G$ (see $(19,\alpha)$ and $(22,\gamma)$). Define $Z_{\gamma+1} = \langle Z_\gamma, c_\gamma \rangle$ and verify the property $(23,\gamma+1)$ (the other two properties are trivial). Let $z + rc_\gamma \in p^\beta G$, where $z \in Z_\gamma$ and r is an integer. If $p|r$ then $z + rc_\gamma \in Z_\gamma \cap p^\beta G = B$. If $(p, r) = 1$ then $c_\gamma \in p^\beta G + Z_\gamma$ and according to the definition of c_γ, $c_\gamma \in Z_\gamma$. Hence $z + rc_\gamma \in Z_\gamma \cap p^\beta G = B$.

b_3) Suppose that γ is a limit ordinal. If for each ordinal $\gamma' < \gamma$ a subgroup $Z_{\gamma'}$ with the properties $(22,\gamma')$-$(24,\gamma')$ has been defined then the subgroup $Z_\gamma = \bigcup_{\gamma'<\gamma} Z_{\gamma'}$ has the propertires $(22,\gamma)$-$(24,\gamma)$.

Now, the subgroup $Y_{\alpha+1} = Z_\sigma$ has the properties $(19,\alpha+1)$-$(21,\alpha+1)$.

c) Let α be a limit ordinal. If for each ordinal $\alpha' < \alpha$ a subgroup $Y_{\alpha'}$ with the properties $(19,\alpha')$-$(21,\alpha')$ has been defined then the subgroup $Y_\alpha = \bigcup_{\alpha'<\alpha} Y_{\alpha'}$ has obviously the properties $(19,\alpha)$-$(21,\alpha)$.

Finally put $Y = Y_\beta$; the properties (i), (ii) arise from $(19,\beta)$-$(21,\beta)$ and (18) with respect to the equality $B = B_1 + B_2$.

THEOREM 2. Let G be a group, p a prime, β a nonzero ordinal and A a subgroup of $p^\beta G$. The following are equivalent:

(i) $r(p^{\beta+1} G \cap A/pA) \leqslant r_{(p,\beta)}(G)$.

(ii) There is a $p^\beta G - A$-high subgroup P of G such that $p^\alpha P = P \cap p^\alpha G$ for each ordinal $\alpha \leqslant \beta$.

(iii) There is a subgroup P of G such that $A = p^\beta P = P \cap p^\beta G$.

PROOF. Suppose (i). Let B be a p-basic subgroup of A. Since $A = B + pA$ and $A \subset p^\beta G$,

$$p^{\beta+1} G \cap A/pA \cong p^{\beta+1} G \cap B/pB$$

and hence

$$r(p^{\beta+1} G \cap B/pB) \leqslant r_{(p,\beta)}(G) .$$

By lemma 3, there is a subgroup Y of G such that

(24) $$Y \cap p^\beta G = B ,$$

(25) $$\forall b \in B , \quad \forall \alpha < \beta , \quad \exists y \in Y \cap p^\alpha G , \quad b = py .$$

Obviously $(A+Y) \cap p^\beta G = A$. Let P be a $p^\beta G - A$-high subgroup of G containing $A+Y$. We prove that $p^\alpha P = P \cap p^\alpha G$ for every ordinal $\alpha \leqslant \beta$. It is sufficient to show that if this equality holds for α $(0 \leqslant \alpha < \beta)$ then it holds for $\alpha + 1$. Let $x \in P \cap p^{\alpha+1}G$, i.e. $x = pg$, where $g \in p^\alpha G$; obviously $x \in P \cap p^\alpha G = p^\alpha P$. If $g \in P$ then $g \in P \cap$ $\cap\, p^\alpha G = p^\alpha P$ and $x = pg \in p^{\alpha+1}P$. If $g \notin P$ then there are an element $x' \in P$ and an integer r such that $rg + x' \in p^\beta G \setminus A$; obviously $x' \in p^\alpha G$ and $(p, r) = 1$. Further

$$prg + px' = rx + px' \in P \cap p^\beta G = A = B + pA;$$

there are elements $b \in B$ and $a \in A$ such that $rx + px' = b + pa$. By (25), there is an element $y \in Y \cap p^\alpha G$ such that $b = py$. Hence $rx = p(y + a - x')$, where $y + a - x' \in P \cap p^\alpha G = p^\alpha P$, and consequently $rx \in p^{\alpha+1}P$. Since $(p, r) = 1$, $x \in p^{\alpha+1}P$ and assertion (ii) is proved.

Obviously, (ii) implies (iii) and (iii) implies (i) by lemma 1.

REFERENCES

[1] L. FUCHS, *Infinite Abelian groups, I, II*, Acad. Press, 1970, 1973.

[2] J. M. IRWIN - E. A. WALKER, *On isotype subgroups of Abelian groups*, Bull. Soc. Math. France, **89** (1961), pp. 451-460.

[3] F. V. KRIVONOS, *Ob N-vysokich podgruppach abelevoj gruppy*, Vest. Moskov. Univ. (1975), pp. 58-64.

[4] R. W. MITCHELL, *An extension of Ulm's theorem*, Ph. D. Dissertation, New Mexico State University, May 1964.

[5] A. R. MITCHELL - R. W. MITCHELL, *Some structure theorems for infinite Abelian p-groups*, J. Algebra, **5** (1967), pp. 367-372.

[6] F. RICHMAN - C. P. WALKER, *On a certain purification problem for primary Abelian groups*, Bull. Soc. Math. France, **94** (1966), pp. 207-210.

Manoscritto pervenuto in redazione il 19 dicembre 1980.