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Löb Operators and Interior Operators.

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SUMMARY - In the theory of Boolean algebras, many kinds of operators have been studied: for instance the ones called Löb operators (see [3] and [5]) and interior operators (see [3] and [4]). In this paper we analyze some links between these two types of operators and in particular we give a characterization of the interior operator which may be expressed by \( x \cdot \tau x \) where \( \tau \) is a Löb operator. It can be noticed that, using the language of modal logic, this interior operator is nothing else but Smorynski’s modal operator \( s \) (see [6]).

SUNTO - Sono state ampiamente studiate le algebre di Boole con un operatore di Löb (vedi [3] e [5]) e quelle con un operatore di interno (vedi [3] e [4]). In questo lavoro si analizzano alcuni legami tra questi operatori. In particolare si caratterizzano gli operatori di interno esprimibili mediante \( x \cdot \tau x \) (\( \tau \) è un operatore di Löb). È da rilevare che, tradotto nel linguaggio della logica modale, a questo operatore di interno corrisponde l’operatore modale \( s \) di Smorynski (vedi [6]).

Introduction and notations.

We shall indicate by \( \mathfrak{B} = \langle B, +, \cdot, v, 0, 1 \rangle \) a boolean algebra and by \( \tau \) and \( I \) respectively a Löb operator and an interior operator, i.e.

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two functions from $B$ to $B$ respectively satisfying the following axioms:

$$
\begin{align*}
\tau_1) & \quad \tau 1 = 1 \\
\tau_2) & \quad \tau(a \cdot b) = \tau a \cdot \tau b \\
\tau_3) & \quad \tau(\tau a \rightarrow a) < \tau a \\
\end{align*}
$$

$I_1) \quad I 1 = 1$

$I_2) \quad I(a \cdot b) = Ia \cdot Ib$

$I_3) \quad I(Ia) = Ia$

$I_4) \quad Ia < a$

Our purpose is to study some links between these kinds of operators. It is easy to verify that, given a L"{o}b operator $\tau$, if we let $Ix = x \cdot \tau x$, then $I$ is an interior operator, that we will call the interior operator associated to the L"{o}b operator. We can note the fact that the condition $\tau_3$ can be successfully substituted, to this purpose, by the weaker condition $\tau x \ll \tau^2 x$, and, on the other hand, that this condition is used only for demonstrating $I_3$).

As usual we shall denote by $\varphi^n a$ the $n$-th reiterate application of the operator $\varphi$. According to the topological language we shall call open, an element of $B$ such that $x = Ix$.

1. First results.

The first problem we are concerned with, consists of the question: can every interior operator be obtained in the way we have shown before? i.e. for every boolean algebra $\mathcal{B}$ and every interior operator $I$, does there exist a L"{o}b operator $\tau$ on $\mathcal{B}$ such that, for each element $x$ of $B$, there holds: $Ix = x \cdot \tau x$?

The answer is negative.

**Counterexample 1-1.** Let $\mathcal{B}$ be a non-simple boolean algebra (i.e. $B$ consists of more than two elements) and let $I$ be defined in the following way:

$$I(x) = \begin{cases} 
1 & \text{if } x = 1 \\
0 & \text{otherwise}
\end{cases}$$

Then it is easy to see that $I$ is an interior operator on $\mathcal{B}$, while $I$ is not associated to any L"{o}b operator.

In fact let, by absurd, $\tau$ be a L"{o}b operator such that $Ix = x \cdot \tau x$ for each $x \in B$. 
Two cases are possible:

a) $\tau 0 \neq 1$. Then, since it must be $x \cdot \tau x = 0$ for each $x \neq 1$, it should hold $I(\tau 0) = 0$ that is, $0 = I(\tau 0) = \tau 0(\tau^2 0) = \tau 0$ but this result, together with $\tau = 1$), contradicts $\tau 1$).

b) $\tau 0 = 1$. Then $Ix = x$ but, since $\mathcal{B}$ is not simple, taking $a \in B$ with $0 \neq a \neq 1$, we obtain $Ia = a$, which is absurd by the definition of $I$.

One could set the problem in a weaker way: given a boolean algebra $\mathcal{B}$ and an interior operator $I$ on $\mathcal{B}$, do there exist a boolean algebra $\mathcal{B}^*$ and a L"{o}b operator $\tau$ on $\mathcal{B}^*$ such that $\mathcal{B}$ is a subalgebra of $\mathcal{B}^*$ and, for every $x \in B$, there holds $x \cdot \tau x = Ix$. Unfortunately also this second conjecture is wrong.

**Counterexample 1-2.** Let $\mathcal{B}$ be the four elements boolean algebra and $I$ the interior operator as defined in the counterexample 1-1. Let $a \in B$ and $0 \neq a \neq 1$. It should hold $a \cdot \tau a = na \cdot \tau na = 0$; it follows $\tau a = \tau(a \cdot \tau a) = \tau 0 = \tau (na \cdot \tau na) = \tau na$. On the other hand from $a \leq \tau a$ and $na \leq \tau na$ we can deduce:

$$1 = a + na \leq \tau a + \nu \tau na = \nu 0 \quad \text{i.e.} \quad 0 = \tau 0$$

which is absurd. ■

Now we give some simple results.

**Theorem 1-3.** Let $I$ be the interior operator on $\mathcal{B}$ associated with the L"{o}b operator $\tau$. The following conditions are sufficient, for an element $a \in B$, to be open for $I$:

i) $a$ belongs to the range of $\tau$;

ii) $a$ is such that $a \geq \tau a$ or $a \leq \tau a$.

**Proof.** i) is obvious since for every $a \in B$, $\tau a \leq \tau^2 a$;

ii) If $a \geq \tau a$, it easily follows that $a = 1$;

if $a \leq \tau a$ then $Ia = a \cdot \tau a = a$. ■

As we have just seen, the indiscrete topology (the one that admits just 0 and 1 as open elements) does not derive from any $\tau$.

It is obvious, instead, that the discrete topology comes at least
from the Löb operator defined by: \( \tau a = 1 \) for each \( a \in B \). The following theorem states that this is the only one that gives origin to the discrete topology.

**Theorem 1-4.** Let \( \tau \) be a Löb operator on \( B \). Then \( a \cdot \tau a = a \) for each \( a \in B \) iff \( \tau a = 1 \) for each \( a \in B \).

**Proof.** Let \( a \cdot \tau a = a \) for each \( a \in B \). If we set \( a = \nu \tau b \) (with \( b \in B \)) then we obtain \( \tau (\nu \tau b) \cdot \nu \tau b = \nu \tau b \) and since we have \( \tau (\nu \tau b) = \tau 0 \), there holds \( \nu \tau b < \tau 0 \); and then \( \tau b \vdash \tau 0 = 1 \); but, as we know, \( \tau 0 < \tau b \) and so there holds \( \tau b = \tau 0 \vdash \tau b = 1 \).

The other implication is obvious. ■

The next question may be, if this result can be generalized or in other words if it is true that every time that an interior operator come out from a \( \tau \) then this \( \tau \) is unique.

We shall see, in the following paragraphs, that the answer is affirmative. However we can give at once, a direct proof of this fact, in the hypothesis that \( B \) is finite.

**Theorem 1-5.** Let \( B \) be a finite boolean algebra.

If \( \tau_1 \) and \( \tau_2 \) are two Löb operators over \( B \), such that \( x \cdot \tau_1 x = x \cdot \tau_2 x \) for each \( x \in B \), then \( \tau_1 = \tau_2 \).

**Proof.** We shall show that:

i) If \( x \) is a co-atom of \( B \), and \( \tau \) a Löb operator on \( B \) then \( x + \tau x = 1 \).

ii) If \( x \) is a co-atom of \( B \) and \( x \cdot \tau_1 x = x \cdot \tau_2 x \) then \( \tau_1 x = \tau_2 x \).

iii) If \( x = x_1 \cdot x_2 \), where \( x_1 \) and \( x_2 \) are co-atoms such that \( \tau_1 x_1 = \tau_1 x_2 \) and \( \tau_2 x_1 = \tau_2 x_2 \), then \( \tau_1 x = \tau_2 x \).

From these statements the theorem follows quite easily.

i) If \( x \) is a co-atom then, being \( x + \tau x > x \), it may hold \( x + \tau x = x \) or \( x + \tau x = 1 \). If \( x + \tau x = x \) then \( \tau x \vdash x \), from which there follows \( \tau x = 1 \). Therefore \( x + \tau x = 1 \).

ii) If \( x \) is co-atom then \( x + \tau_1 x = x + \tau_2 x \). There holds:

\[
\tau_1 x = \tau_1 x \cdot (x + \tau_2 x) = \tau_1 x \cdot x + \tau_1 x \cdot \tau_2 x = \tau_2 x \cdot x + \tau_2 x \cdot \tau_1 x = \\
= \tau_2 x (x + \tau_1 x) = \tau_2 x .
\]

iii) Quite trivial. ■
2. Characterization.

In this section we give a theorem which characterizes the interior operators that come out from a Löb operator.

Let an interior operator \( I \) on boolean algebra \( \mathcal{B} \) be given and be \( P_z(y) \) the following property:

\[
\text{for each } z \in B \text{ such that } x < z < y, \text{ there holds } z = Iz.
\]

**THEOREM 2-1.** Let \( I \) be an interior operator on \( \mathcal{B} \).

\( I \) is associated to a Löb operator \( \tau \) iff

\[
(*) \text{ for every } x \text{ such that } x \neq 1 \text{ and } x = Ix, \text{ there exists an element of } B, \text{ } y > x \text{ such that } P_z(y) \text{ and for every } t \text{ with } P_z(t) \text{ there holds } t < y; \text{ i.e. the set of the } y \text{ such that } P_z(y) \text{ has a maximum, strictly greater than } x.
\]

**PROOF.** Let \( I \) and \( \tau \) be as before and let \( x \neq 1 \) with \( x = Ix = x \cdot \tau x \). Obviously \( x < \tau x \). We shall show that \( P_z(\tau x) \) and that for each \( y \) such that \( P_z(y) \), there holds \( y < \tau x \). Let \( x < z < \tau x \). Applying \( \tau \), there follows \( \tau z < \tau x < \tau \tau x \) and then \( z < \tau x < \tau z \), namely \( z \cdot \tau z = z \). Hence \( P_z(\tau x) \) holds.

Let now \( y \) be such that \( x < y \) and \( P_z(y) \) and let us show that \( y < \tau x \).

Obviously \( x < x + y \cdot \nu \tau x \leq y \) hence \( x + y \cdot \nu \tau x \) is open and then

\[
x + y \cdot \nu \tau x = (x + y) \cdot (x + \nu \tau x) =
\]

\[
= (x + y) \cdot (x + \nu \tau x) \cdot \tau ((x + y) \cdot (x + \nu \tau x)) =
\]

\[
= (x + y) \cdot (x + \nu \tau x) \cdot \tau (x + y) \cdot \tau x = (x + y) \cdot (x + \nu \tau x) \cdot \tau x =
\]

\[
= (x + y) \cdot x \cdot \tau x = x \cdot \tau x = x.
\]

That is to say that \( y \cdot \nu \tau x \leq x \); and, being \( x < \tau x \) we can conclude \( y \cdot \nu \tau x = 0 \) i.e. \( y < \tau x \).

For the right-to-left direction, let us suppose that \( I \) is an interior operator with the property \((*)\) and let \( u \) be such that \( u = Iu \); let us set \( \mu u = \{ \max y: P_u(y) \} \). Let us define \( \tau \) on \( \mathcal{B} \) in the following way:

\[
\tau x = \mu(Ix).
\]

We want to show that \( \tau \) is a Löb operator, and that \( x \cdot \tau x = Ix \) for every \( x \) of \( B \). These facts follow from the following lemmas:
Lemma 2-2. \( \tau 1 = 1 \).

Proof. Trivial.

Lemma 2-3. \( \tau (a \cdot b) = \tau a \cdot \tau b \).

Proof. For our purpose it is enough to show that, if \( x \) and \( y \) are open, then \( \mu (x \cdot y) = \mu x \cdot \mu y \). Let us show first that \( \mu x \cdot \mu y = \mu (x \cdot y) \).

Let \( z \) be such that \( x \cdot y < z < \mu x \cdot \mu y \). Then there hold:

\[
 x \leq \mu x \cdot (x + z) < \mu x
\]

and

\[
 y \leq \mu y \cdot (y + z) < \mu y;
\]

from which, there follows:

\[
 I(\mu x \cdot (x + z)) = \mu x \cdot (x + z) \quad \text{i.e. } \mu x \cdot (x + z) \text{ is open;}
\]

\[
 I(\mu y \cdot (y + z)) = \mu y \cdot (y + z) \quad \text{i.e. } \mu y \cdot (y + z) \text{ is open;}
\]

\[
 (\mu x \cdot (x + z)) \cdot (\mu y \cdot (y + z)) = \mu x \cdot \mu y \cdot (x + z) \cdot (y + z) =
\]

\[
 = \mu x \cdot \mu y \cdot (z + x \cdot y) = \mu x \cdot \mu y \cdot z = z \quad \text{i.e. } z \text{ is open.}
\]

Hence there holds \( P_z, s (\mu x \cdot \mu y) \) and then \( \mu x \mu y < \mu (x \cdot y) \).

Conversely let now \( z \) be such that \( x < z < \mu (x \cdot y) \); then we have:

\( x \cdot y < z < \mu (x \cdot y) \) i.e. \( z \) is open.

Hence there holds \( P_z (\mu (x \cdot y)) \) and then \( \mu (x \cdot y) < \mu x \).

Analogously it can be shown that \( \mu (x \cdot y) < \mu y \).

It follows that \( \mu (x \cdot y) < \mu x \cdot \mu y \).

Lemma 2-4. \( \tau (\tau x \rightarrow x) = \tau x \).

Proof. It is well known (see [5]) that it is equivalent to prove that:

i) \( \tau x \cong \tau x \) and

ii) If \( \tau x < x \), then \( x = 1 \).

i) It follows to \( \mu I(\mu Ix) > \mu (Ix) \) and this is obvious.

ii) Let \( \tau x < x \). Then \( \mu Ix < x \) therefore \( I(\mu Ix) < Ix \) that is \( \mu Ix = Ix \). Hence \( x = 1 \).

Lemma 2-5. \( Ix = x \cdot \tau x \).
PROOF. Obviously $Ix < x \cdot \mu(Ix)$.

On the other hand, being $Ix < x \cdot \mu(Ix) < \mu(Ix)$, $x \cdot \mu(Ix)$ is open, and there holds $Ix < x \cdot \mu(Ix) < x$.

Applying $I$ we obtain $Ix < I(x \cdot \mu(Ix)) < Ix$ and, since $x \cdot \mu(Ix)$ is open, $Ix = x \cdot \mu(Ix)$.

As a corollary we can easily obtain the generalization of the theorem 1-5.

**COROLLARY 2-6.** If an interior operator $I$ is associated to a Löb operator $\tau$, this is unique.

**Proof.** Let us suppose that the interior operator $I$ comes out from two Löb operators $\tau_1, \tau_2$; then $x \cdot \tau_1 x = x \cdot \tau_2 x$ for every $x$ of $B$.

If $x$ is open, then $x \cdot \tau_1 x = x \cdot \tau_2 x = x$; and, from the previous theorem, $\tau_1 x = \max \{y : P_x(y)\} = \tau_2 x$.

Now, for every $x$ of $B$, $x \cdot \tau_1 x$ is open and, by hypothesis, equal to $x \cdot \tau_2 x$. Therefore, there holds:

$$\tau_1 x = \tau_1 x \cdot \tau_1^2 x = \tau_1 (x \cdot \tau_2 x) = \tau_2 (x \cdot \tau_2 x) = \tau_2 x \cdot \tau_2^2 x = \tau_2 x .$$

3. Final remarks.

In the previous paragraphs we have provided a characterization of the interior operator that may be associated to a Löb operator; however this characterization has not been given in terms of equations.

A complete discussion of this matter in terms of model logic can be found for instance in [6, p. 96] or in [8, p. 40]. The required equations can be get by the algebraic translation of the axioms of $S4Grz$, i.e. the logic obtained by adding to $S4$ the schema

$$\Box(\Box (A \rightarrow \Box A) \rightarrow A) \rightarrow A .$$

REFERENCES


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