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Löb Operators and Interior Operators.

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SUMMARY - In the theory of Boolean algebras, many kinds of operators have been studied: for istance the ones called Löb operators (see [3] and [5]) and interior operators (see [3] and [4]). In this paper we analyze some links between these two types of operators and in particular we give a characterization of the interior operator which may be expressed by $x \cdot \tau x$ where τ is a Löb operator. It can be noticed that, using the language of modal logic, this interior operator is nothing else but Smorynski's modal operator s (see [6]).

SUNTO - Sono state ampiamente studiate le algebre di Boole con un operatore di Löb (vedi [3] e [5]) e quelle con un operatore di interno (vedi [3] e [4]). In questo lavoro si analizzano alcuni legami tra questi operatori. In particolare si caratterizzano gli operatori di interno esprimibili mediante $x \cdot \tau x$ (τ è un operatore di Löb). È da rilevare che, tradotto nel linguaggio della logica modale, a questo operatore di interno corrisponde l'operatore modale s di Smorynski (vedi [6]).

Introduction and notations.

We shall indicate by $\mathfrak{B} = \langle B, +, \cdot, \nu, 0, 1 \rangle$ a boolean algebra and by τ and I respectively a Löb operator and an interior operator, i.e.

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two functions from B to B respectively satisfying the following axioms:

$\tau_1)$ $\tau 1 = 1$	$\mathrm{I}_1)$ $I1 = 1$
$\tau_2)$ $\tau(a \cdot b) = \tau a \cdot \tau b$	$\mathrm{I}_2)$ $I(a \cdot b) = Ia \cdot Ib$
$\tau_3)$ $\tau(\tau a \to a) \leqslant \tau a$	$\mathrm{I}_3)$ $I(Ia) = Ia$
	$\mathrm{I}_4)$ $Ia \leqslant a$.

Our purpose is to study some links between these kinds of operators. It is easy to verify that, given a Löb operator τ, if we let $Ix = x \cdot \tau x$, then I is an interior operator, that we will call the interior operator associated to the Löb operator. We can note the fact that the condition $\tau_3)$ can be successfully substituted, to this purpose, by the weaker condition $\tau x \leqslant \tau^2 x$, and, on the other hand, that this condition is used only for demostrating $\mathrm{I}_3)$.

As usual we shall denote by $\varphi^n a$ the n-th reiterate application of the operator φ. According to the topological language we shall call open, an element of B such that. $x = Ix$.

1. First results.

The first problem we are concerned with, consists of the question: can every interior operator be obtained in the way we have shown before? i.e. for every boolean algebra $\mathfrak{B}$ and every interior operator I, does there exist a Löb operator τ on $\mathfrak{B}$ such that, for each element x of B, there holds: $Ix = x \cdot \tau x$?

The answer is negative.

COUNTEREXAMPLE 1-1. Let $\mathfrak{B}$ be a non-simple boolean algebra (i.e. B consists of more than two elements) and let I be defined in the following way:

$$I(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

Then it is easy to see that I is an interior operator on $\mathfrak{B}$, while I is not associated to any Löb operator.

In fact let, by absurd, τ be a Löb operator such that $Ix = x \cdot \tau x$ for each $x \in B$.

Two cases are possible:

a) $\tau 0 \neq 1$. Then, since it must be $x \cdot \tau x = 0$ for each $x \neq 1$, it should hold $I(\tau 0) = 0$ that is, $0 = I(\tau 0) = \tau 0 (\tau^2 0) = \tau 0$ but this result, together with τ_3), contradicts τ_1).

b) $\tau 0 = 1$. Then $Ix = x$ but, since $\mathfrak{B}$ is not simple, taking $a \in B$ with $0 \neq a \neq 1$, we obtain $Ia = a$, which is absurd by the definition of I. ■

One could set the problem in a weaker way: given a boolean algebra $\mathfrak{B}$ and an interior operator I on $\mathfrak{B}$, do there exist a boolean algebra $\mathfrak{B}^*$ and a Löb operator τ on $\mathfrak{B}^*$ such that $\mathfrak{B}$ is a subalgebra of $\mathfrak{B}^*$ and, for every $x \in B$, there holds $x \cdot \tau x = Ix$? Unfortunately also this second conjecture is wrong.

COUNTEREXAMPLE 1-2. Let $\mathfrak{B}$ be the four elements boolean algebra and I the interior operator as defined in the counterexample 1-1. Let $a \in B$ and $0 \neq a \neq 1$. It should hold $a \cdot \tau a = \nu a \cdot \tau \nu a = 0$; it follows $\tau a = \tau(a \cdot \tau a) = \tau 0 = \tau(\nu a \cdot \tau \nu a) = \tau \nu a$. On the other hand from $a \leqslant \tau a$ and $\nu a \leqslant \nu \tau \nu a$ we can deduce:

$$1 = a + \nu a \leqslant \nu \tau a + \nu \tau \nu a = \nu \tau 0 \quad \text{i.e. } 0 = \tau 0$$

which is absurd. ■

Now we give some simple results.

THEOREM 1-3. *Let I be the interior operator on $\mathfrak{B}$ associated with the Löb operator τ. The following conditions are sufficient, for an element $a \in B$, to be open for I:*

i) *a belongs to the range of τ;*

ii) *a is such that $a \geqslant \tau a$ or $a \leqslant \tau a$.*

PROOF. i) is obvious since for every $a \in B$, $\tau a \leqslant \tau^2 a$;

ii) If $a \geqslant \tau a$, it easily follows that $a = 1$;
if $a \leqslant \tau a$ then $Ia = a \cdot \tau a = a$. ■

As we have just seen, the indiscrete topology (the one that admits just 0 and 1 as open elements) does not derive from any τ.

It is obvious, instead, that the discrete topology comes at least from the Löb operator defined by: $\tau a = 1$ for each $a \in B$. The following theorem states that this is the only one that gives origin to the discrete topology.

THEOREM 1-4. *Let* τ *be a Löb operator on* $\mathfrak{B}$. *Then* $a \cdot \tau a = a$ *for each* $a \in B$ *iff* $\tau a = 1$ *for each* $a \in B$.

PROOF. Let $a \cdot \tau a = a$ for each $a \in B$. If we set $a = \nu\tau b$ (with $b \in B$) then we obtain $\tau(\nu\tau b) \cdot \nu\tau b = \nu\tau b$ and since we have $\tau(\nu\tau b) = \tau 0$, there holds $\nu\tau b \leqslant \tau 0$; and then $\tau b + \tau 0 = 1$; but, as we know, $\tau 0 \leqslant \tau b$ and so there holds $\tau b = \tau 0 + \tau b = 1$.

The other implication is obvious. ∎

The next question may be, if this result can be generalyzed or in other words if it is true that every time that an interior operator come out from a τ then this τ is unique.

We shall see, in the following paragraphs, that the answer is affirmative. However we can give at once, a direct proof of this fact, in the hypothesis that B is finite.

THEOREM 1-5. *Let* $\mathfrak{B}$ *be a finite boolean algebra.*

If τ_1 *and* τ_2 *are two Löb operators over* $\mathfrak{B}$, *such that* $x \cdot \tau_1 x = x \cdot \tau_2 x$ *for each* $x \in B$, *then* $\tau_1 = \tau_2$.

PROOF. We shall show that:

i) If x is a co-atom of $\mathfrak{B}$, and τ a Löb operator on $\mathfrak{B}$ then $x + \tau x = 1$.

ii) If x is a co-atom of $\mathfrak{B}$ and $x \cdot \tau_1 x = x \cdot \tau_2 x$ then $\tau_1 x = \tau_2 x$.

iii) If $x = x_1 \cdot x_2$, where x_1 and x_2 are co-atoms such that $\tau_1 x_1 = \tau_2 x_1$ and $\tau_1 x_2 = \tau_2 x_2$, then $\tau_1 x = \tau_2 x$.

From these statements the theorem follows quite easily.

i) If x is a co-atom then, being $x + \tau x \geqslant x$, it may hold $x + \tau x = x$ or $x + \tau x = 1$. If $x + \tau x = x$ then $\tau x \leqslant x$, from which there follows $\tau x = 1$. Therefore $x + \tau x = 1$.

ii) If x is co-atom then $x + \tau_1 x = x + \tau_2 x$. There holds:

$$\tau_1 x = \tau_1 x \cdot (x + \tau_2 x) = \tau_1 x \cdot x + \tau_1 x \cdot \tau_2 x = \tau_2 x \cdot x + \tau_2 x \cdot \tau_1 x = \\ = \tau_2 x (x + \tau_1 x) = \tau_2 x \,.$$

iii) Quite trivial. ∎

2. Characterization.

In this section we give a theorem which characterizes the interior operators that come out from a Löb operator.

Let an interior operator I on boolean algebra $\mathfrak{B}$ be given and be $P_x(y)$ the following property:

« for each $z \in B$ such that $x \leqslant z \leqslant y$, there holds $z = Iz$ ».

THEOREM 2-1. *Let I be an interior operator on $\mathfrak{B}$.*

I is associated to a Löb operator τ iff

(*) *for every x such that $x \neq 1$ and $x = Ix$, there exists an element of B, $y > x$ such that $P_x(y)$ and for every t with $P_x(t)$ there holds $t \leqslant y$; i.e. the set of the y such that $P_x(y)$ has a maximum, strictly greater than x.*

PROOF. Let I and τ be as before and let $x \neq 1$ with $x = Ix = x \cdot \tau x$. Obviously $x < \tau x$. We shall show that $P_x(\tau x)$ and that for each y such that $P_x(y)$, there holds $y \leqslant \tau x$. Let $x \leqslant z \leqslant \tau x$. Applying τ, there follows $\tau x \leqslant \tau z \leqslant \tau^2 x$ and then $z \leqslant \tau x \leqslant \tau z$, namely $z \cdot \tau z = z$. Hence $P_x(\tau x)$ holds.

Let now y be such that $x < y$ and $P_x(y)$ and let us show that $y < \tau x$.

Obviously $x < x + y \cdot \nu\tau x \leqslant y$ hence $x + y \cdot \nu\tau x$ is open and then

$$\begin{aligned} x + y \cdot \nu\tau x &= (x + y)\cdot(x + \nu\tau x) = \\ &= (x+y)\cdot(x+\nu\tau x)\cdot\tau\big((x+y)\cdot(x+\nu\tau x)\big) = \\ &= (x+y)\cdot(x+\nu\tau x)\cdot\tau(x+y)\cdot\tau x = (x+y)\cdot(x+\nu\tau x)\cdot\tau x = \\ &= (x+y)\cdot x\cdot\tau x = x\cdot\tau x = x\,. \end{aligned}$$

That is to say that $y \cdot \nu\tau x \leqslant x$; and, being $x < \tau x$ we can conclude $y \cdot \nu\tau x = 0$ i.e. $y \leqslant \tau x$.

For the right—to—left direction, let us suppose that I is an interior operator with the property (*) and let u be such that $u = Iu$; let us set $\mu u = \{\max y \colon P_u(y)\}$. Let us define τ on $\mathfrak{B}$ in the following way: $\tau x = \mu(Ix)$.

We want to show that τ is a Löb operator, and that $x \cdot \tau x = Ix$ for every x of B. These facts follow from the following lemmas:

LEMMA 2-2. $\tau 1 = 1$.

PROOF. Trivial.

LEMMA 2-3. $\tau(a \cdot b) = \tau a \cdot \tau b$.

PROOF. For our purpose it is enough to show that, if x and y are open, then $\mu(x \cdot y) = \mu x \cdot \mu y$. Let us show first that $\mu x \cdot \mu y \leqslant \mu(x \cdot y)$.

Let z be such that $x \cdot y \leqslant z \leqslant \mu x \cdot \mu y$. Then there hold:

$$x \leqslant \mu x \cdot (x + z) \leqslant \mu x$$

and

$$y \leqslant \mu y \cdot (y + z) \leqslant \mu y ;$$

from which, there follows:

$$I(\mu x \cdot (x + z)) = \mu x \cdot (x + z) \quad \text{i.e.} \quad \mu x \cdot (x + z) \quad \text{is open;}$$

$$I(\mu y \cdot (y + z)) = \mu y \cdot (y + z) \quad \text{i.e.} \quad \mu y \cdot (y + z) \quad \text{is open;}$$

$$\begin{aligned}(\mu x \cdot (x + z)) \cdot (\mu y \cdot (y + z)) &= \mu x \cdot \mu y \cdot (x + z) \cdot (y + z) = \\ &= \mu x \cdot \mu y \cdot (z + x \cdot y) = \mu x \cdot \mu y \cdot z = z \quad \text{i.e.} \quad z \text{ is open}.\end{aligned}$$

Hence there holds $P_{x,y}(\mu x \cdot \mu y)$ and then $\mu x \mu y \leqslant \mu(x \cdot y)$.

Conversely let now z be such that $x \leqslant z \leqslant \mu(x \cdot y)$; then we have: $x \cdot y \leqslant z \leqslant \mu(x \cdot y)$ i.e. z is open.

Hence there holds $P_x(\mu(x \cdot y))$ and then $\mu(x \cdot y) \leqslant \mu x$.

Analogously it can be shown that $\mu(x \cdot y) \leqslant \mu y$.

It follows that $\mu(x \cdot y) \leqslant \mu x \cdot \mu y$.

LEMMA 2-4. $\tau(\tau x \to x) \leqslant \tau x$.

PROOF. It is well known (see [5]) that it is equivalent to prove that:

i) $\tau\tau x \geqslant \tau x$ and

ii) If $\tau x \leqslant x$, then $x = 1$.

i) It follows to $\mu I(\mu(Ix)) \geqslant \mu(Ix)$ and this is obvious.

ii) Let $\tau x \leqslant x$. Then $\mu(Ix) \leqslant x$ therefore $I(\mu(Ix)) \leqslant Ix$ that is $\mu(Ix) = Ix$. Hence $x = 1$.

LEMMA 2-5. $Ix = x \cdot \tau x$.

PROOF. Obviously $Ix \leqslant x \cdot \mu(Ix)$.

On the other hand, being $Ix \leqslant x \cdot \mu(Ix) \leqslant \mu(Ix)$, $x \cdot \mu(Ix)$ is open, and there holds $Ix \leqslant x \cdot \mu(Ix) \leqslant x$.

Applying I we obtain $Ix \leqslant I(x \cdot \mu(Ix)) \leqslant Ix$ and, since $x \cdot \mu(Ix)$ is open, $Ix = x \cdot \mu(Ix)$. ∎

As a corollary we can easily obtain the generalization of the theorem 1-5.

COROLLARY 2-6. *If an interior operator I is associated to a Löb operator τ, this is unique.*

PROOF. Let us suppose that the interior operator I comes out from two Löb operators τ_1, τ_2; then $x \cdot \tau_1 x = x \cdot \tau_2 x$ for every x of B.

If x is open, then $x \cdot \tau_1 x = x \cdot \tau_2 x = x$; and, from the previous theorem, $\tau_1 x = \max \{y \colon P_x(y)\} = \tau_2 x$.

Now, for every x of B, $x \cdot \tau_1 x$ is open and, by hypothesis, equal to $x \cdot \tau_2 x$. Therefore, there holds:

$$\tau_1 x = \tau_1 x \cdot \tau_1^2 x = \tau_1(x \cdot \tau_2 x) = \tau_2(x \cdot \tau_2 x) = \tau_2 x \cdot \tau_2^2 x = \tau_2 x\,. \qquad \blacksquare$$

3. Final remarks.

In the previous paragraphs we have provided a characterization of the interior operator that may be associated to a Löb operator; however this characterization has not been given in terms of equations.

A complete discussion of this matter in terms of model logic can be found for istance in [6, p. 96] or in [8, p. 40]. The required equations can be get by the algebraic translation of the axioms of S4Grz, i.e. the logic obtained by adding to S4 the schema

$$\Box(\Box(A \to \Box A) \to A) \to A\,.$$

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