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Finite Groups with a Standard Component of Type $L_3(4)$, - I

CHENG KAI-NAH - DIETER HELD (*)

1. Introduction.

Following Aschbacher, a quasi-simple subgroup A of a group G is called a standard subgroup of G , if $K = C_G(A)$ is tightly-embedded in G , $N_G(A) = N_G(K)$, and $[A, A^g] \neq 1$ for every $g \in G$. Here, K is tightly-embedded in G , if $|K| \equiv 0 \pmod{2}$ and $|K \cap K^g| \equiv 1 \pmod{2}$ hold for all $g \in G \setminus N_G(K)$. Assume that K is tightly-embedded in G and let x be an involution in K . If $y \in C_G(x)$, then $x = x^y \in K \cap K^y$, and so, $y \in N(K)$; it follows $C_G(x) \subseteq N_G(K)$ for every involution $x \in K$. If y is another involution of K and $x^g = y$ for some $g \in G$, then $x^g = y \in K^g \cap K$, and so, $g \in N(K)$, since K is tightly-embedded in G . This implies that the fusion of the 2-elements of K takes place only in $N(K)$.

The objective of this series of papers is to prove the following result:

THEOREM. Let G be a nonabelian, finite, simple group which possesses a standard subgroup A such that $A/Z(A) \cong L_3(4)$. Then the following two assertions hold:

- (1) If $|Z(A)|_2 = 1$, then $2^{11} \mid |G|$;
- (2) If $|Z(A)|_2 > 1$, then G is isomorphic to He or $O'N$.

REMARK. From the work done by Cheng, Held, and Reifart it seems very likely that in case (1) we have $G \cong Sz$.

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In this first paper we prove the following

THEOREM 1. Let G be a nonabelian, finite, simple group which possesses a standard subgroup A such that $A/\mathbf{Z}(A) \cong L_3(4)$ and that the 2-rank of $\mathbf{Z}(A)$ is greater than 1. Then G is isomorphic to the sporadic simple group He .

2. Some facts about $L_3(4)$.

We shall state here some facts about $L_3(4)$ which are required in later sections. Throughout this section we set $L = L_3(4)$.

(2.1) Denote by « $\Rightarrow$ » the canonical homomorphism from $SL_3(4)$ onto L . Let $\alpha \neq 1$ be an element of $GF(4)^*$. Following the notation introduced by A. Reifart [10], we define the elements $\pi, \tau, \mu, \lambda, \zeta, \xi$ of L as follows:

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} &\Rightarrow \pi, & \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & \alpha^2 \\ 0 & \alpha^2 & \alpha \end{bmatrix} &\Rightarrow \tau, & \begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \alpha & 0 & 1 \end{bmatrix} &\Rightarrow \mu, \\ \begin{bmatrix} 1 & 0 & 0 \\ \alpha^2 & 1 & 0 \\ \alpha^2 & 0 & 1 \end{bmatrix} &\Rightarrow \lambda, & \begin{bmatrix} 1 & \alpha^2 & \alpha^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} &\Rightarrow \zeta, & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} &\Rightarrow \xi. \end{aligned}$$

Then

$$\pi^2 = \tau^2 = \mu^2 = \lambda^2 = \zeta^2 = \xi^2 = 1,$$

$$\begin{aligned} [\pi, \tau] &= [\mu, \lambda] = [\zeta, \xi] = [\mu, \pi] = [\mu, \tau] = [\lambda, \pi] = \\ &= [\lambda, \tau] = [\zeta, \pi] = [\zeta, \tau] = [\xi, \pi] = [\xi, \tau] = 1, \end{aligned}$$

$$[\mu, \zeta] = \pi, \quad [\mu, \xi] = \pi\tau, \quad [\lambda, \zeta] = \pi\tau, \quad [\lambda, \xi] = \tau.$$

Set $P = \langle \pi, \tau, \mu, \lambda, \zeta, \xi \rangle$, $E_1 = \langle \pi, \tau, \mu, \lambda \rangle$, and $E_2 = \langle \pi, \tau, \zeta, \xi \rangle$. Then, $P \in \text{Syl}_2(L)$, and E_1 and E_2 are the only elementary abelian subgroups of order 16 of P . We have $P = E_1 E_2$ and $\mathbf{Z}(P) = P' = \mathbf{D}(P) = \langle \pi, \tau \rangle$. Every involution of P is contained in E_1 or E_2 . The subgroup P possesses precisely three subgroups of type $(4, 4)$, namely, $\langle \mu\lambda\xi, \lambda\zeta \rangle$, $\langle \mu\zeta, \lambda\xi \rangle$, and $\langle \mu\xi, \lambda\zeta\xi \rangle$; these subgroups are self-

centralizing in P . There are precisely 27 involutions in P . Thus, every element of order 4 lies in one of the subgroups of type $(4, 4)$ of P .

The element g of L with

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 1 & \alpha^2 \end{bmatrix} \Rightarrow g$$

has order 3 and operates on P in the following way:

$$g: \pi \rightarrow \pi\tau \rightarrow \tau, \mu \rightarrow \mu\lambda \rightarrow \lambda, \zeta \rightarrow \zeta\xi \rightarrow \xi.$$

One has $N_L(P) = P\langle g \rangle$. The subgroups $E_i \dots, i \in \{1, 2\}, \dots$ are self-centralizing in L and $N_L(E_i)$ is a transitive splitting extension of E_i by a subgroup isomorphic to A_5 . The group L possesses precisely one class of involutions and we have $C_L(\pi) = P$.

(2.2) It is well known that

$$\mathbf{Aut}(L) = \mathbf{Inn}(L) \cdot \Sigma \quad \text{with} \quad \Sigma \cap \mathbf{Inn}(L) = 1 \quad \text{and} \quad \Sigma \cong \Sigma_3 \times Z_2.$$

As a complement Σ of L in $\mathbf{Aut}(L)$ we may choose $\langle \varphi, \kappa, r \rangle$, where φ is induced by the field automorphism of $GL(3, 4)$, κ is induced by the transpose-inverse automorphism of $GL(3, 4)$, and r is induced by the element

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

of $GL(3, 4)$.

Then, $\varphi^2 = \kappa^2 = r^3 = 1$, $[\varphi, \kappa] = 1$, $\langle r, \varphi \rangle \cong \Sigma_3$, and $[r, \varphi\kappa] = 1$. Thus, $\Sigma = \langle r, \varphi \rangle \times \langle \varphi\kappa \rangle \cong \Sigma_3 \times Z_2$.

The subgroup P of L as described in (2.1) is Σ -invariant. The operations of φ, κ , and r on P are as follows:

$$\begin{aligned} \varphi: & \pi \rightarrow \pi, \tau \rightarrow \pi\tau, \mu \rightarrow \lambda, \lambda \rightarrow \mu, \zeta \rightarrow \zeta\xi, \xi \rightarrow \xi; \\ \kappa: & \pi \rightarrow \pi, \tau \rightarrow \tau, \mu \rightarrow \zeta\xi, \lambda \rightarrow \zeta, \zeta \rightarrow \lambda, \xi \rightarrow \mu\lambda; \\ r: & \pi \rightarrow \pi, \tau \rightarrow \tau, \mu \rightarrow \lambda \rightarrow \mu\lambda, \zeta \rightarrow \zeta\xi \rightarrow \xi. \end{aligned}$$

The subgroups E_1 and E_2 are normalized by $\langle r, \varphi \rangle$ and permuted by the action of κ .

(2.3) Let U be a subgroup of $\mathbf{Aut}(L)$ with $U \supseteq \mathbf{Inn}(L)$. Then, obviously, a S_2 -subgroup of U is isomorphic to $P, P\langle\varphi\rangle, P\langle\kappa\rangle, P\langle\varphi\kappa\rangle$ or $P\langle\varphi, \kappa\rangle$.

Set $U = \mathbf{Inn}(L) \cdot \Sigma_0$ with $\Sigma_0 = \Sigma \cap U$. We identify L with $\mathbf{Inn}(L)$. We have $P = E_1 E_2$, where $E_1 \cong E_2 \cong E_2$, and $C_U(E_i) = E_i$, for $i \in \{1, 2\}$. Let $N = N_U(E_i)/E_i$. Then, depending on Σ_0 , one of the following cases arises:

- (1) $N \cong A_5$, if $\Sigma_0 \cap \langle\varphi, r\rangle = 1$;
- (2) $N \cong \Sigma_5$, if $r \notin \Sigma_0$ and $\varphi, \varphi r$, or $\varphi r^2 \in \Sigma_0$;
- (3) $N \cong A_5 \times Z_3$ if $r \in \Sigma_0$ and $\varphi \notin \Sigma_0$;
- (4) $N \cong (A_5 \times Z_3)Z_2$ with $A_5 Z_2 \cong \Sigma_5$ and $Z_3 Z_2 \cong Z_3$ if $\langle r, \varphi \rangle \subseteq \Sigma_0$.

(2.4) By direct computation one obtains

$$\begin{aligned} C_P(\varphi) &= \langle \pi, \mu\lambda, \xi \rangle \cong D_8, \\ C_P(\kappa) &= \langle \pi, \tau \rangle \cong E_{2^4}, \\ C_P(\varphi\kappa) &= \langle \pi, \mu\lambda\xi\tau, \mu\zeta\tau \rangle \cong Q_8. \end{aligned}$$

Furthermore, for each $y \in \{\varphi, \kappa, \varphi\kappa\}$, all the involutions from the coset Py are conjugate to y under the action of P .

(2.5) The rank of $P\langle\varphi, \kappa\rangle$ is equal to 4. Further, E_1 and E_2 are the only elementary abelian subgroups of order 16 of $P\langle\varphi, \kappa\rangle$.

3. Some notations.

Throughout this paper let G denote a fixed group which satisfies the assumptions of Theorem 1. Let $a, b \in G$. Then $a: b \rightarrow c$ means $a^{-1}ba = c$. We shall write $N(A)$ and $C(A)$ for $N_G(A)$ and $C_G(A)$, respectively.

Let A denote a fixed standard subgroup of G where $A/Z(A) \cong \cong L_3(4)$. Then, A is isomorphic to a homomorphic image of the rep-

representation group of $L_3(4)$. Note that the representation group of a simple group is uniquely determined up to isomorphism.

Set $K = \mathbf{C}(A)$. Then $AK/K \cong L_3(4) \cong A/Z(A)$. Note that $[x, A] \subseteq \mathbf{Z}(A)$ for some $x \in G$ implies $x \in K$. Let $Q \in \text{Syl}_2(K)$. Since $\mathbf{Aut}(L)$ splits over L , and since $N(A)/K$ is isomorphic to a subgroup of $\mathbf{Aut}(L)$, we obtain the existence of a subgroup C of $N(A)$ such that $N(A)/K = C/K \cdot AK/K$ with $C \cap AK = K$. By Frattini's argument, we get $N_C(Q)K = C$. Clearly, $N_C(Q)/N_K(Q) \cong C/K$. We may choose a S_2 -subgroup S of A such that $N_C(Q)/N_K(Q)$ operates on $S/Z(A)_2$ in the same way as Σ_0 on P (in the former notation). Using the isomorphism $P \cong S/Z(A)_2$, we may put $S = \mathbf{Z}(A)_2 \langle \pi, \tau, \mu, \lambda, \zeta, \xi \rangle$, where the generators are all 2-elements satisfying the relations modulo $\mathbf{Z}(A)_2$ which we had derived for P above. We may «identify» $N_C(Q)/N_K(Q)$ with $\Sigma_0 \subseteq \Sigma = \langle \varphi, r, \kappa \rangle$, thus a S_2 -subgroup of $N_C(Q)$ could be Q , $Q\langle\varphi\rangle$, $Q\langle\kappa\rangle$, $Q\langle\varphi\kappa\rangle$, or $Q\langle\varphi, \kappa\rangle$, and if r is present, we have $r^3 \in N_K(Q)$. We get that a S_2 -subgroup of $N(A)$ is one of the following types: QS , $QS\langle\varphi\rangle$, $QS\langle\kappa\rangle$, $QS\langle\varphi\kappa\rangle$, $QS\langle\varphi, \kappa\rangle$.

In what follows, we denote by X a fixed S_2 -subgroup of $N(A)$. Furthermore, X is of type QS , $QS\langle\varphi\rangle$, $QS\langle\kappa\rangle$, $QS\langle\varphi\kappa\rangle$, $QS\langle\varphi, \kappa\rangle$. Hence, $X \cap A = S$ and $X \cap K = Q$. One notes that X/Q is isomorphic to P , $P\langle\varphi\rangle$, $P\langle\kappa\rangle$, $P\langle\varphi\kappa\rangle$, or $P\langle\varphi, \kappa\rangle$, where $P \in \text{Syl}_2(L_3(4))$ as given in (2.1).

(3.1) LEMMA. Let $y \in X \setminus QS$ with $y \in \{\varphi, \kappa, \varphi\kappa\}$. Let $z = uvy$ be an involution with $u \in Q$ and $v \in S$. Then $C_S(z) \cong C_S(y)$.

PROOF. We put $z = uvy$, $u \in Q$, $v \in S$. Note that $[Q, S] = 1$. We get

$$1 = z^2 = uvyuvy = uvyvy^{-1}uy = u(vy)^2u^v$$

and

$$u^{-1}u^{-v} = (vy)^2 = vy^2v^v = y^2(vv^v),$$

since $y^2 \in Q$. It follows

$$v(v^v) = y^{-2}u^{-1}u^{-v} \in Q \cap S.$$

We look now at $S\langle z \rangle/Q \cap S = S\langle uvy \rangle/Q \cap S$. Since $[u, S] = 1$, the element vy induces the same automorphism of S as z . Thus,

$$S\langle z \rangle/Q \cap S \cong S\langle vy \rangle/\langle Q \cap S, (vy)^2 \rangle = S\langle y \rangle/\langle Q \cap S, (vy)^2 \rangle,$$

where $\langle (vy)^2, y^2 \rangle \subseteq \langle Q \cap S, (vy)^2 \rangle \subseteq Q$; note that we have $y^2(vv^v) = u^{-1}u^{-v} = (vy)^2$ with $vv^v \in Q \cap S$. We see that $S\langle z \rangle / Q \cap S$ is isomorphic to $P\langle \varphi \rangle$, $P\langle \kappa \rangle$, or $P\langle \varphi\kappa \rangle$. From (2.4) we get that $\langle Q \cap S, (vy)^2 \rangle y$ is conjugate to $\langle Q \cap S, (vy)^2 \rangle vy$ under the action of $S\langle (vy)^2 \rangle / \langle S \cap Q, (vy)^2 \rangle$; this means that there is $s \in S$ such that $(vy)^s = yx$, $x \in \langle Q \cap S, (vy)^2 \rangle$. Hence, $C_s((vy)^s) = C_s(yx) = C_s(y)$, since $x \in Q$, and so, $C_s(z) = C_s(uvy) = C_s(vy) \sim_S C_s(y)$.

4. The structure of $N(A)$.

We follow the notation of the last section and look at $X \in \text{Syl}_2(N(A))$ with $X \cap K = Q \in \text{Syl}_2(K)$ and $X \cap A = S \in \text{Syl}_2(A)$.

(4.1) LEMMA. Q is elementary abelian and $Q \cap S$ is a four-group.

PROOF. By assumption we have $m(Q \cap S) \geq 2$. A result of Aschbacher [1] yields that Q is elementary abelian. This implies that $Q \cap S$ is a four-group, as the Schur-multiplier of $L_3(4)$ is isomorphic to $Z_4 \times Z_4 \times Z_3$.

Throughout this paper we set $|Q| = 2^n$ and $Q \cap S = \langle q_1, q_2 \rangle$.

(4.2) The structure of S .

It is well known that the sporadic simple group He contains a standard subgroup B , where $C(B) = Z(B) \cong Z_2 \times Z_2$, and $B/Z(B) \cong L_3(4)$. Hence, the S_2 -subgroup S of A in our case is isomorphic to a S_2 -subgroup H of B . The subgroup H contains exactly two elementary abelian subgroups H_1 and H_2 of order 2^6 ; we have $H = H_1H_2$ and $H_1 \cap H_2 = Z(H) = H' \cong E_{2^4}$.

Set $S = R_1R_2$ with $R_1 \cong E_{2^6} \cong R_2$ and $Z(S) = R_1 \cap R_2 \cong E_{2^4} \cong S'$. Let $\bar{R}_i$ for $i \in \{1, 2\}$ be the image of R_i under the homomorphism from S onto $S/\langle q_1, q_2 \rangle = \bar{S}$. The group $\bar{S}$ is isomorphic to a S_2 -subgroup of $L_3(4)$. It is clear that R_1 and R_2 both contain $\langle q_1, q_2 \rangle = Q \cap S$. Thus, $\bar{R}_1$ and $\bar{R}_2$ are the only elementary abelian subgroups of order 2^4 of $\bar{S}$. We may put

$$R_1 = \langle q_1, q_2, \pi, \tau, \mu, \lambda \rangle \quad \text{and} \quad R_2 = \langle q_1, q_2, \pi, \tau, \zeta, \xi \rangle,$$

where the nontrivial commutator relations mod $\langle q_1, q_2 \rangle$ are given by $[\mu, \zeta] = \pi$, $[\mu, \xi] = \pi\tau$, $[\lambda, \zeta] = \pi\tau$, $[\lambda, \xi] = \tau$. Since A is the epimorphic image of the full covering group of $L_3(4)$ modulo a charac-

teristic subgroup of the full covering group, we see that A possesses automorphisms which are «lifted-induced» by the field-, transpose-inverse, and field times transpose-inverse automorphism of $L_3(4)$; we use here the fact that every automorphism of a perfect group can be lifted to an automorphism of the full covering group (see Griess [7]).

We want to determine the multiplication table of $S = R_1 R_2$. Since the cosets $\langle q_1, q_2 \rangle \pi$ and $\langle q_1, q_2 \rangle \tau$ consist of involutions only, we may put without loss of generality $[\mu, \xi] = \pi\tau$ and $[\lambda, \xi] = \tau$. Since $S' = \langle q_1, q_2, \pi, \tau \rangle$, we have $[\mu, \zeta] = q\pi$ and $[\lambda, \zeta] = p\pi\tau$, where $q, p \in \langle q_1, q_2 \rangle$. Note that $[\mu\lambda, \xi] = [\mu\lambda, \xi]^\varphi = \pi^\varphi = \pi$ where φ comes from the field automorphism of $L_3(4)$. If $[q, \varphi] = 1$, then we get

$$q\pi = [\mu, \zeta] \xrightarrow{\varphi} [\lambda, \zeta\xi] = q\pi = \lambda\xi\zeta\lambda\zeta\xi = \lambda\xi\lambda(\lambda\zeta\lambda\zeta)\xi = \tau p\pi\tau = p\pi;$$

thus $p = q$. Hence, in this case, $[\mu, \zeta] = q\pi$ and $[\lambda, \zeta] = q\pi\tau$ which implies $|S'| = 8$ against $|S'| = 16$. Hence, $[q, \varphi] \neq 1$. In particular, $q \neq 1$ and $\langle q, q^\varphi \rangle = \mathbf{Z}(A)_2$.

Computing $[\mu, \zeta] = q\pi \xrightarrow{\varphi} [\lambda, \zeta\xi] = q^\varphi\pi = \lambda\xi\lambda(\lambda\zeta\lambda\zeta)\xi = p\pi$, we get $p = q^\varphi$. Put $q_2 = q$ and $q_1 q_2 = q^\varphi$. Then $[\mu, \zeta] = q_2\pi$ and $[\lambda, \zeta] = q_1 q_2 \pi\tau$. In this way we have obtained the multiplication table of S :

$$R_1 = \langle q_1, q_2, \pi, \tau, \mu, \lambda \rangle \cong R_2 = \langle q_1, q_2, \pi, \tau, \zeta, \xi \rangle \cong E_{2^*};$$

$$R_1 \cap R_2 = \langle q_1, q_2, \pi, \tau \rangle = \mathbf{Z}(R_1 R_2) = S' = \mathbf{D}(S),$$

$$[\mu, \xi] = \pi\tau, \quad [\lambda, \xi] = \tau, \quad [\mu, \zeta] = q_2\pi, \quad [\lambda, \zeta] = q_1 q_2 \pi.$$

We know already that $[q_1, \varphi] = 1$, $q_2^\varphi = q_1 q_2$, and $[\pi, \varphi] = 1$. Since $[\mu, \xi] = \pi\tau$ is mapped under φ onto $[\lambda, \xi] = \tau$, we get $\tau^\varphi = \pi\tau$. Hence, the action of φ on $\langle q_1, q_2, \pi, \tau \rangle$ is known.

We have $\pi^{\varphi\kappa} = [\mu\lambda, \xi]^{\varphi\kappa} = [\xi, \mu\lambda] = \pi$; $\tau^{\varphi\kappa} = [\lambda, \xi]^{\varphi\kappa} = [\zeta\xi, \mu\lambda] = q_1 \pi\tau$; $[\mu\lambda, \zeta]^{\varphi\kappa} = (q_1 \tau)^{\varphi\kappa} = [\xi, \mu] = \pi\tau$ and $q_1^{\varphi\kappa} \tau^{\varphi\kappa} = \pi\tau = q_1^{\varphi\kappa} q_1 \pi\tau$ which implies $q_1^{\varphi\kappa} = q_1$. Further, $[\mu, \zeta]^{\varphi\kappa} = (q_2 \pi)^{\varphi\kappa} = [\zeta, \mu] = q_2 \pi$ which implies $q_2^{\varphi\kappa} = q_2$. Hence, the action of $\varphi\kappa$ on $\mathbf{Z}(R_1 R_2)$ is known.

Acting with κ on appropriate commutators we get $\pi^\kappa = \pi$, $q_1^\kappa = q_1$, $\tau^\kappa = q_1 \tau$, $q_2^\kappa = q_1 q_2$. Hence, the action of κ on $\mathbf{Z}(R_1 R_2)$ is known.

We get $[\mu\lambda\xi, \lambda\zeta] = q_1 \neq 1$, $[\mu\zeta, \lambda\xi] = q_1 q_2 \neq 1$ and $[\mu\xi, \lambda\zeta\xi] = q_2 \neq 1$. In particular, we note that S does not contain any subgroup of type $(4, 4)$. Clearly

$$\langle q_1, q_2 \rangle \cap \mathbf{Z}(X) \supseteq \langle q_1 \rangle.$$

(4.3) LEMMA. We have—depending on X —

$$\varphi: \quad q_1 \rightarrow q_1, \quad q_2 \rightarrow q_1 q_2, \quad \pi \rightarrow \pi, \quad \tau \rightarrow \pi \tau;$$

$$\kappa: \quad q_1 \rightarrow q_1, \quad q_2 \rightarrow q_1 q_2, \quad \pi \rightarrow \pi, \quad \tau \rightarrow q_1 \tau;$$

$$\varphi\kappa: \quad q_1 \rightarrow q_1, \quad q_2 \rightarrow q_2, \quad \pi \rightarrow \pi, \quad \tau \rightarrow q_1 \pi \tau.$$

If $3 \mid |N(A)/AK|$, then there is an element $r \in N(A) \setminus AK$ which induces an outer automorphism of order 3 of $A/Z(A) \cong L_3(4)$ and of A . The element r operates fixed-point-free on $\langle q_1, q_2 \rangle$ and $r: \pi \rightarrow \rightarrow q_2 \pi \rightarrow q_1 q_2 \pi$.

PROOF. This follows immediately from the operations of $\varphi, \kappa, \varphi\kappa$, and r on $S/Q \cap S \cong P$.

(4.4) LEMMA. Set $S_i = QR_i$ for $i \in \{1, 2\}$. Then, S_1 and S_2 are the only elementary abelian subgroups of order 2^{n+4} of X .

PROOF. This follows from (4.1) and (4.2).

(4.5) LEMMA. The involutions in $\langle q_1, q_2 \rangle$ have no roots in S . If i is an involution of QS , then i lies in S_1 or S_2 . Further, i is conjugate to an involution in $Q\langle\pi\rangle$ under A .

PROOF. Let $s \in S^\#$ with $s^2 \in \langle q_1, q_2 \rangle$. Then Qs is an involution of QS/Q . Since QS/Q is of type $L_3(4)$, the involution Qs lies in QR_1/Q or QR_2/Q . Thus, s lies in QR_1 or QR_2 . But QR_i is elementary abelian. Hence $s^2 = 1$.

Let $i = qs$ with $q \in Q$ and $s \in S$ be an involution of QS . Then, $1 = i^2 = q^2 s^2$. Since Q is elementary abelian, we get $q^2 = s^2 = 1$. It follows that $s \in R_1$ or $s \in R_2$, so that $i = qs$ lies in QR_1 or QR_2 . Assume that i does not lie in Q . Then Qi is an involution of AQ/Q , and so, Qi is conjugate to $Q\pi$ under AQ/Q . The lemma is proved.

(4.6) LEMMA. Let $y \in X \setminus QS$ with $y \in \{\varphi, \kappa, \varphi\kappa\}$. We have $C_S(\varphi) \cong \cong Z_2 \times D_8$ with $\mathfrak{F}^1(C_S(\varphi)) = \langle \pi \rangle$ and $C_S(\varphi) \subseteq \langle q_1, q_2, \pi, \mu\lambda, \xi \rangle$; $C_S(\kappa) = = C_{Z(S)}(\kappa) = \langle q_1, \pi, q_2 \tau \rangle \cong E_{2^3}$; and $C_S(\varphi\kappa) = C_{Z(S)}(\varphi\kappa) = \langle q_1, q_2, \pi \rangle \cong \cong E_{2^3}$. Let z be an involution from QSy . Then $\mathfrak{F}^1(C_S(z)) = \langle \pi \rangle$, if $y = \varphi$; $C_S(z) = C_S(\kappa)$, if $y = \kappa$; and $C_S(z) = C_S(\varphi\kappa)$, if $y = \varphi\kappa$. For each $y \in \{\varphi, \kappa, \varphi\kappa\}$, all involutions from Sz are conjugate to z under the action of S .

PROOF. Since the field automorphism of $L_3(4)$ centralizes $L_3(2)$, we get $C_s(\varphi) \cong Z_2 \times D_8$, and, obviously, we must have $C_s(\varphi) \subseteq \langle q_1, q_2, \pi, \mu\lambda, \xi \rangle$. The latter group has exponent 4 and $(\mu\lambda\xi)^2 = \pi$. The assertion about $C_s(\kappa)$ is clear as the contragradient automorphism of $L_3(4)$ centralizes A_5 . We have $C_s(\varphi\kappa) \subseteq \langle q_1, q_2, \pi, \mu\lambda\xi\tau, \mu\zeta\tau \rangle$. Compute

$$(\mu\zeta\tau)^{\varphi\kappa} = q\zeta q^{\varphi\kappa} \mu q_1 \pi \tau = q q^{\varphi\kappa} \mu \zeta[\zeta, \mu] q_1 \pi \tau = q_1 q_2 \mu \zeta \tau,$$

and also

$$(\mu\lambda\xi\tau)^{\varphi\kappa} = \xi q \mu \lambda q^{\varphi\kappa} q_1 \pi \tau = q q^{\varphi\kappa} q_1 \mu \lambda \xi[\xi, \mu\lambda] \pi \tau = q_1 \mu \lambda \xi \tau,$$

where $q \in \langle q_1, q_2 \rangle$; and similarly one sees that $\mu\lambda\xi\tau\mu\zeta\tau$ is not centralized by $\varphi\kappa$. All other assertions follow from counting the involutions in Sz using our knowledge about the involutions of $P\varphi, P\kappa, P\varphi\kappa$ and the action of the outer automorphisms of A on $\langle q_1, q_2 \rangle$. The lemma is proved.

(4.7) LEMMA. The subgroup $N_A(R_i)$ for each $i \in \{1, 2\}$ is a splitting extension of $C_A(R_i)$ by a subgroup U_i isomorphic to A_5 . Furthermore, U_i operates transitively on $R_i / \langle q_1, q_2 \rangle$.

PROOF. This is a direct consequence of the structure of $L_3(4)$ and A .

5. The case $|X| = |G|_2$.

In this section all results will be proved under the title assumption.

(5.1) LEMMA. The group $\langle q_1, q_2 \rangle$ is strongly closed in QS with respect to G . If i is an involution of S and $i^x \in QS$, then $i^x \in S$ for any $x \in G$.

PROOF. We know that every involution of QS is conjugate to an involution of $Q\langle\pi\rangle$ under the action of $N(A)$. Note that $Q \times \langle\pi, \tau\rangle = Z(QS) = S_1 \cap S_2$, where S_1 and S_2 are the only elementary abelian subgroups of order 2^{n+4} of X ; clearly $S_1 S_2 = QS$.

We prove first that $\langle q_1, q_2 \rangle$ is strongly closed in QS with respect to G . Assume that this is not the case. Then, there is $u \in \langle q_1, q_2 \rangle^\#$ and $g \in G$ such that $u^g \in QS$ but $u^g \notin \langle q_1, q_2 \rangle$. Hence, $u^g = ws$, $w \in Q$,

$s \in S$, $w^2 = s^2 = 1$. If $s \in Q$, then $u^s = ws \in Q \cap Q^s \neq 1$ implying $g \in N(A)$, against $\langle q_1, q_2 \rangle \triangleleft N(A)$. Thus, we must have $s \notin Q$. Under the action of A , we have that Qs is conjugate to $Q\pi$. Thus, we may put $u^s = w\pi$, $w \in Q$. Obviously, $g \notin N(A)$. We have $\langle u, u^s \rangle \subseteq Z(QS)$ char QS char X , and $QS \subseteq C(u) \cap C(u^s)$; note that QS is generated by the maximal elementary abelian subgroups of X . Clearly, $(QS)^s \subseteq C(u^s)$. Hence, there is $x \in C(u^s)$ such that $QS = (QS)^{sx}$ and, obviously, $u^s = u^{sx}$. Thus, we may assume $g \in N(QS)$. But $g \notin N(A) = N(K)$. Put $V = Z(QS) = Q \times \langle \pi, \tau \rangle$. We have $V^s = V$ and $Q^s \cap Q \subseteq K \cap K^s = \langle 1 \rangle$. Thus, $2^{2n} = |Q \times Q^s| \leq |V| = 2^{n+2}$ implying $|Q| = 4$. Hence, $Q = \langle q_1, q_2 \rangle$, and so $S_i = R_i$ and $QS = R_1 R_2 = S$.

We know that $V = \langle q_1, q_2, \pi, \tau \rangle$ has order 16 and that $u^s = w\pi$ for some $g \in N(QS) \setminus N(A)$. We also know that $N(QS)$ controls fusion of the elements of V . Since $q_1 \in Z(X)$ and $X \subseteq N(QS)$, we see that q_1 has an odd number of conjugates under $N(QS)$ and $N(V)$, and since $C(q_1) \subseteq N(A)$, $g \notin N(A)$, we see that this number is not 1. Clearly, $q_1^s \notin Q = \langle q_1, q_2 \rangle$ as otherwise $g \in N(A)$ because of $q_1^s \in Q \cap Q^s \subseteq K \cap K^s$ and $|K \cap K^s| \equiv 1 \pmod{2}$. In a similar way one sees that each element of $Q^\#$ is conjugate under $N(QS)$ and $N(V)$ to an element of $V \setminus Q$. Note that no element of $\langle q_1, q_2 \rangle^\#$ has a root in QS , however, $\pi, q_2\pi$, and $q_1q_2\pi$ have the roots $\mu\lambda\xi$, $\mu\xi$, and $\lambda\xi\xi$ respectively, in QS ; note that $q_1\pi$ has no root in QS , because if $(\mu^m\lambda^l\xi^z\xi^x)^2 = q_1\pi$, then we would get $l = z = 1 = m = x$, but $(\mu\lambda\xi\xi)^2 = q_1\pi\tau$. Assume first that q_1 has no conjugate other than itself in Q . By the above remarks, we see that $q_1 \sim q_1\pi$ holds in $N(QS)$ and that q_1 is not conjugate to another element of $Q\pi$; note that $Q\pi \sim Q\tau \sim Q\pi\tau$ holds in A/Q . Thus, q_1 would have precisely four G -conjugates in V which is not possible. Hence, there is a conjugate of q_1 in Q other than q_1 . As the number of conjugates of q_1 under the action of $N(A)$ is odd, we get that all three elements of $Q^\#$ are conjugate; note that fusion of the elements of Q takes place precisely in $N(K) = N(A)$. Thus, q_1 has precisely 6 G -conjugates in V as $q_1 \sim q_2 \sim q_1q_2 \sim q_1\pi$ holds and $Q\pi \sim Q\tau \sim Q\pi\tau$ under A/Q . Since $6 \not\equiv 1 \pmod{2}$, we have obtained a contradiction which proves that $\langle q_1, q_2 \rangle$ is strongly closed in QS with respect to G .

Finally, let i be an involution of $S \setminus Q$ and assume that there is $x \in G$ such that $i^x \in QS$ but $i^x \notin S$. Conjugating with elements of A , we may and shall assume that i and i^x lie in $Z(QS) = V$. Since $N(QS)$ controls fusion of the involutions of V , there is $y \in N(V)$ such that $i = i^{xy}$. Clearly $y \in N(V) \setminus N(A)$. Thus, $Q \times Q^y \subseteq Z(QS)$ and $Q \cap Q^y = 1$; and so, we get $|Q| = 4$. This means that $Q = \langle q_1, q_2 \rangle \subseteq S = QS$. Thus, $i^x \in S$ which is a contradiction to our assumption. The lemma is proved.

(5.2) LEMMA. The assumption of this section is not possible.

PROOF. Assume false. We have $|X| = |G|_2$, and $\langle q_1, q_2 \rangle$ is strongly closed in QS with respect to G . Application of a result of Goldschmidt [5] yields that there is a conjugate of some element of $\langle q_1, q_2 \rangle$ in $X \setminus QS$; in particular, we get $X \supset QS$. Glauberman's Z^* -theorem [4] yields that q_1 has a conjugate in $X \setminus \langle q_1 \rangle$. We want to show that q_1 is conjugate to an element of $X \setminus QS$. Assume that this is false. Then q_1 is conjugate to an element of $\langle q_1, q_2 \rangle$ different from q_1 , say q . The conjugation $q_1 \sim q$ is performed in $N(A)$, and as q_1 is 2-central, we see that all involutions of $\langle q_1, q_2 \rangle$ are conjugate. Thus, q_1 must be conjugate to an element of $X \setminus QS$.

Let $z \in QSy$, where $z \sim q_1$ and $y \in \{\varphi, \kappa, \varphi\kappa\}$. Let $\tilde{X} \in \text{Syl}_2(C(z))$ such that $\tilde{X} \supseteq C_x(z)$. Denote by A_z the unique standard subgroup in $C(z)$ isomorphic to A . Put $\tilde{Q} = \tilde{X} \cap C(A_z)$ and $\tilde{S} = \tilde{X} \cap A_z$. Since $z \sim q_1$, we have $Q \sim \tilde{Q}$, $S \sim \tilde{S}$, and $z \in \tilde{Q} \cap \tilde{S}$. Clearly, $C(z) \cap N(A)$ does not possess a subgroup isomorphic to QS , since such a subgroup lies in KA , and so, z would centralize a S_2 -subgroup of A which is not the case. If there were a $q \in Q^\#$ such that $q \in \tilde{Q}\tilde{S}$, then $q \in \tilde{Q}$ and $C(q) \cap C(z) \supseteq \tilde{Q}\tilde{S}$ which is against $C(q) \subseteq N(A)$, namely: If $q \in Q$ would be conjugate to an element of $Q\pi \setminus Q$, then the conjugation $q \sim w\pi$, $w \in Q$, would be performed in $N(QS) \subseteq N(V)$; obviously, $w\pi \notin K$, and so, $N(QS) \not\subseteq N(A)$; this implies $Q = \langle q_1, q_2 \rangle$; but $\langle q_1, q_2 \rangle$ is strongly closed in QS with respect to G ; hence q is not conjugate to an element of $Q\pi \setminus Q$. We have shown that $q \notin \tilde{Q}\tilde{S}$ for each $q \in Q^\#$.

Let $z \in QS\varphi$. Then, we have $\mathfrak{U}^1(C_s(z)) = \langle \pi \rangle$. Because of $\mathfrak{U}^1(\tilde{X}/\tilde{Q}\tilde{S}) = \langle 1 \rangle$, we get $\pi \in \tilde{Q}\tilde{S}$, and this yields $\pi \in \tilde{S}$, since $\pi \in S$ and $S \sim \tilde{S}$. By an above remark, we have $q_1 \notin \tilde{Q}\tilde{S}$. Consider the coset $\tilde{Q}\tilde{S}q_1$. Obviously, $\tilde{Q}\tilde{S}q_1$ is conjugate in G to QSy for some $y \in \{\varphi, \kappa, \varphi\kappa\}$; note that $\tilde{X}$ contains $\tilde{Q}\tilde{S}\langle q_1 \rangle$ as a subgroup of index 1 or 2. An earlier result yields that all involutions of $\tilde{S}q_1$ are conjugate to q_1 under the action of $\tilde{S}$. Thus, $q_1 \sim_{\tilde{S}} q_1\pi$. But $\langle q_1, q_2 \rangle$ is strongly closed in QS with respect to G , and we have obtained a contradiction.

Let $z \in QS\kappa$. We have $C_s(z) = \langle q_1, \pi, q_2\tau \rangle$. Since $|\tilde{X}/\tilde{Q}\tilde{S}| \leq 4$, we get $C_s(z) \cap \tilde{Q}\tilde{S} \neq \langle 1 \rangle$. We know that $q_1 \notin \tilde{Q}\tilde{S}$. Hence, there exists s in $\langle q_1, \pi, q_2\tau \rangle \setminus \langle q_1 \rangle$ such that $s \in \tilde{Q}\tilde{S}$. Because of $s \in S \sim \tilde{S}$, we get $s \in \tilde{S}$. As q_1 and q_1s are involutions of $\tilde{S}q_1$, we get $q_1 \sim_{\tilde{S}} q_1s$. This is a contradiction, since $q_1s \in QS \setminus \langle q_1, q_2 \rangle$ and $\langle q_1, q_2 \rangle$ is strongly closed in QS with respect to G .

Let finally $z \in QS\varphi\kappa$. Then, $C_s(z) = \langle q_1, q_2, \pi \rangle$. If $|\tilde{X}/\tilde{Q}\tilde{S}| = 2$,

then a four-subgroup of $C_S(z)$ lies in $\tilde{Q}\tilde{S}$, and so, $\tilde{Q}\tilde{S}$ would contain an element of $Q^\#$ which is not possible. Suppose that $|\tilde{X}/\tilde{Q}\tilde{S}| = 4$. Then, $\tilde{X} = \tilde{Q}\tilde{S}\langle q_1, q_2 \rangle$ and $\tilde{Q}\tilde{S} \cap \pi\langle q_1, q_2 \rangle \neq \emptyset$. Let $q\pi$ be an element of $\tilde{Q}\tilde{S}$ with $q \in \langle q_1, q_2 \rangle$. Since $q\pi$ is an involution of S , we conclude $q\pi \in \tilde{S}$. Clearly, $\tilde{Q}\tilde{S}q_1$ is conjugate to QSy for some $y \in \{\varphi, \kappa, \varphi\kappa\}$. Thus, $q_1 \sim q_1q\pi$ under the action of $\tilde{S}$; note that $q_1q\pi \in QS \setminus \langle q_1, q_2 \rangle$. This contradicts the strong closure of $\langle q_1, q_2 \rangle$ in QS with respect to G . The lemma is proved.

6. The identification of G with He .

From the result of section 5 we know that X is not a S_2 -subgroup of G .

(6.1) LEMMA. We have $Q = \langle q_1, q_2 \rangle$.

PROOF. Let T be a subgroup of G containing X as a subgroup of index 2. Let $t \in T \setminus X$. Clearly, $t \notin N(A)$ and t normalizes QS and $Z(QS)$ which is equal to $Q \times \langle \pi, \tau \rangle$. This gives $Q \cap Q^t = \langle 1 \rangle$ and $Q \times Q^t \subseteq Z(QS)$, and so, $|Q| = 4$. The lemma is proved.

(6.2) LEMMA. The case $X = S\langle \kappa \rangle$ does not arise.

PROOF. Assume that $X = S\langle \kappa \rangle$. One computes $X'' = \langle [\mu\lambda\xi, \lambda\xi] \rangle = \langle q_1 \rangle$. This implies $N(X) \subseteq C(q_1) \subseteq N(A)$. This, however, is not the case.

(6.3) LEMMA. The case $X = S\langle \varphi\kappa \rangle$ does not arise.

PROOF. Assume $X = S\langle \varphi\kappa \rangle$. Then $Z(X) = \langle q_1, q_2, \pi \rangle$. Denote by T a subgroup of G which contains X as a subgroup of index 2. Let $t \in T \setminus X$. Then, $Q \cap Q^t = \langle 1 \rangle$ and $Q \times Q^t \subseteq Z(X)$ which is against $|Q| = 4$ and $|Z(X)| = 8$.

(6.4) LEMMA. We have $3 \mid |N(A)/KA|$. In particular, $q_1 \sim q_2 \sim q_1q_2$, and for $i \in \{1, 2\}$ we have

$$N_{N(A)}(R_i)/C(R_i) \cong \begin{cases} A_5 \times Z_3, & \text{if } X = S \\ (A_5 \times Z_3)Z_2, & \text{if } X \in \{S\langle \varphi \rangle, S\langle \varphi, \kappa \rangle\}, \end{cases}$$

here $A_5 \cdot Z_2 \cong \Sigma_5$ and $Z_3 \cdot Z_2 \cong \Sigma_3$.

PROOF. By way of contradiction assume that $3 \nmid |N(A)/KA|$. Then, $N(A) = AKX$ and $q_1 \in Z(N(A))$. Since K is tightly-embedded in G , we get $q_2 \sim q_1 \sim q_1q_2$ in G . Under the action of $N_A(S)$ the set $\langle q_1, q_2, \pi, \tau \rangle^\#$ splits into 7 conjugate classes with representatives $q_1, q_2, q_1q_2, \pi, q_1\pi, q_2\pi, q_1q_2\pi$. We know that $\pi, q_2\pi$, and $q_1q_2\pi$ have roots in S , whereas q_1, q_2, q_1q_2 , and $q_1\pi$ have no roots in S .

Let T be a subgroup of G containing X as a subgroup of index 2. Let $t \in T \setminus X$. Then, t normalizes $Z(S)$ and maps q_1 onto an element of $V \setminus Q$, where $V = Z(S) = \langle q_1, q_2, \pi, \tau \rangle$. It follows $q_1 \sim q_1\pi$ in G . Also, one gets that q_2 is mapped under t onto an element of $V \setminus Q$ which implies $q_2 \sim q_1\pi$. Hence, $q_1 \sim q_2$ which is not possible. Thus, we have that 3 divides the order of $N(A)/KA$.

From (2.3) and (4.3) we get $q_1 \sim q_2 \sim q_1q_2$ and $N_{N(A)}(R_i)/C(R_i)$ has the stated structure.

(6.5) LEMMA. The S_2 -subgroup X of $N(A)$ splits over S .

PROOF. By (6.1), (6.2), and (6.3) we have $X \in \{S, S\langle\varphi\rangle, S\langle\varphi, \kappa\rangle\}$. We may assume $X \supset S$. If $X = S\langle\varphi\rangle$, then $\langle q_1, q_2, \varphi \rangle \cong D_8$, and there is an involution in $X \setminus S$.

Let $X = S\langle\varphi, \kappa\rangle$. Clearly, $K = H \times Q$, where $H = O(N(A))$. We know that $N(A)/K = C/K \cdot KA/K \cong \mathbf{Aut}(L_3(4))$ and that $C/K \cong \mathbf{Out}(L_3(4))$. We look at C/H and determine the normalizer in C/H of an element r of order 3 in C/H . We get $|N(\langle r \rangle) \cap C/H| = 3 \cdot 4$, since r acts fixed-point-free on QH/H . Since $\langle r \rangle \in \text{Syl}_3(C/H)$ and $HQ\langle r \rangle/H \triangleleft C/H$, we get that a S_2 -subgroup of $N(\langle r \rangle) \cap C/H$ is a four-group. This means that a S_2 -subgroup of C splits over Q . Thus, $Q\langle\varphi, \kappa\rangle$ splits over Q , and so, $S\langle\varphi, \kappa\rangle$ splits over Q . The lemma is proved.

(6.6) LEMMA. The involution π is not conjugate to q_1 in G .

PROOF. We know that $X \in \text{Syl}_2(C(q_1))$, and we know that S is the Thompson-subgroup of X , i.e., S is generated by the elementary abelian subgroups of X of greatest possible order. Assume that $q_1^g = \pi$ for some $g \in G$. Then, $\langle S, S^g \rangle \subseteq C(q_1^g) = C(\pi)$, and so there is $x \in C(\pi)$ such that $S = S^{gx}$ and $q_1^{gx} = \pi$. It follows that q_1 and π are conjugate in $N(S)$. But q_1 has no root in S , whereas π has a root in S . Thus, $q_1 \sim \pi$.

(6.7) LEMMA. The case $X = S$ does not arise.

PROOF. Assume $X = S$. We have $\mathbf{Z}(X) = \mathbf{Z}(S) = \langle q_1, q_2, \pi, \tau \rangle$. We know that under the action of $N(S) \cap N(A)$ the set $\mathbf{Z}(S)^\#$ splits into conjugate classes in the following way: $3q_1, 3q_1\pi, 9\pi$; q_1 and $q_1\pi$ have no roots in S and a 3-element acts non-trivially on the coset $\langle q_1, q_2 \rangle \pi$. Clearly, $N(X) \supset N_{N(A)}(X)$, as $|X| < |G|_2$. Since $q_1 \sim \pi$ in G , we get $q_1 \sim q_1\pi$ in $N(X)$, as $q_1 \sim q_1q_2 \sim q_2$ takes place only in $N(A)$. Thus, q_1 has precisely 6 conjugates under the action of $N(X)$. Thus, $|N_{N(A)}(X) : N(X) \cap \mathbf{C}(q_1)| = 3$ and $|N(X) : N_{N(A)}(X)| = 2$.

Recall that R_1 and R_2 are the only elementary abelian subgroups of order 2^6 of X . We shall determine $N_G(R_i)$ for $i \in \{1, 2\}$. We have $N_{N(A)}(R_i)/\mathbf{C}(R_i) \cong A_5 \times Z_3$, where $N_A(R_i)/\mathbf{C}_A(R_i) \cong A_5$ operates transitively on R_i/Q . From (4.3) and (6.4) we get that under $N_{N(A)}(R_i)$ the involutions of R_i split into exactly three classes: $3q_1, 15q_1\pi, 45\pi$.

We have shown earlier that $q_1 \sim q_1\pi$ in $N(X)$ and $\pi \not\sim q_1$. Since $X \notin \text{Syl}_2(G)$, there is a 2-element t in $N(X) \setminus X$. Since $t \notin N(A)$, we get $q_1^t \in \langle q_1, q_2, \pi, \tau \rangle \setminus \langle q_1, q_2 \rangle$. If $R_1^t = R_2$ and $q_1 \sim q_1\pi$ in $N(R_i)$, a class of three elements would be mapped onto a class of 15 elements by t which is not possible. Hence, $q_1 \sim q_1\pi$ in $N(R_i)$ in any case for $i \in \{1, 2\}$. Thus, q_1 has precisely 18 conjugates and π precisely 45 conjugates in $N(R_i)$, $i \in \{1, 2\}$, and so, $N(R_i) \supset N(R_i) \cap N(A)$. Since

$$|N_{N(A)}(R_i) : \mathbf{C}_{N(R_i)}(q_1)| = 3 \quad \text{and} \quad |N(R_i) : \mathbf{C}_{N(R_i)}(q_1)| = 18,$$

we get $|N(R_i) : N_{N(A)}(R_i)| = 6$. Since $X \subseteq N(R_i)$, we get from an above result that $|N(X)|_2 = 2|X| = |N(R_i)|_2$.

Set $N_0 = N_{N(A)}(R_1)$, $N = N(R_1)$, $C = \mathbf{C}(R_1)$ and $O = \mathbf{O}(N(R_1))$. Then, $O \subseteq C$. Hence, $C = O \times R_1 \subseteq \mathbf{C}(q_1) \subseteq N(A)$ and $K = O \times Q$. We recall that $N_0/C \cong A_5 \times Z_3$. Denote by $\bar{x}, \bar{H}$ the images of $x \in N$, $H \subseteq N$, respectively, under the epimorphism $N \rightarrow N/O$. Let w be an element of order 5 of $A \cap N(R_1)$. Then, $\bar{w} \in \bar{N}_0$, $o(\bar{w}) = 5$. Note that $|\bar{N}/\bar{R}_1| = 2^3 \cdot 3^3 \cdot 5$. We have $\mathbf{C}_{\bar{R}_1}(\bar{w}) = \bar{Q}$ and $\bar{Q} \in \text{Syl}_2(\mathbf{C}(\bar{w}) \cap \bar{N}_0)$. Since $|\mathbf{C}(w) \cap \mathbf{C}(q_1)|_2 = 4$, and since the fusion of involutions of $\langle q_1, q_2 \rangle$ is controlled by $N(A)$, we see that $Q \in \text{Syl}_2(\mathbf{C}_O(w))$. We want to show that $|\mathbf{C}(\bar{w}) \cap \bar{N}|_2 = 4$. Note that $O \triangleleft N(R_1)$. Assume that there is a subgroup Q_1 of $N(R_1)$ which contains Q as a subgroup of index 2 such that $[w, Q_1] \subseteq O$. As $|Q_1 : Q| = 2$, we get $OQ_1 \langle w \rangle \subseteq N(A)$. But in $N(A)$, an element of order 5 of A does not centralize a subgroup of order 8. Thus, $|\mathbf{C}(\bar{w}) \cap \bar{N}|_2 = 4$. From the structure of $N(A)$ and $\bar{N}_0/\bar{C}$ we get $\mathbf{C}_{\bar{N}_0}(\bar{w}) \cong A_4 \times Z_5$. Obviously, $\bar{Q} \triangleleft \mathbf{C}_{\bar{N}}(\bar{w})$, and so,

$$O \times Q \triangleleft \mathbf{C}_X(w \bmod O) \subseteq N(A),$$

since $N(Q) \subseteq N(A)$. Hence,

$$C_{\overline{N}}(\overline{w}) \subseteq \overline{N(A)} \cap C(\overline{w})$$

which implies $|C_{\overline{N}}(\overline{w})| = 2^2 \cdot 3 \cdot 5$.

Let $\mathfrak{X}$ be a minimal normal subgroup of $\mathfrak{N} = N/C$; note that $|\mathfrak{N}| = 2^3 \cdot 3^3 \cdot 5$. If $\mathfrak{X}$ is not solvable, then $\mathfrak{X} \cong A_5$ or $\mathfrak{X} \cong A_6$. The fact that 3^2 does not divide $|C_{\overline{N}}(\overline{w})|$ yields $\mathfrak{X} \cong A_6$. Thus, $\mathfrak{X}$ is centralized by a group of order 3. This gives $O(\mathfrak{N}) \neq \langle 1 \rangle$. If $\mathfrak{X}$ is solvable, then $\mathfrak{X}$ is not a 2-group because of $|C_{\overline{N}}(\overline{w})|_2 = 4$. So, in any case, we must have $O(\mathfrak{N}) \neq \langle 1 \rangle$. The structure of $C_{\overline{N}}(\overline{w})$ yields $|O(\mathfrak{N})| = 3$. In particular, $\mathfrak{N}$ contains a chief factor isomorphic to A_6 .

Let $T \in \text{Syl}_2(N(R_1))$ with $T \supset X$. We have $|T:X| = 2$ and by assumption $R_1 R_2 = S = X$.

Hence, $T = S\langle t \rangle$ for any $t \in T \setminus S$. Since $N(R_1)/C(R_1)$ contains a chief factor isomorphic to A_6 and $|N(R_1)/C(R_1)|_2 = 2^3$, we get that T/R_1 is isomorphic to a dihedral group of order 8. We may therefore choose the element t so that $t^2 \in R_1$. Consider now the action of t on R_1 . We have $|C_{R_1}(\zeta)| = 2^4 = |C_{R_1}(\xi)|$, where $\zeta, \xi \in T \setminus R_1$. Since A_6 has only one class of involutions, we have $|C_{R_1}(t)| = 2^4$. Since K is tightly-embedded in G , we have $C(t) \cap Q = \langle 1 \rangle$ as $t \notin N(A)$. But t acts on $Z(S) = \langle Q, \pi, \tau \rangle$ as an involution, and so, by the Jordan-canonical-form, we must have $|C_{Z(S)}(t)| = 2^2$. Since $t \in N(R_1)$, we have $t \in N(R_2)$; note that $Z(S) = R_1 \cap R_2$. It follows

$$R_1 = Q \times C_{R_1}(t), \quad R_1 R_2 = R_2 C_{R_1}(t) \quad \text{and} \quad T = R_1 R_2 \langle t \rangle = R_2 C_{R_1}(t) \langle t \rangle.$$

Thus, T/R_2 is abelian of order 8. Working with $N(R_2)$ in the same way as we did with $N(R_1)$, we get $T/R_2 \cong D_8$. This is a contradiction proving the lemma.

(6.8) LEMMA. The case $X = S\langle \varphi, \kappa \rangle$ is not possible.

PROOF. Assume that $X = S\langle \varphi, \kappa \rangle$. From (6.5) we get that X splits over S . Hence, there are elements $\varphi' \in S\varphi$ and $\kappa' \in S\kappa$ such that $\langle \varphi', \kappa' \rangle$ is a four-group. Note that $S = R_1 R_2 \text{ char } X$, $Z(X) = \langle q_1, \pi \rangle$ and $N(X) \supset N_{N(A)}(X)$. We know that under $N_{N(A)}(S)$ the set $Z(S)^\#$ splits into exactly three conjugate classes with three conjugates of q_1 and $q_1 \pi$, each, and nine conjugates of π ; here $q_1 \pi$ is the only involution from $\langle q_1, q_2 \rangle \pi$ which has no roots in S . Since $q_1 \sim \pi$ in G , we get $q_1 \sim q_1 \pi$ in $N(X)$; since $N(X)$ normalizes S , we get $q_1 \sim$

$\sim q_1\pi$ in $N(S)$. It follows $|N(X):N_{N(A)}(X)| = 2$; note that $N_{N(A)}(X) \subseteq \langle C\langle q_1, \pi \rangle \rangle$. Also, we get $|N(S):N_{N(A)}(S)| = 2$. As in the proof of (6.7) and by the presence of κ , we get $|N(R_i):N_{N(A)}(R_i)| = 6$ for $i \in \{1, 2\}$. We remark that $S\langle\varphi\rangle \in \text{Syl}_2(N_{N(A)}(R_i))$ and

$$N_{N(A)}(R_i)/C(R_i) \cong (A_5 \times Z_3) \cdot Z_2.$$

Let $T \in \text{Syl}_2(N(X))$. Then, $T \in \text{Syl}_2(N(S))$, and obviously, $X \subset T$ with $|T:X| = 2$. Since $R_1^\kappa = R_2$, we have $|T:N_T(R_i)| = 2$. Hence, $|X| = |N_T(R_i)| = 2^{10} = |N(R_i)|_2$ and $N_T(R_i) \in \text{Syl}_2(N(R_i))$.

Consider $N(R_1)$. Set $\mathfrak{N} = N(R_1)/C(R_1)$. As in the proof of (6.7), we can show that $\mathfrak{N}$ possesses a chief factor isomorphic to A_6 and that $|\mathbf{O}(\mathfrak{N})| = 3$. It follows from the structure of $N(A)$ that a generator for $\mathbf{O}(\mathfrak{N})$ acts fixed-point-free on R_1 , and that $\mathbf{O}(\mathfrak{N})$ acts on Q . It is now possible to show that $\mathfrak{N}'$ is isomorphic to the tripple cover of A_6 and that $\mathfrak{N}/\mathbf{O}(\mathfrak{N}) \cong \Sigma_6$. Further, we get that a S_2 -subgroup of $N(R_1)$ is of type M_{24} . Let $Y \in \text{Syl}_2(N(R_1))$ with $Y \supset S\langle\varphi\rangle$. There is an involution $t \in Y$ such that $Y = S\langle\varphi, t\rangle$, since Y is generated by involutions. We have $X = S\langle\varphi, \kappa\rangle$ and we may put $T = S\langle\varphi, \kappa, t\rangle$. We show that $T \in \text{Syl}_2(G)$. Let R be an elementary abelian subgroup of T of order 2^6 such that $T = XR$. Then, $|X \cap R| = 2^5$. From the action of φ and κ on S we get that $|S \cap R| = 2^4$ is not possible. From the structure of X we get that there is no four-group in X intersecting S in $\langle 1 \rangle$ which centralizes an elementary abelian subgroup of order 8 of S . It follows $S = R_1 R_2 \text{ char } T$, and since $T \in \text{Syl}_2(N(S))$, we get that $T \in \text{Syl}_2(G)$. Remember that $|N(S):N_{N(A)}(S)| = 2$, and so, $N_{N(A)}(S) \triangleleft N(S)$. We know that mod H , $H = \mathbf{O}(N(A))$, the group $N(S) \cap N(A)$ is an extension of S by a group of type $(3,3)$ and by a four-group. Thus, $N(S) = \mathbf{O}_{2',2,3,2}(N(S))$ and $N(S)/\mathbf{O}_{2',2,3}(N(S))$ is a group of order 8 containing a four-subgroup. Clearly, $N(S)/\mathbf{O}_{2',2}(N(S))$ is faithful extension of an elementary abelian group of order 9 by a dihedral group of order 8, as otherwise we would get $T \subseteq N(A)$ which is not the case. By the fixed-point-free action of the 3-layer on OS/O , we see that T splits over S . Thus, there is a dihedral subgroup $\langle\varphi', t'\rangle\langle\kappa'\rangle$ of order 8 in T such that $S\langle\varphi', t'\rangle\langle\kappa'\rangle = T$; $\varphi' \in S\varphi$, $\kappa' \in S\kappa$, $t' \in St$; the elements φ', t', κ' are involutions, $S\langle\varphi', t'\rangle \in \text{Syl}_2(N(R_1))$, and $Z(\langle\varphi', t', \kappa'\rangle) = \langle\varphi'\rangle$; note that φ acts invertingly on $\mathbf{O}_{2',2,3}(N(S))/\mathbf{O}_{2',2}(N(S))$. Obviously, $\kappa' \sim \varphi'\kappa'$ by t' . We have that $C_T(\varphi'\kappa') = \langle q_1, q_2, \pi, \varphi'\kappa', \varphi' \rangle = W$ by (4.6). Now, $W' = \langle q_1 \rangle$ and this implies that W is a S_2 -subgroup of $C_G(\varphi'\kappa')$. However, by a transfer

lemma of J. Thompson, we get that in G the involution $\varphi'\kappa'$ must be conjugate to an involution of $S\langle\kappa't'\rangle$, and so, $\varphi'\kappa'$ must be conjugate to an element of $S\langle\varphi'\rangle = S\langle\varphi\rangle$. Representatives for the G -classes of involutions of $S\langle\varphi\rangle$ are π , q_1 , and φ . But the centralizers of these elements have larger S_2 -subgroups. This contradiction proves the lemma.

(6.9) LEMMA. The group G is isomorphic to He .

PROOF. From the preceding results, we have to consider finally the case in which $X = S\langle\varphi\rangle$, $\varphi^2 = 1$ and $C_G(q_1) = AKX$. Clearly, $H = O(C(q_1))$. Set $\mathfrak{C} = C(q_1)/H$. Then, $\mathfrak{C}$ is isomorphic to the centralizer of a non-central involution of He . A characterization of Deckers and Held leads to $G \cong He$.

The theorem is proved.

REFERENCES

- [1] M. ASCHBACHER, *On finite groups of component type*, Ill. J. M., **19** (1975), pp. 87-116.
- [2] B. BEISIEGEL, *Über einfache, endliche Gruppen mit Sylow-2-Gruppen der Ordnung höchstens 2^{10}* , Dissertation, Mainz University, 1975.
- [3] M. DECKERS, *On groups related to Held's simple group*, Arch. der Math., **25** (1974), pp. 23-28.
- [4] G. GLAUBERMAN, *Central elements in core-free groups*, J. Alg., **4** (1966), pp. 403-420.
- [5] D. M. GOLDSCHMIDT, *2-Fusion in finite groups*, Ann. of Math., **99** (1974), pp. 70-117.
- [6] D. GORENSTEIN, *Finite Groups*, Harper and Row, New York, 1968.
- [7] R. GRIESS, *Schur multipliers of finite simple groups of Lie type*, Trans. A.M.S., **183** (1973), pp. 355-421.
- [8] D. HELD, *The simple groups related to M_{24}* , J. Alg., **13** (1969), pp. 253-296.
- [9] B. HUPPERT, *Endliche Gruppen, I*, Springer, Berlin, New York, 1967.
- [10] A. REIFART, *A characterization of the sporadic simple group of Suzuki*, J. Alg., **33** (1975), pp. 288-305.
- [11] I. SCHUR, *Untersuchung über die Darstellungen der endlichen Gruppen durch gebrochene lineare Substitutionen*, J. Math., **132** (1907), pp. 85-137.
- [12] R. STEINBERG, *Automorphisms of finite linear groups*, Can. J. Math., **12** (1960), pp. 606-615.