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Supplemented nilpotent groups

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Supplemented Nilpotent Groups.

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1. Introduction.

In 1966, the same class of modules was characterized in two different papers using different terminology and notation [8, 10]. Collecting results from [8, 9, 10] and using terminology from [5], one obtains:

(1.1) THEOREM. The following properties of the projective module G are equivalent.

(1) G is supplemented.

(2) G is amply supplemented.

(3) The Jacobson radical $J(G)$ of G is small in G, $G/J(G)$ is semisimple, and decompositions of $G/J(G)$ can be lifted to G.

In this note we consider analogous conditions for a not necessarily abelian group G. Clearly, the analogue to the Jacobson radical is the Frattini subgroup $\Phi(G)$, both being the intersection of all maximal subobject. A normal subgroup N of G is *small* if the only subgroup S of G satisfying $G = N \cdot S$ is $S = G$. If, given any normal subgroup N of G, there exists a subgroup S of G such that $G = N \cdot S$ and $G \neq N \cdot T$ for all proper subgroups T of S, then G is called *supplemented*; G is *amply supplemented* if, given any pair of subgroups N and H of G with N normal and $G = N \cdot H$, there exists a subgroup S of H which is minimal with respect to $G = N \cdot S$. Replacing « subgroup » by « submodule », these definitions yield the module concepts of (1.1).

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The purpose of this article is to show that (1) and (2) above are equivalent for any nilpotent group G, and to relate this property to the smallness of $\Phi(G)$. We will prove that, for such G, the equivalent properties (1) and (2) are equivalent to each of the following; here G' denotes the commutator subgroup of G.

(i) G/G' is supplemented.

(ii) $G/\Phi(G)$ and $\Phi(G)$ are supplemented.

(iii) G is a torsion group whose maximal radicable subgroup $\varrho(G)$ is supplemented and has the property that $\Phi(G)/\varrho(G)$ is small in $G/\varrho(G)$.

(Cf. (4.7).) If, in addition, G is reduced and periodic, then the equivalent conditions (1) and (2) are equivalent to $\Phi(G)$ being small in G, thus providing an analogue to (1.1). Note that, for G a nilpotent torsion group, $G/\Phi(G)$ is always the direct sum of simple groups. However, even if G is abelian and supplemented, decompositions of $G/\Phi(G)$ cannot, in general, be lifted to G.

These results will follow from our structure theorem (4.5) for nilpotent supplemented groups: the nilpotent group G is supplemented if and only if G is a torsion group whose Sylow p-subgroups are extensions of abelian groups of finite rank by groups of finite exponent. It follows that the class of nilpotent supplemented groups is closed with respect to subgroups. This is not the case for supplemented groups in general since every simple group is supplemented and not all such groups are periodic; as was announced by P. A. Birjukov in [3], every abelian supplemented group is torsion.

The material is organized as follows. Section 2 contains an auxiliary result we were unable to locate in the literature. In Section 3 we consider a certain class $\mathfrak{X}$ of groups which contains all nilpotent as well as some centerless groups, and furnish a criterion which is sufficient for an $\mathfrak{X}$-group to be supplemented. Section 4 is devoted to our structure theorem and various equivalent characterizations of the nilpotent supplemented groups.

2. Preliminaries.

Throughout, G will be a multiplicative group. Notation and terminology will follow [14] and [4]. Center, commutator, and Frattini subgroup of G will be denoted by $Z(G)$, G', and $\Phi(G)$, respectively.

We write $S \leqslant G$ if S is a subgroup, and $N \lhd G$ if N is a normal subgroup of G. A group is called *artinian* if it satisfies the minimum condition for subgroups. For the structure of abelian artinian groups see [4; p. 110, 25.1].

The following result will be needed. Here, $Z_n(G)$ denotes the n-th term in the ascending central series of G [14; p. 2].

(2.1) LEMMA. For any non-negative integer n, there exists an imbedding

$$Z_{n+2}(G)/Z_{n+1}(G) \hookrightarrow \operatorname{Hom}\big(G/G', Z_{n+1}(G)/Z_n(G)\big) .$$

PROOF. For $a \in Z_{n+2}(G)$, the map

$$\varphi_a \colon G/G' \to Z_{n+1}(G)/Z_n(G)$$

defined by

$$\varphi_a(xG') = [x, a]Z_n(G) , \qquad x \in G ,$$

is a homomorphism into $Z_{n+1}(G)/Z_n(G)$ [14; p. 5, 1.13]; and

$$\psi \colon a \mapsto \varphi_a , \qquad a \in Z_{n+2}(G) ,$$

is a homomorphism from $Z_{n+2}(G)$ into $\operatorname{Hom}\big(G/G', Z_{n+1}(G)/Z_n(G)\big)$. Since $Z_{n+1}(G)$ is the kernel of ψ, the lemma follows.

3. Groups with small commutator subgroups.

Baer [1, 2] calls a normal subgroup N of a group G « Frattini imbedded » if $G = N \cdot S$ for S a subgroup of G implies $S = G$. We will use module theoretical terminology instead and call such N *small*, in symbol

$$N \ll G .$$

It is well known that every small normal subgroup of G is contained in the Frattini subgroup $\Phi(G)$. However, $\Phi(G)$ need not be small [1; p. 612], nor need it be the product of all small normal subgroups [2; p. 219].

For convenience, we collect a few simple results on small normal subgroups.

If N, M, and K are normal subgroups of the group G then

(3.1) $N \ll G,\ M \leqslant N \Rightarrow M \ll G$ [10; p. 87, 1.1];

(3.2) $N \ll G \Rightarrow (N \cdot K)/K \ll G/K$ [1; p. 611, 1.1];

(3.3) $N/M \ll G/M,\ M \ll G \Rightarrow N \ll G$ [1; p. 611, 1.1].

If G is nilpotent then $G' \ll G$ [14; p. 4, 1.8]. For the most part, it is this property, rather than nilpotency, which we will need in our proofs.

Throughout the following, p is some fixed prime.

(3.4) LEMMA. Let G be a p-group such that $G' \ll G$. Then $\Phi(G) = G^p G'$. If N is a normal subgroup of G then $N \ll G$ is and only if $N \leqslant G^p G'$ and $(NG')/G'$ has finite exponent.

PROOF. The smallness of G' implies that every maximal subgroup of G contains G'. Hence [4; p. 18, Exercise 4]

$$\Phi(G)/G' = \Phi(G/G') = (G/G')^p = (G^p G')/G' ,$$

proving the first part. For the second part observe that, by (3.2) and (3.3),

$$N \ll G \Leftrightarrow (NG')/G' \ll G/G' .$$

The latter being equivalent to

$$(NG')/G' \leqslant (G/G')^p = (G^p G')/G'$$

and

$$[(NG')/G']^{p^k} = 1 \qquad \text{for some } k \in \mathbf{Z}$$

[6; 5.5] completes the proof.

(3.5) COROLLARY. If G is a p-group such that $G' \ll G$ then $G^p G' \ll G$ if and only if G/G' has finite exponent.

Combining this with [14; p. 4, 1.8, and p. 9, 3.2] we obtain

(3.6) COROLLARY. If G is a nilpotent p-group then $G^p G' \ll G$ if and only if G has finite exponent.

Let N be a normal subgroup of G. A supplement of N (in G) is any subgroup S of G such that $G = N\cdot S$; a *minimal supplement* of N is a minimal element in the poset of all supplements of N. The following connection between minimal supplements and small subgroups will be called upon heavily.

(3.7) [10; p. 87, 1.3] $S \leqslant G$ is a minimal supplement of $N \lhd G$ if and only if $N\cdot S = G$ and $N \cap S \ll S$.

As mentioned earlier, a group G is called *amply supplemented* if, given any normal subgroup N of G, every supplement of N in G contains a minimal supplement. Clearly, amply supplemented groups are supplemented. Concerning the converse, we have

(3.8) LEMMA. If all subgroups of the group G are supplemented then G is amply supplemented.

The simple proof is left to the reader.

A criterion for a group to be amply supplemented is contained in

(3.9) PROPOSITION. Let G be a p-group such that, for every subgroup X of G, $X^pX' \ll X$. Then G is amply supplemented.

REMARK. By (3.1) and (3.4), $X^pX' \ll X$ implies $\Phi(X) \ll X$. However, these two properties are not equivalent: the « Tarski Monster » whose existence was recently established by E. Rips [11] is an infinite group of exponent p which is equal to its own commutator subgroup even though the Frattini subgroup of each of its subgroups is trivial.

PROOF OF (3.9). Let $N \lhd G$ and let $S \leqslant G$ such that $G = N\cdot S$. Put $G^pG' = \Phi$. Since G/Φ is elementary abelian, there exist $x_i \in S$ such that

$$G/\Phi = (N\Phi)/\Phi \oplus \bigoplus_{i\in I} \langle x_i\Phi\rangle , \qquad x_i\Phi \neq 1 .$$

Let $X = \langle x_i | i \in I\rangle$. Then $X \leqslant S$ and, furthermore,

(3.10) $G = N\Phi X = NX$,

(3.11) $N \cap X \leqslant N \cap X \cap \Phi \leqslant X \cap \Phi$.

Let $x \in X \cap \Phi$. Then

$$x = x_{i_1}^{n_1} \dots x_{i_k}^{n_k} y , \qquad i_j \in I,\ n_j \in \mathbf{Z} ,$$

where $y \in X'$ and the i_j are pairwise distinct. Since $x \in \Phi$, each ex ponent is a multiple of p. Let $m_j \in \mathbb{Z}$ such that

$$n_j = pm_j\,, \quad j = 1, \ldots, k\,.$$

Then, for some $y' \in X'$,

$$x = x_{i_1}^{pm_1} \ldots x_{i_k}^{pm_k} y = (x_{i_1}^{m_1} \ldots x_{i_k}^{m_k})^p y' y \in X^p X'\,.$$

We have shown that $X \cap \Phi = X^p X'$ which, using the hypothesis together with (3.1) and (3.11), implies $N \cap X \ll X$. The conclusion now follows from (3.10) and (3.7).

(3.12) COROLLARY. If G is a p-group of finite exponent such that $X' \ll X$ for every subgroup X of G then G is (amply) supplemented.

PROOF. (3.5) and (3.9).

For the following, let $\mathfrak{X}$ denote the class of all groups G such that, for every subgroup X of G, $X' \ll X$. Clearly, every nilpotent group belongs to $\mathfrak{X}$ [14; p. 4, 1.8]. That $\mathfrak{X}$ properly contains the class of nilpotent groups can be seen from the group constructed by Heineken and Mohamed [7]: this is a metabelian p-group (p any prime) with small commutator subgroup and trivial center all proper subgroups of which are nilpotent.

A consequence of (3.2) is:

(3.13) $\mathfrak{X}$ is closed with respect to subgroups and epimorphic images.

If $S \in \mathfrak{X}$ such that S/S' is cyclic then

$$S = S' \cdot \langle s \rangle$$

for some $s \in S$ and, hence, $S = \langle s \rangle$ is abelian. By [13; p. 1408, Theorem 2], an artinian group G is nilpotent if, for every non-abelian subgroup S of G, S/S' is non-cyclic. Thus:

(3.14) Artinian $\mathfrak{X}$-groups are nilpotent.

We are ready to prove the main result of this section.

(3.15) THEOREM. Let G be a p-group such that (i) $X' \ll X$ for every subgroup X if G; and (ii) there exists an integer k such the G^{p^k} is artinian. Then G is supplemented.

PROOF. A nilpotent artinian group is a finite extension of a radicable abelian group of finite rank [12; p. 69, 3.14]. Thus, by (3.14), any group G satisfying the hypothesis of (3.15) contains a radicable abelian artinian subgroup $\varrho(G)$ such that $G/\varrho(G)$ has finite exponent; and $\varrho(T) \leqslant \varrho(G)$, for every subgroup T of G. Set $R = \varrho(G)$ and let n denote the rank of R. We will induct on n. If $n = 0$ the conclusion follows form (3.12). Suppose $n \geqslant 1$ and the proposition holds for groups with maximal radicable subgroups of rank less then n. Let $N \lhd G$. By (3.13) and (3.12), G/R is supplemented. Thus, there exists $R \leqslant M \leqslant G$ such that

$$G/R = (NR)/R \cdot M/R$$

and

$$[(N \cap M)R]/R = (NR)/R \cap M/R \ll M/R \,, \tag{3.16}$$

using (3.7). In particular,

$$G = NRM = NM \,.$$

If $N \cap M \ll M$, (3.7) completes the proof. Suppose that $N \cap M$ is not small in M. Then there exists $T \leqslant M$ such that

$$M = (N \cap M) \cdot T \,, \qquad T \lneqq M \,.$$

Because of (3.16), $R \nleqslant T$. Hence

$$\varrho(T) \lneqq \varrho(G) = R \,,$$

and $\varrho(T)$ has rank less than n. The induction hypothesis implies the existence of $Q \leqslant T$ such that

$$T = (N \cap M \cap T) \cdot Q = (N \cap T) \cdot Q$$

and

$$N \cap Q = N \cap T \cap Q \ll Q$$

which, because of (3.7) and

$$G = N \cdot M = N(N \cap M)T = N(N \cap T)Q = NQ$$

completes the proof.

Combining (3.8) and (3.15), we obtain

(3.17) COROLLARY. Let G be a p-group such that $X' \ll X$ for every subgroup X of G. If G is the extension of an artinian group by a group of finite exponent then G is amply supplemented.

4. Nilpotent supplemented groups.

Throughout the following, G is a nilpotent group. In [6; 4.9] we proved that supplemented nilpotent groups are periodic. Hence [14; p. 19, 4.3], G is supplemented if and only if

$$G = \bigoplus_p G_p$$

is the direct sum (i.e. weak direct product) of supplemented nilpotent p-groups, one for each prime p. In order to characterize the nilpotent supplemented groups it therefore suffices to consider p-groups.

For convenience of notation, fix p and denote by $\mathfrak{P}$ the class of all nilpotent p-groups which are the extension of an artinian abelian group by a group of finite exponent. Let $\varrho(G)$ denote the unique maximal radicable subgroup of G [14; p. 20 *f*]. Then $G \in \mathfrak{P}$ if and only if (i) $G^{p^n} = \varrho(G)$ for some integer n, and (ii) $\varrho(G)$ is a direct sum of finitely many copies of $Z(p^\infty)$.

Closure properties again will be of interest.

(4.1) PROPOSITION. $\mathfrak{P}$ is closed with respect to subgroups, epimorphic images, and nilpotent extensions.

PROOF. Closure with respect to subgroups and quotient groups follows from the corresponding properties of nilpotent and abelian artinian groups. For closure under extensions, let $N \lhd G$ such that both N and G/N are in $\mathfrak{P}$. Then, by the part just proved,

$$N \cap Z(G) \in \mathfrak{P}\,,$$

$$G/N \geqslant (Z(G)N)/N \simeq Z(G)/[N \cap Z(G)] \in \mathfrak{P}\,,$$

$$(NG')/G' \in \mathfrak{P}\,,$$

and

$$(G/G')/[(NG')/G'] \simeq G/(NG') \in \mathfrak{P}\,.$$

The class of abelian $\mathfrak{P}$-groups is closed with respect to (abelian) extensions. Consequently, $Z(G) \in \mathfrak{P}$ and $G/G' \in \mathfrak{P}$. The conlusion will follow once we show that $G/Z(G)$ has finite exponent. Since G is nilpotent, $G = Z_m(G)$ for some integer m; and $G/G' \in \mathfrak{P}$ implies that, for every abelian group A, all torsion subgroups of Hom $(G/G', A)$ are bounded [4; p. 182 *f*, *F*, 43.1, 43.3]. Combining this with (2.1) completes the proof.

(4.2) Corollary. If G is a nilpotent group such that $G/G' \in \mathfrak{P}$ then $G \in \mathfrak{P}$.

Proof. The class of abelian $\mathfrak{P}$-groups satisfies the hypotheses of [14; p. 10, 3.8]. Thus, G is a poly-$\mathfrak{P}$-group, and $G \in \mathfrak{P}$ by (4.1).

We can now determine the supplemented nilpotent p-groups. Note that, by (3.2) and (3.7).

(4.3) Epimorphic images of supplemented groups are supplemented.

(4.4) Theorem. A nilpotent p-group G is supplemented if and only if G is the extension of an abelian artinian group by a group of finite exponent.

Proof. $G \in \mathfrak{P}$ implies G supplemented by (3.15) and [4; p. 4, 1.8]. Conversely, using (4.3) and [6; 5.8], $G/G' \in \mathfrak{P}$ if G is supplemented. Apply (4.2).

Using the remark at the beginning of this section, we can now state our structure theorem.

(4.5) Theorem. The nilpotent group G is supplemented if and only if G is a torsion group whose Sylow p-subgroups are extensions of abelian artinian groups by groups of finite exponent.

(4.6) Corollary. The class of nilpotent supplemented groups is closed with respect to subgroups, epimorphic images, and nilpotent extensions.

Different characterizations of nilpotent supplemented groups are contained in

(4.7) Theorem. The following properties of the nilpotent group G are equivalent.

(1) G is amply supplemented.

(2) G is supplemented.

(3) G/G' is supplemented.

(4) $G/\varrho(G)$ and $\varrho(G)$ are supplemented.

(5) $\Phi(G)$ and $G/\Phi(G)$ are supplemented.

(6) G is a torsion group such that $\varrho(G)$ is supplemented and $\Phi(G)/\varrho(G)$ is small in $G/\varrho(G)$.

(7) G is a torsion group whose Sylow p-subgroups are extensions of abelian artinian groups by groups of finite exponent.

PROOF. (1) and (2) are equivalent by (3.8) and (4.6); the equivalence of (2), (4), (5), and (7) follows from (4.6) and (4.5); clearly, (3) follows from (2), and (7) follows from (3) because of [6; 5.17], [4; p. 11, 3.13] and (4.2). Only the equivalence of (4) and (6) remains to be shown. For this, note that by (4.5) in either case G is a torsion group. If G_p denotes the Sylow p-subgroup of G then

$$G = \bigoplus_p G_p ,$$

and there exists a natural isomorphism

$$G/\varrho(G) \simeq \bigoplus_p [G_p/\varrho(G_p)] \tag{4.8}$$

inducing an isomorphism

$$\Phi(G)/\varrho(G) \simeq \bigoplus_p [(G_p^p G_p')/\varrho(G_p)] .$$

Thus, $\Phi(G)/\varrho(G) \ll G/\varrho(G)$ if and only if, for each prime p,

$$(G_p^p G_p')/\varrho(G_p) \ll G_p/\varrho(G_p) .$$

By (3.6), this is equivalent to $G_p/\varrho(G_p)$ having finite exponent. Observing (4.8) and (4.5), we have shown that $\Phi(G)/\varrho(G) \ll G/\varrho(G)$ if and only if $G/\varrho(G)$ is supplemented. Thus, (4) and (6) are equivalent and the theorem is proven.

Under the additional hypothesis that G be reduced and periodic we can state the following result containing the group version of (1.1).

(4.9) COROLLARY. The following properties of the reduced nilpotent torsion group G are equivalent.

(1) G is supplemented.

(2) G is amply supplemented.

(3) $\Phi(G)$ is small in G.

(4) $\Phi(G)$ is supplemented.

(5) The Sylow p-subgroups of G have finite exponents.

Given any subgroup B of the abelian supplemented group A, there exists a minimal supplement of B which is a direct summand of A (However, not every minimal supplement of B is necessarily a direct summand.) [6; 5.17]. This raises the question whether, in nilpotent supplemented groups, minimal supplements can be chosen to be (if not direct summands) at least normal subgroups. That this need not be the case can be seen from the following

EXAMPLE. Let $A = \langle a \rangle \oplus \langle b \rangle$ with a and b of order p^2, p a prime, let $\gamma \in \operatorname{Aut} A$ such that

$$x^\gamma = x^{1+p} \quad \text{for all } x \in A ,$$

and let $G = A \cdot \langle \gamma \rangle$ semi-directly. Suppose that M is any minimal supplement of A. Then M is cyclic [6; 3.4], and $M = \langle x\gamma \rangle$ for some $x \in A$. Assume $M \lhd G$. Then, for each $y \in A$,

$$y(x\gamma)y^{-1} = x\gamma y^p \in M ,$$

which implies $y^p \in M$ for all $y \in A$. Hence $A^p \leqslant M$, contradicting the fact that M is cyclic.

REFERENCES

[1] R. BAER, *An essay on Frattini imbedded normal subgroups*, Comm. Pure Appl. Math., **26** (1973), pp. 609-658.

[2] R. BAER, *Frattini-eingebettete Normalteiler II*, Math. Z., **139** (1974), pp. 205-223.

[3] P. A. BIRJUKOV, *On a certain class of Abelian torsion groups*, Notices Amer. Math. Soc., **25** (1978), p. A-418.

[4] L. FUCHS, *Infinite Abelian Groups*, Vol. I, Academic Press, New York, 1970.

[5] J. S. GOLAN, *Quasiperfect modules*, Quart. J. Math. Oxford, **22** (1971), pp. 173-182.

[6] J. Hausen, *Groups whose normal subgroups have minimal supplements*, Arch. Math. (Basel), **32** (1979), pp. 213-222.
[7] H. Heineken - I. J. Mohamed, *A group with trivial centre satisfying the normalizer condition*, J. Algebra, **10** (1968), pp. 368-376.
[8] F. Kasch - E. A. Mares, *Eine Kennzeichnung semi-perfekter Moduln*, Nagoya Math. J., **27** (1966), pp. 525-529.
[9] E. A. Mares, *Semi-perfect modules*, Math. Z., **82** (1963), pp. 347-360.
[10] Y. Miyashita, *Quasi-projective modules, perfect modules and a theorem for modular lattices*, J. Fac. Sci. Hokkaido Univ., **19** (1966), pp. 86-110.
[11] E. Rips, Lectures at Warwick, 17-19 June '78, unpublished notes by F. Pickel.
[12] D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups*, Vol. I, Springer-Verlag, Heidelberg, 1972.
[13] H. Simon, *Artinian and noetherian hypercentral groups*, Proc. Amer. Math. Soc., **17** (1966), pp. 1407-1409.
[14] R. B. Warfield Jr., *Nilpotent Groups*, Lecture Notes in Mathematics, Vol. 513, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

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