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# Bifurcation of Closed Orbits from a Limit Cycle in $R^2$.

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## 1. Introduction.

The theory of bifurcation for differential equations is motivated, as is well known, by the problems which arise in Celestial Mechanics, Fluidodynamics, Nonlinear Oscillations, Biomathematics and many other fields. After Poincaré and Liapunov set the bases, many scientists like Hopf, Andronov, Leontovich, Bogoliubov developed very extensively such theory by giving results which found many important uses in physics and engineering (for references see e.g. [1]).

In this paper we will be concerned with a particular bifurcation problem considered by Andronov et al. in [2]. Precisely let

$$(A_\mu) \qquad \begin{aligned} \dot{x} &= P(\mu, x, y)\,, \\ \dot{y} &= Q(\mu, x, y)\,, \end{aligned}$$

be a one parameter family of differential systems in $R^2$, $\mu \in (-\bar{\mu}, \bar{\mu})$, $\bar{\mu} > 0$, $P, Q \in C^h[(-\bar{\mu}, \bar{\mu})\times R^2, R^2]$, $h > 4$, and let us suppose that

$$(A_0) \qquad \begin{aligned} \dot{x} &= P(0, x, y) =: P(x, y)\,, \\ \dot{y} &= Q(0, x, y) =: Q(x, y)\,, \end{aligned}$$

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has a closed orbit $L_0$ which is not an equilibrium position. We want to consider the problem of existence of closed orbits for $(A_\mu)$, which tend, in a suitable sense, to $L_0$ as $\mu$ tends to 0. This is the so called problem of bifurcation of periodic solutions from a periodic solution. Such problem is not analogous to Hopf bifurcation (bifurcation of periodic solutions from an equilibrium position) essentially because

1) The existence of a closed orbit of $(A_0)$ does not imply, in general, the existence of a closed orbit of $(A_\mu)$, in the neighborhood of $L_0$, for $|\mu|$ small enough, whereas in Hopf bifurcation one can suppose that the origin is an equilibrium position also for the perturbed systems;

2) The orbit $L_0$ could be semistable, that is attracting for inside orbits and repulsing for outside orbits or viceversa.

Andronov et al. [2] consider the previous problem for two kinds of analytical systems $(A_\mu)$ and they look for the zeros of the so called displacement function relative to $L_0$, by using Newton's polygon. For these two kinds of systems, different conditions, expressed by means of the coefficients in Taylor development of the displacement function, are satisfied. Such conditions lead them to obtain interesting results expressed by means of Ths. 71, 72, 73. The first two theorems are concerned with one kind of systems and are relative to the cases in which $L_0$ is an attracting, repulsing or semistable limit cycle for the umperturbed system. The third one is concerned with the other kind of systems and only the case in which $L_0$ is an attracting or repulsing limit cycle is considered.

In [3] a general definition [III, Def. 1.1] of bifurcation for a family $\{M_\mu\}$ of invariant sets is given, that is a bifurcation occurs at $\mu = 0$ when a new family of invariant sets $\{M'_\mu\}$ arises such that $M'_\mu \cap M_\mu$ is empty and $M'_\mu$ tends, in a suitable sense, to $M_0$ as $\mu$ tends to 0. Also in [3] a theorem [III, Th. 1.3] is given in which bifurcation phenomenon is related to a drastic change of stability properties of the family $\{M_\mu\}$ at $\mu = 0$. This result was used in [4] to study the attractivity properties of bifurcating periodic orbits in Hopf bifurcation. There, the sets $M_\mu$ are taken coincident with the origin for every $\mu$ and the trasversality condition assures the change of stability properties at $\mu = 0$.

The aim of this paper is to show how it is possible to approach the problem of bifurcation of periodic orbits from a periodic orbit by using the point of view adopted in [3, 4]. Therefore we prove that under suitable conditions (which are satisfied in the cases considered in [2]) there exist families $\{M_\mu\}$ of invariant sets for which there is a drastic change of stability properties at $\mu = 0$. Then, the mentioned theorem, given in [3], ensures the existence of bifurcating sets $\{M'_\mu\}$ which are, in our case, annular regions, close to $L_0$, which have as boundary periodic orbits of $(A_\mu)$. Also the same theorem gives us informations about attractivity properties of the sets $M'_\mu$. Such properties will coincide with the properties of bifurcating closed orbits if the annular regions, which constitue $M'_\mu$, shrink to just one closed orbit. This happens when the hypotheses of Ths. 71, 72, 73 of [2] are satisfied. Thus, the results given in [2] can be interpreted as particular bifurcations of families of invariant sets under drastic changes of stability properties through $\mu = 0$. Our procedure seems, therefore, to replace fruitfully the computational methods used in [2]; further it allows us to complete the analysis made in [2]. In fact we are able to solve also the bifurcation problem which arises when, for the second kind of systems considered in [2], $L_0$ is supposed to be a semistable limit cycle.

## 2. Preliminaries.

Let

$$\begin{aligned} x &= \varphi(t) \\ y &= \psi(t) \end{aligned} \tag{2.1}$$

be the equations of the periodic solution of $(A_0)$ corresponding to $L_0$ and $\tau > 0$ the smallest positive period of (2.1). Let us introduce as in [2] a system of curvilinear coordinates $(n, s)$ in the neighborhood of $L_0$ by setting

$$\begin{aligned} x &= \varphi(s) + n\psi'(s) \\ y &= \psi(s) - n\varphi'(s) \end{aligned} \qquad s \in (-\infty, +\infty)\,, \ n \in (-\delta, \delta)\,,$$

with $\delta > 0$ sufficiently small. Under the condition $(\varphi'(s))^2 + (\psi'(s))^2 > 0$ for every $s \in [0, \tau)$, the orbits of $(A_\mu)$, for $|\mu|$ small enough, near to $L_0$ have equations $n = n(s)$ with $n(s)$ satisfying the equation

$$\frac{dn}{ds} = R(\mu, n, s)\,, \tag{2.2}$$

where $R(\mu, n, s) \in C^h$ is a function, periodic in $s$ with period $\tau$, such that $R(0, 0, s) \equiv 0$. In particular the orbits of $(A_0)$ near to $L_0$ will be given by the solutions of equation

$$(2.3) \qquad \frac{dn}{ds} = R(0, n, s) = R_1(s)n + R_2(s)n^2 + \ldots + R_h(s)n^h + o(n^h)\,.$$

Let $n(\mu, n_0, s)$ denote the solution of (2.2) satisfying the condition $n(\mu, n_0, 0) = n_0$. If $|\mu|$, $|n_0|$ are small enough, $n(\mu, n_0, s)$ is defined on $[0, \tau]$ and therefore one can introduce the *displacement function* $V(\mu, n_0)$, relative to $L_0$, by putting

$$V(\mu, n_0) = n(\mu, n_0, \tau) - n_0\,.$$

Then the orbit of $(A_\mu)$ corresponding to $n_0$ will be periodic if and only if $\mu$, $n_0$ satisfy the equation

$$(2.4) \qquad V(\mu, n_0) = 0\,.$$

As $V \in C^h$, we can write

$$(2.5) \qquad V(\mu, n_0) = a_{0,1}\mu + a_{1,0}n_0 + \ldots + a_{h,0}n_0^h + \Phi(\mu, n_0)$$

with $\Phi$ of order higher than $h$ in $(\mu, n_0)$.

DEFINITION 2.1 [2]. *The closed orbit $L_0$ of $(A_0)$ is said to be a limit cycle of multiplicity $k \in \{1, \ldots, h\}$ if the coefficients of the development* (2.5) *satisfy the condition*

$$a_{1,0} = \ldots = a_{k-1,0} = 0\,, \qquad a_{k,0} \neq 0\,.$$

*If $k = 1$, $L_0$ is said to be a simple limit cycle.*

It is easy to see that a limit cycle of odd multiplicity is attracting or repulsing, whereas it is semistable if its multiplicity is even. To evaluate the multiplicity of a limit cycle, let us develop the solution $n(0, n_0, s)$ in terms of $n_0$:

$$(2.6) \qquad n(0, n_0, s) = u_1(s)n_0 + \ldots + u_h(s)n_0^h + o(n_0^h)$$

and substitute (2.6) in (2.3). Then the functions $u_l(s)$, $l = 1, \ldots, h$, have to satisfy the system of differential equations

$$\frac{du_1}{ds} = R_1(s)u_1 ,$$

$$\frac{du_l}{ds} = U_l(s, u_1, \ldots, u_l) \qquad l \geqslant 2 ,$$

($U_l$ is the coefficient of $n_0^l$ in the right hand side of (2.3) after substitution of (2.6)) with the initial conditions

$$u_1(0) = 1 , \quad u_l(0) = 0 \qquad l = 2, \ldots, h .$$

The multiplicity of $L_0$ will be the index of the first function $u_k(s)$, $k = 1, \ldots, h$, which is not periodic of period $\tau$, if it exists. In particular we have

$$u_1(s) = \exp\left[\int_0^s R_1(s)\, ds\right] = \exp\left[\int_0^s [P'_x(\varphi(s), \psi(s)) + Q'_y(\varphi(s), \psi(s))]\, ds\right]$$

and it will be periodic of period $\tau$ if

$$\int_0^\tau [P'_x(\varphi(s), \psi(s)) + Q'_y(\varphi(s), \psi(s))]\, ds = 0 ,$$

that is, if the characteristic index of $L_0$ is null. In such a case we have

$$a_{1,0} = u_1(\tau) - 1 = 0 . \tag{2.7}$$

In the following we will suppose that (2.7) holds, that is $L_0$ is not a simple limit cycle. In fact if $a_{1,0} \neq 0$, then equation (2.4) can be solved with respect to $n_0$ in the neighborhood of (0, 0) and, for $|\mu|$ small enough, $(A_\mu)$ has in the neighborhood of $L_0$ just one closed orbit which is a simple limit cycle with the same attractivity property of $L_0$.

In [2] some results (Ths. 71, 72, 73) are given about the existence and attractivity properties of periodic orbits of $(A_\mu)$ when $(A_\mu)$ coincides with one of the two following systems:

$$
\begin{aligned}
\dot{x} &= P(x, y) - \mu Q(x, y)\,,\\
\dot{y} &= Q(x, y) + \mu P(x, y)\,,
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
\dot{x} &= P(x, y) + \mu p_1(x, y) + \mu^2 p_2(x, y) + \ldots\,,\\
\dot{y} &= Q(x, y) + \mu q_1(x, y) + \mu^2 q_2(x, y) + \ldots\,.
\end{aligned}
\tag{2.9}
$$

The functions $P$, $Q$ are supposed to be analytic and in (2.9) the functions $p_1$, $q_1$ are defined as follows:

$$p_1(x, y) = F(x, y) F'_x(x, y)\,, \qquad q_1(x, y) = F(x, y) F'_y(x, y)\,,$$

where $F(x, y)$ is an analytic function such that

$$F(\varphi(s), \psi(s)) \equiv 0\,, \quad [F'_x(\varphi(s), \psi(s))]^2 + [F'_y(\varphi(s), \psi(s))]^2 > 0 \qquad \forall s \in [0, \tau)\,.$$

Such results are obtained by using the Newton's polygon in the analysis of the zeros of the corresponding displacement function and they hold because for system (2.8) we have

$$a_{0,1} = \frac{\partial V}{\partial \mu}(0, 0) \neq 0\,, \tag{2.10}$$

whereas for system (2.9) the condition

$$a_{0,1} = 0\,, \qquad a_{1,1} = \frac{\partial V}{\partial n_0\, \partial \mu}(0, 0) \neq 0 \tag{2.11}$$

is satisfied.

REMARK 2.2. In Hopf bifurcation the displacement function relative to the origin satisfies condition (2.11) because of trasversality condition.

We want now to recall the results given in [3] about bifurcation from a family of invariant sets for systems $(A_\mu)$. Let us denote by $J(\bar{\mu})$ one of the two intervals $[0, \bar{\mu})$, $(-\bar{\mu}, 0]$ and consider a family of compact sets $\{M_\mu\}_{\mu\mu\in J(\bar{\mu})}$, $M_\mu \subset R^2$, such that

1) $\forall \mu \in J(\bar{\mu})$, $M_\mu$ is $A_\mu$-invariant;

2) $\max\{\varrho(\boldsymbol{x}, M_0): \boldsymbol{x} \in M_\mu\} \to 0$ as $\mu \to 0$, where $\boldsymbol{x} \equiv (x, y)$ and $\varrho$ denotes the usual distance.

DEFINITION 2.3 [3, III, Def. 1.1]. *$\mu = 0$ is said to be a bifurcation point for the family $\{M_\mu\}_{\mu\in J(\bar{\mu})}$ if there exists $\mu^* \in (0, \bar{\mu})$ and a new family of compact sets $\{M'_\mu\}_{\mu\in J(\mu^*)\setminus\{0\}}$, $J(\mu^*) \subset J(\bar{\mu})$, such that*

*$\alpha$) $\forall \mu \in J(\mu^*)\setminus\{0\}$, $M'_\mu$ is $A_\mu$-invariant, $M'_\mu \cap M_\mu = \emptyset$;*

*$\beta$) $\max\{\varrho(\boldsymbol{x}, M_0): \boldsymbol{x} \in M'_\mu\} \to 0$ as $\mu \to 0$.*

THEOREM 2.4 [3, III, Th. 1.3]. *Let $\{M_\mu\}_{\mu\in J(\bar{\mu})}$ be a family of compact sets satisfying conditions* 1) *and* 2). *Suppose that $M_0$ is $A_0$-asymptotically stable [resp. $A_0$-completely unstable, that is $A_0$-asymptotically stable in the past] and $M_\mu$ is $A_\mu$-completely unstable [resp. $A_\mu$-asymptotically stable] for every $\mu \in J(\bar{\mu})\setminus\{0\}$. Then $\mu = 0$ is a bifurcation point for the family $\{M_\mu\}$ and the family $\{M'_\mu\}$ of Def. 2.3 can be determined such that $M'_\mu$ is $A_\mu$-asymptotically stable [resp. $A_\mu$-completely unstable] for every $\mu \in J(\mu^*)\setminus\{0\}$.*

REMARK 2.5. Theorem 2.4 holds also if the sets $M_\mu$ are not compact but their complements $R^2/M_\mu$ are compact for every $\mu \in J(\bar{\mu})$.

Let us suppose now that

$$\text{(2.12)}\qquad \begin{aligned} &\forall \mu \in J(\bar{\mu}) \text{ there exists a closed orbit } L_\mu \text{ for } (A_\mu) \text{ such that} \\ &\max\{\varrho(\boldsymbol{x}, L_0): \boldsymbol{x} \in L_\mu\} \to 0 \text{ as } \mu \to 0\,. \end{aligned}$$

Then, denoting by $C_\mu$ the disk whose boundary is $L_\mu$ and by $\bar{C}_\mu$ its complement $R^2/C_\mu$, we can identify in Th. 2.4 the family $\{M_\mu\}$ with one of the families $\{C_\mu\}$, $\{\bar{C}_\mu\}$, $\{L_\mu\}$. In this way, it is possible to prove the following corollaries.

COROLLARY 2.6. *Suppose* (2.12) *holds. Let $L_0$ be $A_0$-attracting [resp. $A_0$-repulsing] for outside orbits and $\forall \mu \in J(\bar{\mu})\setminus\{0\}$ let $L_\mu$ be $A_\mu$-repulsing [resp. $A_\mu$-attracting] for outside orbits. Then $\mu = 0$ is a bifurcation point for the family $\{C_\mu\}_{\mu\in J(\bar{\mu})}$ and the family $\{M'_\mu\}$ can be determined such that $\forall \mu \in J(\mu^*)\setminus\{0\}$ $M'_\mu$ is $A_\mu$-asymptotically stable [resp. $A_\mu$-completely unstable] and $M'_\mu$ is an annular region outside of $L_\mu$ bounded by two closed orbits of $(A_\mu)$.*

COROLLARY 2.7. *Suppose* (2.12) *holds. Let* $L_0$ *be* $A_0$*-attracting* [*resp.* $A_0$*-repulsing*] *for inside orbits and* $\forall \mu \in J(\bar{\mu}) \setminus \{0\}$ *let* $L_\mu$ *be* $A_\mu$*-repulsing* [*resp.* $A_\mu$*-attracting*] *for inside orbits. Then* $\mu = 0$ *is a bifurcation point for the family* $\{\bar{C}_\mu\}_{\mu \in J(\bar{\mu})}$ *and the family* $\{M'_\mu\}$ *can be determined such that* $\forall \mu \in J(\mu^*) \setminus \{0\}$ $M'_\mu$ *is* $A_\mu$*-asymptotically stable* [*resp.* $A_\mu$*-completely unstable*] *and* $M'_\mu$ *is an annular region inside of* $L_\mu$ *bounded by two closed orbits of* $(A_\mu)$.

COROLLARY 2.8. *Suppose* (2.12) *holds. Let* $L_0$ *be* $A_0$*-attracting* [*resp.* $A_0$*-repulsing*] *and* $\forall \mu \in J(\bar{\mu}) \setminus \{0\}$ *let* $L_\mu$ *be* $A_\mu$*-repulsing* [*resp.* $A_\mu$*-atttracting*]. *Then* $\mu = 0$ *is a bifurcation point for the family* $\{L_\mu\}_{\mu \in J(\bar{\mu})}$ *and the family* $\{M'_\mu\}$ *can be determined such that* $\forall \mu \in J(\mu^*) \setminus \{0\}$ $M'_\mu$ *is* $A_\mu$*-asymptotically stable* [*resp.* $A_\mu$*-completely unstable*] *and* $M'_\mu$ *is constituted by two annular regions, one inside of* $L_\mu$ *the other one outside of* $L_\mu$*, each of them bounded by two closed orbits of* $(A_\mu)$.

The proof of Corollaries 2.6, 2.7, 2.8 can be obtained by using the same arguments as in [3, III, Th. 2.2].

## 3. Results.

In this section we will consider families of systems $(A_\mu)$ for which either condition (2.10) or condition (2.11) is satisfied. As we observed in Remark 2.2, in Hopf bifurcation the displacement function relative to the origin satisfies condition (2.11) because of trasversality condition. Such condition implies an exchange of attıactivity properties of the origin at a critical value of the parameter and, by Hopf's theorem [5], the existence of periodic orbits bifurcating from the origin. Hereafter we will show how (2.10) and (2.11) play in our bifurcation problem the same rôle as Hopf's trasversality condition. Precisely (2.10) assures the existence of a *bifurcation function* $\mu(n_0)$ which gives us the solutions of equation (2.4). On the other hand (2.11) assures, except a critical case, the existence of a family of closed oıbits $\{L_\mu\}$, $L_\mu \to L_0$ as $\mu \to 0$, whose attractivity properties change at $\mu = 0$. The periodic orbits of $(A_\mu)$, near to $L_0$ and different from $L_\mu$, are given by a new bifurcation function, whose existence is assured always by (2.11). The number of bifurcating orbits from $L_0$ and their attractivity properties will depend, as in Hopf bifurcation [4], on the properties of the above bifurcation functions.

3.1. Suppose first that $(A_\mu)$ satisfies condition (2.10). Then (2.4) can be solved with respect to $\mu$ in the neighborhood of (0, 0) and we have the following theorem which is analogous to Hopf's theorem.

THEOREM 3.1. *Suppose that* $(A_\mu)$ *satisfies* (2.10). *Then, there exist* $\gamma > 0$, $\delta > 0$ *and a function* $\mu(n_0) \in C^h[(-\gamma, \gamma), (-\delta. \delta)]$ *with* $\mu(0) = \mu'(0) = 0$ *such that for* $\mu \in (-\delta, \delta)$ *and* $n_0 \in (-\gamma, \gamma)$ *the orbit of* $(A_\mu)$ *corresponding to* $n_0$ *is periodic if and only if* $\mu = \mu(n_0)$.

It is easy now to give a more direct and easy proof of Ths. 71, 72 of [1]. Let us denote by $U_\varepsilon(L_0)$, $\varepsilon > 0$, the $\varepsilon$-neighborhood of $L_0$.

THEOREM 3.2 [2, Th. 71]. *Suppose that* $(A_\mu)$ *satisfies* (2.10) *and* $L_0$ *is a limit cycle with even multiplicity* $k$. *Then there exist* $\varepsilon_0 > 0$, $\mu_0 > 0$ *such that one of the two following situations occurs:*

*a)* $\forall \mu > 0$, $\mu < \mu_0$, $(A_\mu)$ *has exactly two closed orbits in* $U_{\varepsilon_0}(L_0)$ *and* $\forall \mu < 0$, $|\mu| < \mu_0$, $(A_\mu)$ *has no closed orbits in* $U_{\varepsilon_0}(L_0)$;

*b)* $\forall \mu > 0$, $\mu < \mu_0$, $(A_\mu)$ *has no closed orbits in* $U_{\varepsilon_0}(L_0)$ *and* $\forall \mu < 0$, $|\mu| < \mu_0$, $(A_\mu)$ *has exactly two closed orbits in* $U_{\varepsilon_0}(L_0)$.

*Further,* $\forall \varepsilon \in (0, \varepsilon_0)$ *there exists* $\mu^* \in (0, \mu_0)$ *such that* $\forall \mu \in (-\mu^*, \mu^*)$ *the above periodic orbits belong to* $U_\varepsilon(L_0)$ *and they are simple limit cycles for* $\mu \neq 0$.

PROOF. Let $\mu(n_0)$ be the bifurcation function which exists because of Th. 3.1. By deriving $k$ times the identity $V(\mu(n_0), n_0) \equiv 0$ we get

$$\mu^{(s)}(0) = 0\,, \quad s = 1, \ldots, k-1\,; \qquad \mu^{(k)}(0) = -\frac{a_{k,0}}{a_{0,1}} \neq 0\,. \tag{3.1}$$

Therefore the function $\mu(n_0)$ has a minimum [resp. a maximum] at $n_0 = 0$ if $a_{k,0}/a_{0,1} < 0$ [resp. $a_{k,0}/a_{0,1} > 0$] and there exists $\varepsilon_0 \in (0, \gamma)$ such that $\mu(n_0)$ is strictly increasing [resp. decreasing] in $[0, \varepsilon_0)$ and strictly decreasing [resp. increasing] in $(-\varepsilon_0, 0]$. Thus situation *a*) [resp. *b*)] occurs if $a_{k,0}/a_{0,1} < 0$ [resp. $a_{k,0}/a_{0,1} > 0$] with $\mu_0 \in (0, \mu(\varepsilon_0))$. Also the above monotonicity properties of $\mu(n_0)$ imply that for $|\mu|$ small enough the periodic orbits of $(A_\mu)$ are as near as we want to $L_0$. Finally, let us prove that for $|\mu|$ small enough, $\mu \neq 0$, the periodic orbits of $(A_\mu)$ near to $L_0$ are simple limit cycles. Suppose, for example, that $a_{k,0}/a_{0,1} < 0$ so that situation *a*) occurs (the case $a_{k,0}/a_{0,1} > 0$ can be discussed in a similar way). Let $\mu \in (0, \mu_0)$ and let $\nu > 0$ be the $k$-th positive root of $\mu$. We want to look for the zeros of $V(\mu, n_0)$ of the type $n_0(\mu) = \nu c$. We have

$$V(\mu, n_0) = V(\nu^k, \nu c) = \nu^k d(\nu, c)$$

with

$$d(\nu, c) = a_{0,1} + a_{1,1}\nu c + a_{k,0} c^k + \nu^2 d_1(\nu, c) .$$

The equation $d(\nu, c) = 0$ can be solved with respect to $c$ in the neighborhoods of $(0, \bar{c})$ and $(0, -\bar{c})$ with $\bar{c}$ positive $k$-th root of $-a_{0,1}/a_{k,0}$. Therefore there exist two functions $c_1(\nu)$, $c_2(\nu)$ defined in a neighborhood of $\nu = 0$ such that $c_1(0) = \bar{c}$, $c_2(0) = -\bar{c}$, $d(\nu, c_1(\nu)) \equiv 0$, $d(\nu, c_2(\nu)) \equiv 0$. Then, for $\mu$ positive and small enough we have two zeros of $V(\mu, n_0)$, that is $n_0^{(1)}(\mu) = \nu c_1(\nu)$, $n_0^{(2)}(\mu) = \nu c_2(\nu)$ with $\nu$ positive $k$-th root of $\mu$. Such zeros have to correspond to the two bifurcating closed orbits which exist when situation *a*) occurs. By evaluating the derivative $\partial V/\partial n_0$ for $n_0 = n_0^{(i)}(\mu)$, $i = 1, 2$, one easily proves that it is non null for $\mu > 0$ small enough. Therefore such orbits are simple limit cycles and this completes the proof.

THEOREM 3.3 [2, Th. 72]. *Suppose that* $(A_\mu)$ *satisfies* (2.10) *and* $L_0$ *is a limit cycle with odd multiplicity* $k$. *Then there exist* $\varepsilon_0 > 0$, $\mu_0 > 0$ *such that*

(3.3) $\quad \forall\mu \in (-\mu_0, \mu_0)$, $(A_\mu)$ *has exactly one closed orbit in* $U_\varepsilon(L_0)$.

*Further, for every* $\varepsilon \in (0, \varepsilon_0)$ *there exists* $\mu^* \in (0, \mu_0)$ *such that* $\forall\mu \in \in (-\mu^*, \mu^*)$ *the previous periodic orbits belong to* $U_\varepsilon(L_0)$ *and they are, for* $\mu \neq 0$, *simple limit cycles.*

PROOF. As $k$ is odd and (3.2) holds, there exists $\varepsilon_0 \in (0, \gamma)$ such that the bifurcation function $\mu(n_0)$, existing because of Th. 3.1, is strictly increasing [resp. decreasing] in $(-\varepsilon_0, \varepsilon_0)$ if $a_{k,0}/a_{0,1} < 0$ [resp. $a_{k,0}/a_{0,1} > 0$]. Therefore (3.3) holds for a suitable $\mu_0 \in (0, \mu(\varepsilon_0))$. The rest of the proof is analogous to that of Th. 3.2 by taking here $\nu$ equal to the $k$-th root of $\mu$ for every $\mu \in (-\mu_0, \mu_0)$.

Now we want to interpret these results by using Corollaries 2.6, 2.7 of Th. 2.4. Suppose that $k$ is even and situation *a*) of Th. 3.2 occurs. The two closed orbits which we have for $\mu > 0$ have to be one, say $L_\mu$, repulsing, the other one, $\bar{L}_\mu$, attracting. If $L_0$ is attracting from outside, [resp. from inside], we have from Corollary 2.6 [resp. Corollary 2.7] that there exists an attracting annular region outside [resp. inside] of $L_\mu$ bounded by two closed orbits of $(A_\mu)$. In our case, this annular region has to shrink to just one closed orbit coinciding with $\bar{L}_\mu$.

When $k$ is odd, say $L_\mu$ the unique closed orbit which we have for every $\mu$, $|\mu|$ small enough, in the neighborhood of $L_0$. Such orbits have to have the same attractivity properties of $L_0$, otherwise other closed orbits should arise because of Corollaries of Th. 2.4. Therefore, in such case $\mu = 0$ is not a bifurcation point for the families $\{C_\mu\}$, $\{\bar{C}_\mu\}$, $\{L_\mu\}$.

3.2. Suppose now that for the family of systems $(A_\mu)$ condition (2.11) is satisfied. As we pointed out, such a condition is verified in Hopf bifurcation. Therefore one can think that it is possible to proceed as in Negrini and Salvadori's paper [4] to study the existence and the attractivity properties of bifurcating orbits. Actually, this can be done when $L_0$ has a multiplicity bigger than 2. In fact, in such a case we can prove the existence of a family of closed orbits for which there is a drastic change of stability properties at $\mu = 0$.

THEOREM 3.4. *Suppose that condition* (2.11) *holds and* $L_0$ *has multiplicity bigger than* 2. *Then there exists* $\hat{\mu} \in (0, \bar{\mu})$ *such that for every* $\mu \in (-\hat{\mu}, \hat{\mu}) \setminus \{0\}$, $(A_\mu)$ *has a simple limit cycle* $L_\mu$ *which tends to* $L_0$ *as* $\mu$ *tends to* 0 *and for which the attractivity properties change through* $\mu = 0$.

PROOF. Because of (2.11) and $a_{2,0} = 0$, there exists $\mu^* \in (0, \bar{\mu})$ such that for every $\mu \in (-\mu^*, \mu^*)$ $V(\mu, n_0)$ has a root of the type $n_0(\mu) = \mu c(\mu)$. Indeed, we have $V(\mu, \mu c) = \mu^2 d(\mu, c)$, with $d(\mu, c) = a_{1,1} c + a_{0,2} + \mu d_1(\mu, c)$, and the equation $d(\mu, c) = 0$ can be solved in the neighborhood of $(0, -a_{0,2}/a_{1,1})$. Of course, $n_0(\mu)$ tends to 0 as $\mu$ tends to 0 and it is easy to show that $(\partial V/\partial n_0)_{n_0 = n_0(\mu)} = a_{1,1}\mu + o(\mu)$.

REMARK 3.5. Under condition (2.11), if $L_0$ has multiplicity equal to 2, then the displacement function has the form

$$V(\mu, n_0) = a_{1,1}\mu n_0 + a_{0,2}\mu^2 + a_{2,0} n_0^2 + \psi(\mu, n_0)$$

with $\psi(\mu, n_0)$ of order higher than two in $(\mu, n_0)$ and $a_{2,0} \neq 0$. Let us consider the three cases $a_{1,1}^2 - 4a_{0,2}a_{2,0} \lesseqgtr 0$. In the first case $V(\mu, n_0)$ is sign-definite in the neighborhood of $(0, 0)$ and therefore $(A_\mu)$ has no closed orbits for $\mu \neq 0$ and $|\mu|$ small enough. In the second case by looking for the zeros of $V(\mu, c)$ of the type $\mu c(\mu)$, we can prove the existence of two closed orbits of $(A_\mu)$ for $\mu \neq 0$ and $|\mu|$ small enough which tends to $L_0$ as $\mu \to 0$. As $L_0$ has multiplicity two, Theorems 42 and 43 of [2] imply that they are the only closed orbits of $(A_\mu)$ in the neighborhood of $L_0$ and they are simple limit cycles. Finally, the third case is critical and the existence of closed orbits of $(A_\mu)$ for $\mu \neq 0$ will depend upon the function $\psi(\mu, n_0)$.

REMARK 3.6. Under the hypotheses of Th. 3.4, condition (2.12) holds with $\bar{\mu}$ replaced by $\hat{\mu}$. Further, the attractivity properties of $L_\mu$ change through $\mu = 0$. Therefore, if we suppose that $L_0$ is attracting, repulsing or semistable, we can use Corollaries 2.6, 2.7, 2.8 to state the existence of periodic orbits of $(A_\mu)$, bifurcating from $\{L_\mu\}$, in the neighborhood of $L_0$.

Now under the hypotheses of Th. 3.4, we want to get results to establish the exact number of closed orbits of $(A_\mu)$ in the neighborhood of $L_0$ and their attractivity properties. For $\mu \in (-\hat{\mu}, \hat{\mu})$, we have

$$V(\mu, n_0) = [n_0 - n_0(\mu)]\,\tilde{V}(\mu, n_0)$$

where $\tilde{V} \in C^{h-1}$ is such that $\tilde{V}(\mu, n_0(\mu)) \neq 0$ for $\mu \neq 0$. Therefore the periodic orbits of $(A_\mu)$ different from $L_\mu$ will correspond to the zeros of $\tilde{V}(\mu, n_0)$. As we have by (2.11)

$$\tilde{V}(0,0) = 0 \qquad \frac{\partial \tilde{V}}{\partial \mu}(0,0) \neq 0\,,$$

the implicit function theorem yields the following extension to our problem of Hopf's theorem.

THEOREM 3.7. *In the same hypotheses of Th.* 3.4, *there exist* $\delta > 0$, $\varepsilon > 0$ *and a function* $\tilde{\mu} \in C^{h-1}[(-\varepsilon, \varepsilon), (-\delta, \delta)]$ *with* $\tilde{\mu}(0) = \tilde{\mu}'(0) = 0$ *such that for any* $\mu \in (-\delta, \delta)$ *and* $n_0 \in (-\varepsilon, \varepsilon)$, $n_0 \neq n_0(\mu)$, *the orbit of* $(A_\mu)$ *corresponding to* $n_0$ *is closed if and only if* $\mu = \tilde{\mu}(n_0)$.

The existence of closed orbits of $(A_\mu)$ in the neighborhood of $L_0$, different from $L_\mu$, depends upon the properties of the *bifurcation function* $\tilde{\mu}(n_0)$, given by Th. 3.7. If we suppose that $L_0$ is a multiple limit cycle with finite multiplicity bigger than two, then we are able to say exactly how many closed orbits there are. Also the attractivity properties of such orbits can be settled.

THEOREM 3.8. *Let* (2.11) *hold and* $L_0$ *be an attracting* [*resp. repulsing*] *limit cycle with finite* (*odd*) *multiplicity* $k \in \{3, \ldots, h-1\}$. *Then there exist* $\varepsilon_0 > 0$, $\mu_0 > 0$ *such that one of the two following situations occurs*:

*a*) *for* $\mu > 0$, $\mu < \mu_0$, *exactly two closed orbits of* $(A_\mu)$, *different from* $L_\mu$, *exist in* $U_{\varepsilon_0}(L_0)$, *one inside the other one outside of* $L_\mu$, *whereas for* $\mu < 0$, $|\mu| < \mu_0$, *we don't have closed orbits of* $(A_\mu)$ *different from* $L_\mu$ *in* $U_{\varepsilon_0}(L_0)$;

*b*) *for* $\mu > 0$, $\mu < \mu_0$, *we don't have closed orbits of* $(A_\mu)$, *different from* $L_\mu$, *in* $U_{\varepsilon_0}(L_\mu)$ *and for* $\mu < 0$, $|\mu| < \mu_0$, *exactly two closed orbits of* $(A_\mu)$, *different from* $L_\mu$, *exist in* $U_{\varepsilon_0}(L_0)$, *one inside the other one outside of* $L_\mu$.

*Situation a*) *occurs when* $a_{1,1} > 0$ [*resp.* $a_{1,1} < 0$], *whereas b*) *occurs when* $a_{1,1} < 0$ [*resp.* $a_{1,1} > 0$]. *Moreover, for every* $\varepsilon \in (0, \varepsilon_0)$ *there exists* $\mu^* \in (0, \mu_0)$ *such that for every* $\mu \in (-\mu^*, \mu^*)$ *the above periodic orbits different from* $L_\mu$ *belong to* $U_\varepsilon(L_0)$ *and they are simple attracting* [*resp. repulsing*] *limit cycles.*

PROOF. Suppose, for instance, that $a_{1,1} > 0$ and $L_0$ be attracting (the other cases can be treated in the same way). Deriving $k$ times the identity $V(\tilde{\mu}(n_0), n_0) \equiv 0$ we have

$$\tilde{\mu}^{(s)}(0) = 0\,, \quad s = 1, \ldots, k-2\,; \qquad \tilde{\mu}^{(k-1)}(0) = -\frac{1}{k}\frac{a_{k,0}}{a_{1,1}} > 0\,. \tag{3.2}$$

Therefore $\tilde{\mu}(n_0)$ has a proper minimum in $n_0 = 0$ and situation *a*) occurs. To prove that the closed orbits of $(A_\mu)$, for $\mu > 0$, different from $L_\mu$, are simple attracting limit cycles, let us set for $\mu > 0$ $\mu = \nu^{k-1}$, $\nu > 0$, and look for the zeros of $V(\mu, n_0)$ of the type $n_0 = \nu c$. We have

$$V(\nu^{k-1}, \nu c) = \nu^k[a_{1,1}c + a_{k,0}c^k + \varphi(\nu, c)]$$

with $\varphi(0, c) = 0$. By setting

$$d(\nu, c) = a_{1,1}c + a_{k,0}c^k + \varphi(\nu, c)\,,$$

we have

$$d(0, c) = 0 \quad \text{for } c = 0\,, \qquad c = \pm\sqrt[k-1]{-\frac{a_{1,1}}{a_{k,0}}}\,;$$

$$\frac{\partial d}{\partial c}(0, 0) = a_{1,1}\,, \qquad \frac{\partial d}{\partial c}\left(0, \pm\sqrt[k-1]{-\frac{a}{a_{k,0}}}\right) = -(k-1)\,a_{1,1}\,.$$

Therefore the equation $d(v, c) = 0$ can be solved in the neighborhood of

$$(0,0), \qquad \left(0, \sqrt[k-1]{-\frac{a_{1,1}}{a_{k,0}}}\right), \qquad \left(0, -\sqrt[k-1]{-\frac{a_{1,1}}{a_{k,0}}}\right)$$

and we can determine, for $\mu > 0$ and small enough, three zeros of $V(\mu, n_0)$ which correspond the first one to $L_\mu$ and the others to the two closed orbits of $(A_\mu)$, different from $L_\mu$, one inside the other one outside of $L_\mu$. Finally, it is easy to prove that $\partial V/\partial n_0 < 0$ corresponding to the last two zeros and therefore the closed orbits of $(A_\mu)$ different from $L_\mu$ are simple attracting limit cycles.

Theorem 3.8 is a reformulation of Th. 73 of [2]. However, as we pointed out in Sec. 1, its proof comes out from a different approach of the bifurcation problem. Such approach suggests a new interpretation of the result and, moreover, allows us to analyze also the case in which $L_0$ has even multiplicity. In fact, by using always (3.2), we can prove in an analogous way the following theorem.

THEOREM 3.9. *Let* (2.11) *hold and* $L_0$ *be a semistable limit cycle with finite* (*even*) *multiplicity* $k \in (3, \dots, h-1)$ *attracting for outside* [*resp. inside*] *orbits. Then, there exist* $\varepsilon_0 > 0$, $\mu_0 > 0$ *such that we have either*

*a'*) *for* $\mu > 0$, $\mu < \mu_0$, *exactly one closed orbit of* $(A_\mu)$, *different from* $L_\mu$, *exists in* $U_{\varepsilon_0}(L_0)$ *and it is outside of* $L_\mu$, *whereas for* $\mu < 0$, $|\mu| < \mu_0$, *exactly one closed orbit of* $(A_\mu)$, *different from* $L_\mu$, *exists in* $U_{\varepsilon_0}(L_0)$ *and it is inside of* $L_\mu$;

or

*b'*) *for* $\mu > 0$, $\mu < \mu_0$, *exactly one closed orbit of* $(A_\mu)$, *different from* $L_\mu$, *exists in* $U_{\varepsilon_0}(L_0)$ *and it is inside of* $L_\mu$, *whereas for* $\mu < 0$, $|\mu| < \mu_0$, *exactly one closed orbit of* $(A_\mu)$, *different from* $L_\mu$, *exists in* $U_{\varepsilon_0}(L_0)$ *and it is outside of* $L_\mu$.

*Situation a'*) *occurs when* $a_{1,1} > 0$ [*resp.* $a_{1,1} < 0$], *whereas b'*) *holds if* $a_{1,1} < 0$ [*resp.* $a_{1,1} > 0$]. *Further, for every* $\varepsilon \in (0, \varepsilon_0)$ *there exists* $\mu^* \in (0, \mu_0)$ *such that for every* $\mu \in (-\mu^*, \mu^*)$ *the above periodic orbits, different from* $L_\mu$, *belong to* $U_\varepsilon(L_0)$ *and they are simple limit cycles, attracting* [*resp. repulsing*] *if they are outside of* $L_0$, *repulsing* [*resp. attracting*] *if they are inside of* $L_0$.

REMARK 3.10. As we pointed out in Remark 3.6, if $L_0$ is attracting, repulsing or semistable, then annular regions bounded by closed orbits of $(A_\mu)$ bifurcate from the family $\{L_\mu\}$. The further hypotheses that $L_0$ has finite multiplicity assures that such annular regions reduce to just one orbit.

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