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Linearly Compact Rings and Strongly Quasi-Injective Modules.

C. MENINI (*)

Introduction.

Throughout this paper, all rings are associative with identity $1 \neq 0$ and all modules are unitary.

Let R be a ring. A left R -module ${}_R K$ is called *strongly quasi-injective* (for short s.q.i.) if given any submodule B of ${}_R K$, a morphism $f: B \rightarrow {}_R K$ and an element $x \in K \setminus B$, f extends to an endomorphism \bar{f} of ${}_R K$ such that $(x)\bar{f} \neq 0$.

The notion of s.q.i. module comes from the study of dualities, induced by topological bimodules, between a category of abstract modules and a category of topological modules, where it plays a central role (cf. [2]).

Investigating on the concept of s.q.i. module, the following question naturally arises. Let ${}_R K$ be a s.q.i. module, $A = \text{End}({}_R K)$. When is K_A s.q.i.? The study of this problem leads to the following characterization of linearly compact rings.

THE MAIN THEOREM. *Let R be a left linearly topologized ring with respect to a ring topology τ , let \mathcal{F} be the filter of open left ideals of R and let $\mathcal{G}_{\mathcal{F}}$ be the hereditary pretorsion class of left R -modules associated with \mathcal{F} . The following statements are equivalent.*

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- (a) R is linearly compact in the topology τ .
- (b) If ${}_R K$ is a cogenerator of $\mathfrak{C}_{\mathcal{F}}$ and $A = \text{End}({}_R K)$, then ${}_R K_A$ is faithfully balanced and K_A is quasi-injective.
- (c) There exists a faithfully balanced module ${}_R K_A$ such that ${}_R K$ is a cogenerator of $\mathfrak{C}_{\mathcal{F}}$ and K_A is quasi-injective.
- (d) Let ${}_R U$ be a minimal cogenerator of $\mathfrak{C}_{\mathcal{F}}$, $T = \text{End}({}_R U)$. Then ${}_R U_T$ is faithfully balanced and both the modules ${}_R U$ and U_T are s.q.i.

Moreover, if condition (d) is fulfilled, T is linearly compact in its U -adic topology

(See below for explained definitions.)

Some results obtained in [4] for discrete linearly compact rings are here extended to the general case.

As an application of our results, we get a quick proof of Leptin's theorem which characterizes a linearly compact ring with zero Jacobson radical as a cartesian product of endomorphism rings of vector spaces.

A structure theorem on faithfully balanced modules ${}_R K_A$ which are s.q.i. both on R and A , obtained as intermediate result, has an intrinsic interest (cf. Theorem 10).

I would like to thank Prof. A. Orsatti for his helpful suggestions.

Some conventions and notations. Let R be a ring. $R\text{-Mod}$ will denote the category of left R -modules and $\text{Mod-}R$ that of right R -modules. The notation ${}_R M$ will be used to emphasize that M is a left R -module. Morphisms between modules will be written on the opposite side to that of the scalars and the composition of morphisms will follow this convention. For every $M \in R\text{-Mod}$, $E_R(M)$, or simply $E(M)$, will denote the injective envelope of M in $R\text{-Mod}$ and $\text{Soc}({}_R M)$, or simply $\text{Soc}(M)$, the socle of M . If L is a subset of ${}_R M \in R\text{-Mod}$, we denote by $\text{Ann}_R(L)$ the annihilator of L in R :

$$\text{Ann}_R(L) = \{r \in R: rx = 0 \text{ for every } x \in L\}.$$

If $L = \{x\}$, we will simply write $\text{Ann}_R(x)$.

If J is a left ideal of R , we define the annihilator of J in M , $\text{Ann}_M(J)$, by setting:

$$\text{Ann}_M(J) = \{x \in M: rx = 0 \text{ for every } r \in J\}.$$

The annihilator in R of $\text{Ann}_M(J)$ will be denoted by $\text{Ann}_R \text{Ann}_M(J)$.

Analogous notations will be used for right modules.

\mathbf{N} will denote the set of positive integers.

1. To begin with, let us recall some definitions.

Let R be a ring and let $M \in R\text{-Mod}$. ${}_R M$ is *quasi-injective* (for short q.i.) if for every submodule $L \leqslant_R M$ and every morphism $f: L \rightarrow {}_R M$, f extends to an endomorphism \bar{f} of ${}_R M$. ${}_R M$ is a *self-cogenerator* if, for every $n \in \mathbf{N}$, given a submodule L of ${}_R M^n$ and an element $x \in M^n \setminus L$, there exists a morphism $f: {}_R M^n \rightarrow {}_R M$ such that $(L)f = 0$ and $(x)f \neq 0$. Clearly if ${}_R M$ is both quasi-injective and selfcogenerator, then ${}_R M$ is strongly quasi-injective. The converse is true as well (cf. [2], Corollary 4.5).

Let ${}_R K_A$ be a bimodule. ${}_R K_A$ is *faithfully balanced* if $A \cong \text{End}({}_R K)$ and $R \cong \text{End}(K_A)$ canonically.

Let R be a ring and let $M \in R\text{-Mod}$. The *M -topology* of R is defined by taking as a basis of neighbourhoods of 0 in R the annihilators in R of the finite subsets of M . It is easy to check that this topology is a left linear ring topology on R .

Finally recall that a linearly topologized left module M over a discrete ring R is said to be *linearly compact* if M is Hausdorff and if any finitely solvable system of congruences $x \equiv x_i \pmod{X_i}$, where the X_i are closed submodules of ${}_R M$, is solvable.

2. PROPOSITION. *Let R be a ring, ${}_R K \in R\text{-Mod}$ a selfcogenerator, $A = \text{End}({}_R K)$. If R is linearly compact in the K -topology, then K_A is quasi-injective.*

PROOF. Cf. [4], Prop. 3.4 a).

Let R be a ring, τ a left linear ring topology on R , \mathcal{F} the filter of open left ideals of R . The left exact preradical in $R\text{-Mod}$ associated with \mathcal{F} , $t_{\mathcal{F}}$, is defined by setting, for every $M \in R\text{-Mod}$:

$$t_{\mathcal{F}}(M) = \{x \in M : \text{Ann}_R(x) \in \mathcal{F}\}.$$

The hereditary pretorsion class of $R\text{-Mod}$ associated with \mathcal{F} is defined by setting

$$\mathcal{C}_{\mathcal{F}} = \{M \in R\text{-Mod} : M = t_{\mathcal{F}}(M)\}.$$

3. LEMMA. *Let R be a left linearly topologized ring with respect to a ring topology τ , let \mathcal{F} be the filter of open left ideals of R and let ${}_R K$*

be a cogenerator of $\mathcal{C}_{\mathcal{F}}$. For every closed left ideal J of R it is

$$\text{Ann}_R \text{Ann}_K(J) = J.$$

PROOF. Let $r \in R \setminus J$. There is an open left ideal L of R such that $L \supseteq J$ and $r \notin L$. Since ${}_R K$ is a cogenerator of $\mathcal{C}_{\mathcal{F}}$, there is a morphism $f: R/L \rightarrow {}_R K$ such that $(r + L)f \neq 0$. Hence there is an $x \in {}_R K$ such that $Lx = 0$ and $rx \neq 0$. Thus $Jx = 0$ and therefore $r \notin \text{Ann}_R \text{Ann}_K(J)$.

4. PROPOSITION. Let ${}_R K_A$ be a faithfully balanced bimodule, let τ be a left linear Hausdorff ring topology on R and let \mathcal{F} be the filter of open left ideals of R . Assume that ${}_R K$ is a cogenerator of $\mathcal{C}_{\mathcal{F}}$ and that K_A is quasi-injective. Then R is linearly compact in the topology τ .

PROOF. The following technique is due to Müller (cf. [3], Lemma 4). Let $(J_i)_{i \in I}$ be a family of closed left ideals of R and let

$$(1) \quad x \equiv r_i \pmod{J_i} \quad (r_i \in R)$$

be a finitely solvable system of congruences in R . Set $L = \sum_{i \in I} \text{Ann}_K(J_i)$. L is a submodule of K_A . Define a morphism $g: L \rightarrow K_A$ by setting $g\left(\sum_{i \in F} x_i\right) = \sum_{i \in F} r_i x_i$, where F is a finite subset of I and, for every $i \in F$, $x_i \in \text{Ann}_K(J_i)$. Since (1) is finitely solvable, g is well defined. Since K_A is quasi-injective, g extends to an endomorphism of K_A . Since ${}_R K_A$ is faithfully balanced, this endomorphism is the left multiplication by an element $r \in R$ so that we have, for every $i \in I$, $r - r_i \in \text{Ann}_R \text{Ann}_K(J_i)$. By Lemma 3, $\text{Ann}_R \text{Ann}_K(J_i) = J_i$ for every $i \in I$, thus (1) is solvable.

Let R be a ring and let τ be a left linear ring topology on R . The *Leptin topology* τ^* of τ is the ring topology on R defined by taking as a basis of neighbourhoods of 0 in R the cofinite open left ideals of R . Recall that a left ideal of R is *cofinite* if it is a finite intersection of completely irreducible left ideals of R . A left ideal I of R is *completely irreducible* if R/I is an essential submodule of the injective envelope $E(S)$ of a left simple R -module S .

Let \mathcal{F} be the filter of open left ideals of R . In the following \mathcal{S} will always denote a system of representatives of the isomorphism classes of the simple left R -modules and $\mathcal{S}_{\mathcal{F}}$ the intersection $\mathcal{S} \cap \mathcal{C}_{\mathcal{F}}$.

Let ${}_R U$ be the minimal cogenerator of $\mathfrak{C}_{\mathcal{F}}$. It is well known that

$${}_R U = t_{\mathcal{F}}\left(\bigoplus_{S \in \mathcal{S}} E(S)\right) = \bigoplus_{S \in \mathcal{S}} t_{\mathcal{F}}(E(S))$$

and hence, in our notations, it is:

$${}_R U = \bigoplus_{S \in \mathcal{S}_{\mathcal{F}}} t_{\mathcal{F}}(E(S)).$$

5. LEMMA. *Let R be a left linearly topologized ring with respect to a ring topology τ , let \mathcal{F} be the filter of open left ideals of R and let ${}_R U$ be the minimal cogenerator of $\mathfrak{C}_{\mathcal{F}}$. Then the U -topology of R coincides with the Leptin topology τ^* of τ .*

PROOF. Let $x \in {}_R U$. Then $\text{Ann}_R(x)$ is open and cofinite in R . Conversely, let $J \in \mathcal{F}$ such that $E(R/J) = E(S)$ where $S \in \mathcal{S}$. Since $J \in \mathcal{F}$, $R/J \in \mathfrak{C}_{\mathcal{F}}$ so that $R/J \leq t_{\mathcal{F}}(E(R/J)) = t_{\mathcal{F}}(E(S)) \leq {}_R U$.

6. LEMMA. *In the hypothesis of Lemma above, let ${}_R K$ be a cogenerator of $\mathfrak{C}_{\mathcal{F}}$. Then the K -topology of R is equivalent to τ (i.e. they have the same closed ideals).*

PROOF. Let J be a left ideal of R which is closed in the K -topology of R . Since ${}_R K \in \mathfrak{C}_{\mathcal{F}}$, J is closed in τ . Conversely assume J closed in τ . J is an intersection of open completely irreducible ideals of \mathcal{F} . Thus, by Lemma 5, J is closed in the U -topology of R . Since ${}_R K$ is a cogenerator of $\mathfrak{C}_{\mathcal{F}}$, it contains the minimal cogenerator ${}_R U$. Hence the U -topology of R is contained in the K -topology and thus J is closed in the K -topology of R .

Let ${}_R K_A$ be a bimodule over the rings R and A . We say that R separates points and (finitely generated) submodules of K_A if for every (finitely generated) submodule L of K_A and for every $x \in K \setminus L$, there is an $r \in R$ such that $r(L) = 0$ and $rx \neq 0$.

7. LEMMA. *Let R be a ring, ${}_R K \in R\text{-Mod}$, $A = \text{End}({}_R K)$. If ${}_R K$ is quasi-injective, then R separates points and finitely generated submodules of K_A .*

PROOF. Let L be a finitely generated submodule of K_A and let $y \in K$. Assume that $\text{Ann}_R(y) \supseteq \text{Ann}_R(L)$ and let $\{x_1, \dots, x_n\}$ be a finite system of generators of L_A . Consider the element $x = (x_1, \dots, x_n) \in K^n$

and define a morphism $f: Rx \rightarrow Ry$ by setting $(rx)f = ry$ ($r \in R$). f is well defined since $rx = 0$ means $r \in \bigcap_{i=1}^n \text{Ann}_R(x_i) = \text{Ann}_R(L) \leq \text{Ann}_R(y)$ by assumption. Since ${}_R K$ is q.i. and by Proposition 6.6 [2], f extends to a morphism $\bar{f}: {}_R K^n \rightarrow {}_R K$. Hence there are $a_1, \dots, a_n \in A$ such that $y = (x)f = (x)\bar{f} = \sum_{i=1}^n x_i a_i \in L$.

8. LEMMA. *Let M be a left linearly topologized R -module over the discrete ring R . Assume that M is linearly compact and let Y be an open submodule of ${}_R M$, $(X_i)_{i \in I}$ a family of closed submodules of ${}_R M$. If M/Y is finitely embedded and $\bigcap_{i \in I} X_i \subseteq Y$, then there is a finite subset F of I such that $\bigcap_{i \in F} X_i \subseteq Y$.*

PROOF. M/Y is finitely embedded means that there is a finite number Y_1, \dots, Y_n of modules of M such that $Y = Y_1 \cap \dots \cap Y_n$ and, for each i , $E(M/Y_i)$ is the injective envelope of a simple left R -module S_i . The same proof of Lemma 2 [3] shows that for every $j = 1, \dots, n$ there is a finite subset F_j of I such that $\bigcap_{i \in F_j} X_i \subseteq Y_j$. Setting $F = \bigcup_{i=1}^n F_j$ we get $\bigcap_{i \in F} X_i \subseteq Y$.

9. Let R and A be rings and let ${}_R K_A$ be a faithfully balanced bimodule such that both ${}_R K$ and K_A are strongly quasi-injective. Let \mathcal{F} be the filter of left ideals of R which are open in the K -topology of R and let ${}_R \mathcal{F}$ be the set of maximal left ideals of R belonging to \mathcal{F} . Let $P \in {}_R \mathcal{F}$. R/P is a left simple R -module belonging to $\mathcal{C}_{\mathcal{F}}$, i.e. $R/P \in \mathcal{S}_{\mathcal{F}}$. Let \mathcal{G} be the filter of right ideals of A which are open in the K -topology of A . By statements *d*) and *e*) in 6.9 [2] it follows that for every $P \in {}_R \mathcal{F}$, set $S = R/P$ and $S^* = \text{Hom}_R(S, K) = \text{Ann}_K P$, S^* is a right simple submodule of K_A and moreover each simple submodule of K_A has this form. Since K_A is strongly quasi-injective, K_A is a cogenerator of $\mathcal{C}_{\mathcal{G}}$ (cf. [2], Theorem 6.7), thus K_A contains a copy of each right simple A -module in $\mathcal{C}_{\mathcal{G}}$. Therefore the right simple A -modules of $\mathcal{C}_{\mathcal{G}}$ are precisely those of the form $\text{Ann}_K P$ where $P \in {}_R \mathcal{F}$. Moreover, by Lemma 3, for each $P \in {}_R \mathcal{F}$, $P = \text{Ann}_R \text{Ann}_K(P)$.

Let $(S_\lambda)_{\lambda \in \mathcal{A}}$ be a system of representatives of the isomorphisms classes of the left simple R -modules of $\mathcal{C}_{\mathcal{F}}$. Note that if $\lambda, \mu \in \mathcal{A}$, $\lambda \neq \mu$ then $S_\lambda^* \neq S_\mu^*$. Let $\sigma(S_\lambda)$, $\lambda \in \mathcal{A}$, be the isotypical component of $\text{Soc}({}_R K)$ with respect to S_λ and write $\sigma(S_\lambda) = S_\lambda^{(\nu_\lambda)}$ where ν_λ is a

suitable cardinal number and $S_\lambda^{(\nu_\lambda)}$ denotes the direct sum of ν_λ copies of S_λ . By Proposition 6.10 [2], $\text{Soc}({}_R K) = \text{Soc}(K_A)$, $\text{Soc}({}_R K)$ is essential in ${}_R K$ and $\text{Soc}(K_A)$ is essential in K_A . Moreover it is $\text{Soc}(K_A) = \bigoplus_{\lambda \in A} \sigma(S_\lambda^*)$ and $\sigma(S_\lambda) = \sigma(S_\lambda^*)$. Finally, for every $\lambda \in A$, $\sigma(S_\lambda^*) = S_\lambda^{*(\mu_\lambda)}$ where μ_λ is a suitable cardinal number. The cardinal numbers ν_λ and μ_λ ($\lambda \in A$) are uniquely determined by ${}_R K_A$.

10. THEOREM. *Let ${}_R K_A$ be a faithfully balanced bimodule over the rings R and A such that ${}_R K$ is strongly quasi-injective. The following statements are equivalent:*

- (a) K_A is strongly quasi-injective.
- (b) R is linearly compact in the K -topology and R separates points and submodules of K_A .
- (c) R is a linearly compact in the K -topology and $\text{Soc}({}_R K)$ is essential in ${}_R K$.

If these conditions hold, then A is linearly compact in the K -topology and moreover, using the notations of 9., it is

$${}_R K \cong \bigoplus_{\lambda \in A} t_{\mathcal{F}}(E_R(\sigma(S_\lambda))) = \bigoplus_{\lambda \in A} [t_{\mathcal{F}}(E_R(S_\lambda))]^{(\nu_\lambda)}$$

and

$$K_A \cong \bigoplus_{\lambda \in A} t_{\mathcal{G}}(E_A(\sigma(S_\lambda^*))) = \bigoplus_{\lambda \in A} [t_{\mathcal{G}}(E_A((S_\lambda^*)))]^{(\mu_\lambda)}.$$

PROOF. (a) \Rightarrow (b) follows by Proposition 4, since, as we remarked in 9., ${}_R K$ is a cogenerator of $\mathcal{C}_{\mathcal{F}}$.

(b) \Rightarrow (a) follows by Proposition 2.

(a) \Rightarrow (c). By (b) R is linearly compact in the K -topology and since ${}_R K_A$ is faithfully balanced with both ${}_R K$ and K_A s.q.i., $\text{Soc}({}_R K)$ is essential in ${}_R K$, as we recalled in 9.

(c) \Rightarrow (b). First of all, let us prove that ${}_R K \leq \bigoplus_{\lambda \in A} t_{\mathcal{F}}(E(S_\lambda))^{(\nu_\lambda)}$.

Let $x \in K$. Rx is linearly compact discrete and hence $\text{Soc}(Rx)$ is a direct sum of a finite number of left simple R -modules S_1, \dots, S_n . By hypothesis, $\text{Soc}({}_R K)$ is essential in ${}_R K$. Hence $\text{Soc}(Rx)$ is essen-

tial in Rx . It follows that

$$(1) \quad Rx \leq \bigoplus_{i=1}^n t_{\mathcal{F}}(E(S_i))$$

and hence the claimed inclusion is proved.

Let us prove that R separates points and submodules of K_A . Let $L \leq K_A$ and let $x \in K$. Assume that $\text{Ann}_R(x) \geq \text{Ann}_R(L)$. Note that, by (1), $R \setminus \text{Ann}_R(x) \cong Rx$ is finitely embedded. Hence, by Lemma 8, there is a finite subset $F \subseteq L$ such that $\text{Ann}_R(x) \geq \bigcap_{l \in F} \text{Ann}_R(l)$. Thus, by Lemma 7, x belongs to the submodule of K_A spanned by F and hence $x \in L$.

Let us assume that the equivalent conditions (a), (b) and (c) hold. We have already seen in the proof of (c) \Rightarrow (b) that

$${}_R K \leq \bigoplus_{\lambda \in A} [t_{\mathcal{F}}(E(S_\lambda))]^{(v_\lambda)}.$$

Obviously, it is clear that for every $\lambda \in A$,

$$[t_{\mathcal{F}}(E(S_\lambda))]^{(v_\lambda)} \leq t_{\mathcal{F}}(E(\sigma(S_\lambda))).$$

Since ${}_R K$ is s.q.i., ${}_R K$ is an injective cogenerator of $\mathcal{C}_{\mathcal{F}}$ (cf. [2], Theorem 6.7). Thus it is straightforward to prove that $\bigoplus_{\lambda \in A} t_{\mathcal{F}}(E(\sigma(S_\lambda))) \leq {}_R K$. Hence we get the following chain of inclusions:

$${}_R K \leq \bigoplus_{\lambda \in A} [t_{\mathcal{F}}(E(S_\lambda))]^{(v_\lambda)} \leq \bigoplus_{\lambda \in A} t_{\mathcal{F}}(E(\sigma(S_\lambda))) \leq {}_R K$$

and therefore the first chain of inclusions is proved.

In view of remarks in 9. and by symmetry, the analogous equalities hold for K_A .

11. COROLLARY. *Let ${}_R K_A$ be a faithfully balanced bimodule such that both ${}_R K$ and K_A are s.q.i. Let ${}_R \mathcal{F}$ be the set of left maximal ideals of R which are open in the K -topology of R and let $J(R)$ be the Jacobson radical of R . Then*

$$J(R) = \bigcap \{P : P \in {}_R \mathcal{F}\}.$$

In particular, $J(R)$ is closed in the K -topology of R .

PROOF. Let Z_A denote the socle of K_A . As we recalled in 9., it is $\text{Ann}_R(Z_A) = \bigcap \{P: P \in {}_R\mathcal{F}\} \supseteq J(R)$.

Let $a \in \text{Ann}_R(Z_A)$ and, by way of contradiction, assume that $a \notin J(R)$. Thus there is a left maximal ideal Q of R such that $a \notin Q$. Hence $Ra + Q = R$ and therefore $1 = ra + q$, where $r \in R$ and $q \in Q$. Since $a \in \text{Ann}_R(Z_A)$, for every $x \in Z_A$ it is $qx = (1 - ra)x = x$. Thus, since Z_A is essential in K_A , q , as endomorphism of K_A , is injective and $\text{Im}(q) = K_A$. Let $q': \text{Im}(q) \rightarrow K$ be the left inverse of the corestriction of q to $\text{Im}(q)$. Since K_A is q.i., q' extends to an endomorphism of K_A . Thus $1 \in Q$. Contradiction.

12. REMARK. Let R be a linearly compact ring with respect to a left linear topology τ and \mathcal{F} be the filter of open left ideals of R . Let ${}_R U = \bigoplus_{S \in \mathcal{S}_{\mathcal{F}}} t_{\mathcal{F}}(E(S))$ be the minimal cogenerator of $\mathcal{C}_{\mathcal{F}}$ and denote by \mathcal{F}^* the filter of left ideals which are open in the U -topology of R , i.e. in the Leptin topology of τ, τ^* (cf. Lemma 5). Clearly, a left simple R -module belongs to $\mathcal{C}_{\mathcal{F}}$ if and only if it belongs to $\mathcal{C}_{\mathcal{F}^*}$. Moreover for each simple left R -module S , $t_{\mathcal{F}}(E(S)) = t_{\mathcal{F}^*}(E(S))$. In fact, since $\mathcal{F}^* \subseteq \mathcal{F}$, $t_{\mathcal{F}^*}(E(S)) \leq t_{\mathcal{F}}(E(S))$. On the other hand,

$$t_{\mathcal{F}}(E(S)) \leq {}_R U \in \mathcal{C}_{\mathcal{F}^*}.$$

In particular ${}_R U$ is also the minimal cogenerator of $\mathcal{C}_{\mathcal{F}^}$.*

PROOF OF THE MAIN THEOREM. (a) \Rightarrow (b). Let ${}_R K$ be a cogenerator of $\mathcal{C}_{\mathcal{F}}$ and let $A = \text{End}({}_R K)$. By Lemma 6, the K -topology of R is equivalent to τ and hence R is linearly compact in the K -topology too. Thus, since ${}_R K$ is a selfcogenerator, by Corollary 7.4 [2], $R = \text{End}(K_A)$ and therefore ${}_R K_A$ is faithfully balanced. By Proposition 2, K_A is q.i.

(b) \Rightarrow (c) is trivial.

(c) \Rightarrow (a). Since ${}_R K$ is faithful, the K -topology of R is Hausdorff. By Lemma 6, τ is equivalent to the K -topology of R and hence τ is Hausdorff too. Thus, by Proposition 4, R is linearly compact.

(d) \Rightarrow (c) is trivial.

(a) \Rightarrow (d). Let us remark, first of all, that in view of Lemma 5, R is linearly compact in the U -topology. Let us now proceed by steps.

1) ${}_R U$ is *q.i.* Set $E = E\left(\bigoplus_{S \in \mathfrak{S}_{\mathcal{F}}} S\right)$. Let us prove that ${}_R U = t_{\mathcal{F}}(E)$.

From this the claim will follow for ${}_R U$ will be a fully invariant submodule of ${}_R E$. Since ${}_R U = \bigoplus_{S \in \mathfrak{S}_{\mathcal{F}}} t_{\mathcal{F}}(E(S))$, it is clear that ${}_R U \leq t_{\mathcal{F}}(E)$.

Conversely, let $x \in t_{\mathcal{F}}(E)$. Then $\text{Ann}_R(x) \in \mathcal{F}$ and hence $Rx \cong \cong R/\text{Ann}_R(x)$ is linearly compact discrete. Thus $\text{Soc}(Rx)$ is a direct sum of a finite number of (non-isomorphic) left simple R -modules. Since $x \in E$, $\text{Soc}(Rx)$ is essential in Rx . Thus $x \in t_{\mathcal{F}}(E(\text{Soc}(Rx))) \leq {}_R U$.

2) ${}_R U$ is *s.q.i.* By Remark 12 and by Theorem 6.7 [2].

3) ${}_R U_{\tau}$ is *faithfully balanced*. Since R is linearly compact in the U -topology, it is complete. Thus, since ${}_R U$ is a selfcogenerator, by Corollary 7.4 [2], $R = \text{End}(U_{\tau})$.

4) U_{τ} is *s.q.i.* Note that $\text{Soc}({}_R U)$ is essential in ${}_R U$. Then the claim follows by Theorem 10.

13. COROLLARY. *Let R be a left linearly compact ring with respect to a ring topology τ , let ${}_R \mathfrak{S}$ be the set of open left maximal ideals of R and let $J(R)$ be the Jacobson radical of R . Then*

$$J(R) = \bigcap \{P : P \in {}_R \mathfrak{S}\}.$$

In particular, $J(R)$ is closed in R .

PROOF. Follows by THE MAIN THEOREM, by Corollary 11 and by Remark 12.

The idea of the following application is due to Prof. A. Orsatti.

14. THEOREM (Leptin [1]). *Let R be a left linearly topologized ring with respect to a ring topology τ . Assume that R is linearly compact and that the Jacobson radical of R , $J(R)$, is zero. Then R , endowed with the Leptin topology of τ , is topologically isomorphic to a topological product $\prod_{\lambda \in \Lambda} \text{End}_{D_{\lambda}}(V_{\lambda})$ where, for every $\lambda \in \Lambda$, V_{λ} is a vector space over the division ring D_{λ} and $\text{End}_{D_{\lambda}}(V_{\lambda})$ is endowed with the finite topology.*

(Zelinsky [5]). *Moreover if τ has two-sided ideals as a basis of neighbourhoods of zero, each V_{λ} has finite dimension over D_{λ} .*

PROOF. Let \mathcal{F} be the filter of open left ideals of τ and let ${}_R U$ be the minimal cogenerator of $\mathfrak{C}_{\mathcal{F}}$. Set $A = \text{End}({}_R U)$. By THE MAIN

THEOREM, ${}_R U_A$ is faithfully balanced and both the modules ${}_R U$ and U_A are s.q.i. Suppose that $\text{Soc}(U_A)$ is strictly contained in U_A . Then, since U_A is s.q.i. and $R = \text{End}(U_A)$, there is a non zero element $r \in R$ such that $r(\text{Soc}(U_A)) = 0$. Thus, by 9., by Remark 12 and by Corollary 13, r belongs to the Jacobson radical of R . Hence $\text{Soc}(U_A) = U_A$. Since $\text{Soc}(U_A) = \text{Soc}({}_R U)$ (cf. 9.), we get ${}_R U = \bigoplus_{\lambda \in A} S_\lambda$,

where $(S_\lambda)_{\lambda \in A}$ is a system of representatives of the isomorphism classes of the left simple R -modules of \mathfrak{F} . Thus each S_λ is fully invariant in ${}_R U$ and hence A is canonically isomorphic to the ring product $\prod_{\lambda \in A} D_\lambda$ where, for each $\lambda \in A$, $D_\lambda = \text{End}_R(S_\lambda)$ is a division ring.

Of course such a product acts componentwise over U so that the action of A over each S_λ naturally identifies with that of D_λ . Recall that, by 9., $S_\lambda = \sigma(S_\lambda^*)$. Moreover, since $R = \text{End}(U_A)$, each S_λ is fully invariant submodule of U_A . Therefore we get the natural algebraic isomorphisms

$$(1) \quad \text{End}(U_A) \cong \prod_{\lambda \in A} \text{End}_A(S_\lambda) = \prod_{\lambda \in A} \text{End}_{D_\lambda}(S_\lambda).$$

Now, since ${}_R U$ is a selfgenerator, by Corollary 7.4 [2], $\text{End}(U_A)$, endowed with the finite topology, is isomorphic to the completion of R in the U -topology. Since R is linearly compact in τ , the first statement follows easily by Lemma 5, as soon as we note that the finite topology of $\text{End}(U_A)$ corresponds, through the isomorphisms (1), to the product topology of the finite topologies on the $\text{End}_{D_\lambda}(S_\lambda)$, $\lambda \in A$.

Assume now that τ has two-sided ideals as a basis of neighbourhoods of 0. Fix $\lambda \in A$ and let $P \in {}_R \mathfrak{F}$ such that $R/P \cong S_\lambda$. Since $P \in \mathfrak{F}$, P contains an open two-sided ideal. Since $\text{Ann}_R(S_\lambda)$ is the largest two-sided ideal contained in P , it follows that $\text{Ann}_R(S_\lambda) \in \mathfrak{F}$. Let $\{e_i\}_{i \in I}$ be a basis of S_λ as a vector space over D_λ . Then $\text{Ann}_R(S_\lambda) = \bigcap_{i \in I} \text{Ann}_R(e_i)$. Since $(\text{Ann}_R(e_i))_{i \in I}$ is a family of open coprimary left ideals of R and R is linearly compact, it is easy to check that the diagonal map $R/\text{Ann}_R(S_\lambda) \rightarrow \prod_{i \in I} R/\text{Ann}_R(e_i)$ of the canonical maps $R/\text{Ann}_R(S_\lambda) \rightarrow R/\text{Ann}_R(e_i)$ ($i \in I$), is an isomorphism. Since $R/\text{Ann}_R(S_\lambda)$ is linearly compact discrete, I must be finite.

15. REMARKS. 1) In the hypothesis of Theorem above, if τ is the discrete topology, then R is semisimple artinian ([5]). In fact, since ${}_R U$ is linearly compact discrete (cf. [3], Th. 1), A is finite.

2) In the hypothesis of Theorem above, if τ has two-sided ideals as a basis of neighbourhoods of zero, then $\tau = \tau^*$. In fact let L be an open two-sided ideal of R . Then R/L is a discrete linearly compact ring with zero Jacobson radical. By 1) above, R/L is artinian. Thus L is cofinite.

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