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Linearly compact rings and strongly quasi-injective modules

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Introduction.

Throughout this paper, all rings are associative with identity $1 \neq 0$ and all modules are unitary.

Let $R$ be a ring. A left $R$-module $_RK$ is called strongly quasi-injective (for short s.q.i.) if given any submodule $B$ of $_RK$, a morphism $f : B \to _RK$ and an element $x \in K \setminus B$, $f$ extends to an endomorphism $\tilde{f}$ of $_RK$ such that $(x)\tilde{f} \neq 0$.

The notion of s.q.i. module comes from the study of dualities, induced by topological bimodules, between a category of abstract modules and a category of topological modules, where it plays a central role (cf. [2]).

Investigating on the concept of s.q.i. module, the following question naturally arises. Let $_RK$ be a s.q.i. module, $A = \text{End} (_RK)$. When is $K_A$ s.q.i.? The study of this problem leads to the following characterization of linearly compact rings.

**The Main Theorem.** Let $R$ be a left linearly topologized ring with respect to a ring topology $\tau$, let $\mathcal{F}$ be the filter of open left ideals of $R$ and let $\mathcal{C}_\mathcal{F}$ be the hereditary pretorsion class of left $R$-modules associated with $\mathcal{F}$. The following statements are equivalent.

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Lavoro eseguito nell'ambito della attività dei gruppi di ricerca matematica del C.N.R.
(a) $\mathcal{R}$ is linearly compact in the topology $\tau$.

(b) If $\mathcal{R}K$ is a cogenerator of $\mathcal{C}_{\mathcal{F}}$ and $A = \text{End} (\mathcal{R}K)$, then $\mathcal{R}K_A$ is faith-fully balanced and $K_A$ is quasi-injective.

(c) There exists a faithfully balanced module $\mathcal{R}K_A$ such that $\mathcal{R}K$ is a cogenerator of $\mathcal{C}_{\mathcal{F}}$ and $K_A$ is quasi-injective.

(d) Let $\mathcal{R}U$ be a minimal cogenerator of $\mathcal{C}_{\mathcal{F}}$, $T = \text{End} (\mathcal{R}U)$. Then $\mathcal{R}U_T$ is faithfully balanced and both the modules $\mathcal{R}U$ and $U_T$ are s.q.i.

Moreover, if condition (d) is fulfilled, $T$ is linearly compact in its $U$-adic topology.

(See below for explained definitions.)

Some results obtained in [4] for discrete linearly compact rings are here extended to the general case.

As an application of our results, we get a quick proof of Leptin’s theorem which characterizes a linearly compact ring with zero Jacobson radical as a cartesian product of endomorphism rings of vector spaces.

A structure theorem on faithfully balanced modules $\mathcal{R}K_A$ which are s.q.i. both on $\mathcal{R}$ and $A$, obtained as intermediate result, has an intrinsic interest (cf. Theorem 10).

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Some conventions and notations. Let $\mathcal{R}$ be a ring. $\mathcal{R}$-Mod will denote the category of left $\mathcal{R}$-modules and Mod-$\mathcal{R}$ that of right $\mathcal{R}$-modules. The notation $\mathcal{R}M$ will be used to emphasize that $M$ is a left $\mathcal{R}$-module. Morphisms between modules will be written on the opposite side to that of the scalars and the composition of morphisms will follow this convention. For every $M \in \mathcal{R}$-Mod, $E_{\mathcal{R}}(M)$, or simply $E(M)$, will denote the injective envelope of $M$ in $\mathcal{R}$-Mod and $\text{Soc} (M)$, or simply $\text{Soc} (M)$, the socle of $M$. If $L$ is a subset of $\mathcal{R}M \in \mathcal{R}$-Mod, we denote by $\text{Ann}_{\mathcal{R}} (L)$ the annihilator of $L$ in $\mathcal{R}$:

$$\text{Ann}_{\mathcal{R}} (L) = \{ r \in \mathcal{R} : rx = 0 \text{ for every } x \in L \}.$$  

If $L = \{ x \}$, we will simply write $\text{Ann}_{\mathcal{R}} (x)$.

If $J$ is a left ideal of $\mathcal{R}$, we define the annihilator of $J$ in $\mathcal{M}$, $\text{Ann}_{\mathcal{M}} (J)$, by setting:

$$\text{Ann}_{\mathcal{M}} (J) = \{ x \in \mathcal{M} : rx = 0 \text{ for every } r \in J \}.$$
The annihilator in \(R\) of \(\text{Ann}_M(J)\) will be denoted by \(\text{Ann}_R \text{Ann}_M(J)\). Analogous notations will be used for right modules.

\(N\) will denote the set of positive integers.

1. To begin with, let us recall some definitions.

Let \(R\) be a ring and let \(M \in R\text{-Mod}\). \(M\) is \textit{quasi-injective} (for short \(q.i.\)) if for every submodule \(L \leq_M M\) and every morphism \(f : L \rightarrow_R M\), \(f\) extends to an endomorphism \(\tilde{f}\) of \(M\). \(M\) is a \textit{self-cogenerator} if, for every \(n \in N\), given a submodule \(L\) of \(M^n\) and an element \(x \in M^n \setminus L\), there exists a morphism \(f : R^n \rightarrow_R M\) such that \((L)f = 0\) and \((x)f \neq 0\). Clearly if \(M\) is both quasi-injective and self-cogenerator, then \(M\) is strongly quasi-injective. The converse is true as well (cf. [2], Corollary 4.5).

Let \(K_A\) be a bimodule. \(K_A\) is \textit{faithfully balanced} if \(A \cong \text{End}(K_A)\) and \(R \cong \text{End}(K_A)\) canonically.

Let \(R\) be a ring and let \(M \in R\text{-Mod}\). The \(M\)-\textit{topology} of \(R\) is defined by taking as a basis of neighbourhoods of 0 in \(R\) the annihilators in \(R\) of the finite subsets of \(M\). It is easy to check that this topology is a left linear ring topology on \(R\).

Finally recall that a linearly topologized left module \(M\) over a discrete ring \(R\) is said to be \textit{linearly compact} if \(M\) is Hausdorff and if any finitely solvable system of congruences \(x \equiv x_i \mod X_i\), where the \(X_i\) are closed submodules of \(M\), is solvable.

2. \textbf{Proposition.} Let \(R\) be a ring, \(K \in R\text{-Mod}\) a self-cogenerator, \(A = \text{End}(K)\). If \(R\) is linearly compact in the \(K\)-topology, then \(K_A\) is quasi-injective.

\textbf{Proof.} Cf. [4], Prop. 3.4 a).

Let \(R\) be a ring, \(\tau\) a left linear ring topology on \(R\), \(\mathcal{F}\) the filter of open left ideals of \(R\). The left exact preradical in \(R\text{-Mod}\) associated with \(\mathcal{F}\), \(t_\mathcal{F}\), is defined by setting, for every \(M \in R\text{-Mod}\):

\[t_\mathcal{F}(M) = \{x \in M : \text{Ann}_R(x) \in \mathcal{F}\}.

The hereditary pretorsion class of \(R\text{-Mod}\) associated with \(\mathcal{F}\) is defined by setting

\[\mathcal{G}_\mathcal{F} = \{M \in R\text{-Mod} : M = t_\mathcal{F}(M)\}.

3. \textbf{Lemma.} Let \(R\) be a left linearly topologized ring with respect to a ring topology \(\tau\), let \(\mathcal{F}\) be the filter of open left ideals of \(R\) and let \(K\)
be a cogenerator of $\mathcal{C}_R$. For every closed left ideal $J$ of $R$ it is
\[ \text{Ann}_R \text{Ann}_R (J) = J. \]

**Proof.** Let $r \in R \setminus J$. There is an open left ideal $L$ of $R$ such that $L \supset J$ and $r \notin L$. Since $\mathcal{R}K$ is a cogenerator of $\mathcal{C}_R$, there is a morphism $f: R/L \to \mathcal{R}K$ such that $(r + L)f \neq 0$. Hence there is an $x \in \mathcal{R}K$ such that $Lx = 0$ and $rx \neq 0$. Thus $Jx = 0$ and therefore $r \notin \text{Ann}_R \text{Ann}_R (J)$.

4. **Proposition.** Let $\mathcal{R}K_A$ be a faithfully balanced bimodule, let $\tau$ be a left linear Hausdorff ring topology on $R$ and let $\mathcal{F}$ be the filter of open left ideals of $R$. Assume that $\mathcal{R}K$ is a cogenerator of $\mathcal{C}_R$ and that $KA$ is quasi-injective. Then $R$ is linearly compact in the topology $\tau$.

**Proof.** The following technique is due to Müller (cf. [3], Lemma 4). Let $(J_i)_{i \in I}$ be a family of closed left ideals of $R$ and let $\mathcal{F}$ be a finitely solvable system of congruences in $R$. Set $L = \sum_{i \in I} \text{Ann}_R (J_i)$. $L$ is a submodule of $KA$. Define a morphism $g: L \to KA$ by setting
\[ g \left( \sum_{i \in F} x_i \right) = \sum_{i \in F} r_i x_i, \]
where $F$ is a finite subset of $I$ and, for every $i \in F$, $x_i \in \text{Ann}_R (J_i)$. Since $1$ is finitely solvable, $g$ is well defined. Since $KA$ is quasi-injective, $g$ extends to an endomorphism of $\mathcal{R}K_A$. Since $\mathcal{R}K_A$ is faithfully balanced, this endomorphism is the left multiplication by an element $r \in R$ so that we have, for every $i \in I$, $r - r_i \in \text{Ann}_R \text{Ann}_R (J_i)$. By Lemma 3, $\text{Ann}_R \text{Ann}_R (J_i) = J_i$ for every $i \in I$, thus (1) is solvable.

Let $R$ be a ring and let $\tau$ be a left linear ring topology on $R$. The **Leptin topology** $\tau^*$ of $\tau$ is the ring topology on $R$ defined by taking as a basis of neighbourhoods of 0 in $R$ the cofinite open left ideals of $R$. Recall that a left ideal of $R$ is **cofinite** if it is a finite intersection of completely irreducible left ideals of $R$. A left ideal $I$ of $R$ is **completely irreducible** if $R/I$ is an essential submodule of the injective envelope $E(S)$ of a left simple $R$-module $S$.

Let $\mathcal{F}$ be the filter of open left ideals of $R$. In the following $S$ will always denote a system of representatives of the isomorphism classes of the simple left $R$-modules and $S_{\mathcal{F}}$ the intersection $S \cap \mathcal{C}_R$. 
Let $rU$ be the minimal cogenerator of $G_\mathcal{F}$. It is well known that

$$rU = t_\mathcal{F}\left(\bigoplus_{s \in S} E(S)\right) = \bigoplus_{s \in S} t_\mathcal{F}(E(S))$$

and hence, in our notations, it is:

$$rU = \bigoplus_{s \in S} t_\mathcal{F}(E(S)).$$

5. **Lemma.** Let $R$ be a left linearly topologized ring with respect to a ring topology $\tau$, let $\mathcal{F}$ be the filter of open left ideals of $R$ and let $rU$ be the minimal cogenerator of $G_\mathcal{F}$. Then the $U$-topology of $R$ coincides with the Leptin topology $\tau^*$ of $\tau$.

**Proof.** Let $x \in rU$. Then $\text{Ann}_R(x)$ is open and cofinite in $R$. Conversely, let $J \in \mathcal{F}$ such that $E(R/J) = E(S)$ where $S \in S$. Since $J \in \mathcal{F}$, $R/J \in G_\mathcal{F}$ so that $R/J \leq t_\mathcal{F}(E(R/J)) = t_\mathcal{F}(E(S)) \leq rU$.

6. **Lemma.** In the hypothesis of Lemma above, let $rK$ be a cogenerator of $G_\mathcal{F}$. Then the $K$-topology of $R$ is equivalent to $\tau$ (i.e. they have the same closed ideals).

**Proof.** Let $J$ be a left ideal of $R$ which is closed in the $K$-topology of $R$. Since $rK \in G_\mathcal{F}$, $J$ is closed in $\tau$. Conversely assume $J$ closed in $\tau$. $J$ is an intersection of open completely irreducible ideals of $\mathcal{F}$. Thus, by Lemma 5, $J$ is closed in the $U$-topology of $R$. Since $rK$ is a cogenerator of $G_\mathcal{F}$, it contains the minimal cogenerator $rU$. Hence the $U$-topology of $R$ is contained in the $K$-topology and thus $J$ is closed in the $K$-topology of $R$.

Let $rK_A$ be a bimodule over the rings $R$ and $A$. We say that $R$ separates points and (finitely generated) submodules of $K_A$ if for every (finitely generated) submodule $L$ of $K_A$ and for every $x \in K \setminus L$, there is an $r \in R$ such that $r(L) = 0$ and $rx \neq 0$.

7. **Lemma.** Let $R$ be a ring, $rK \in R$-Mod, $A = \text{End}(rK)$. If $rK$ is quasi-injective, then $R$ separates points and finitely generated submodules of $K_A$.

**Proof.** Let $L$ be a finitely generated submodule of $K_A$ and let $y \in K$. Assume that $\text{Ann}_R(y) > \text{Ann}_R(L)$ and let $\{x_1, \ldots, x_n\}$ be a finite system of generators of $L_A$. Consider the element $x = (x_1, \ldots, x_n) \in K^n$.
and define a morphism \( f : Rx \to Ry \) by setting \( (rx)f = ry \) \((r \in R)\). \( f \) is well defined since \( rx = 0 \) means \( r \in \bigcap_{i=1}^{n} \text{Ann}_R(x_i) = \text{Ann}_R(L) \subset \text{Ann}_R(y) \) by assumption. Since \( R \) is q.i. and by Proposition 6.6 [2], \( f \) extends to a morphism \( \bar{f} : R^n \to RK \). Hence there are \( a_1, \ldots, a_n \in A \) such that \( y = (x)f = (x)\bar{f} = \sum_{i=1}^{n} x_ia_i \in L \).

8. Lemma. Let \( M \) be a left linearly topologized \( R \)-module over the discrete ring \( R \). Assume that \( M \) is linearly compact and let \( Y \) be an open submodule of \( R M \), \( (X_i)_{i \in I} \) a family of closed submodules of \( R M \). If \( M/Y \) is finitely embedded and \( \bigcap_{i \in F} X_i \subseteq Y \), then there is a finite subset \( F \) of \( I \) such that \( \bigcap_{i \in F} X_i \subseteq Y \).

Proof. \( M/Y \) is finitely embedded means that there is a finite number \( Y_1, \ldots, Y_n \) of modules of \( M \) such that \( Y = Y_1 \cap \cdots \cap Y_n \) and, for each \( i \), \( E(M/Y_i) \) is the injective envelope of a simple left \( R \)-module \( S_i \). The same proof of Lemma 2 [3] shows that for every \( j = 1, \ldots, n \) there is a finite subset \( F_j \) of \( I \) such that \( \bigcap_{i \in F_j} X_i \subseteq Y_i \).

Setting \( F = \bigcup_{i=1}^{n} F_j \) we get \( \bigcap_{i \in F} X_i \subseteq Y \).

9. Let \( R \) and \( A \) be rings and let \( RK_A \) be a faithfully balanced bimodule such that both \( R \) and \( K_A \) are strongly quasi-injective. Let \( F \) be the filter of left ideals of \( R \) which are open in the \( K \)-topology of \( R \) and let \( S \) be the set of maximal left ideals of \( R \) belonging to \( F \). Let \( P \in S \). \( R/P \) is a left simple \( R \)-module belonging to \( C_F \), i.e. \( R/P \in S_F \). Let \( G \) be the filter of right ideals of \( A \) which are open in the \( K \)-topology of \( A \). By statements \( d) \) and \( e) \) in 6.9 [2] it follows that for every \( P \in S \), set \( S = R/P \) and \( S^* = \text{Hom}_R(S, K) = \text{Ann}_K P \), \( S^* \) is a right simple submodule of \( K_A \) and moreover each simple submodule of \( K_A \) has this form. Since \( K_A \) is strongly quasi-injective, \( K_A \) is a cogenerator of \( C_G \) (cf. [2], Theorem 6.7), thus \( K_A \) contains a copy of each right simple \( A \)-module in \( C_G \). Therefore the right simple \( A \)-modules of \( C_G \) are precisely those of the form \( \text{Ann}_K P \) where \( P \in S \). Moreover, by Lemma 3, for each \( P \in S \), \( P = \text{Ann}_K \text{Ann}_K (P) \).

Let \( (S_\lambda)_{\lambda \in \Lambda} \) be a system of representatives of the isomorphisms classes of the left simple \( R \)-modules of \( C_F \). Note that if \( \lambda, \mu \in \Lambda \) and \( \lambda \neq \mu \) then \( S_\lambda^* \neq S_\mu^* \). Let \( \sigma(S_\lambda) \), \( \lambda \in \Lambda \), be the isotypical component of \( \text{Soc} (RK) \) with respect to \( S_\lambda \) and write \( \sigma(S_\lambda) = S_\lambda^{(v_\lambda)} \) where \( v_\lambda \) is a
suitable cardinal number and $S^{(\nu_\lambda)}_\lambda$ denotes the direct sum of $\nu_\lambda$ copies of $S_\lambda$. By Proposition 6.10 [2], $\text{Soc}(rK) = \text{Soc}(K_A)$, $\text{Soc}(rK)$ is essential in $rK$ and $\text{Soc}(K_A)$ is essential in $K_A$. Moreover it is $\text{Soc}(K_A) = \bigoplus_{\lambda \in \Lambda} \sigma(S^*_\lambda)$ and $\sigma(S_\lambda) = \sigma(S^*_\lambda)$. Finally, for every $\lambda \in \Lambda$, $\sigma(S^*_\lambda) = S^{(\mu_\lambda)}_\lambda$ where $\mu_\lambda$ is a suitable cardinal number. The cardinal numbers $\nu_\lambda$ and $\mu_\lambda (\lambda \in \Lambda)$ are uniquely determined by $rK_A$.

10. THEOREM. Let $rK_A$ be a faithfully balanced bimodule over the rings $R$ and $A$ such that $rK$ is strongly quasi-injective. The following statements are equivalent:

(a) $K_A$ is strongly quasi-injective.

(b) $R$ is linearly compact in the $K$-topology and $R$ separates points and submodules of $K_A$.

(c) $R$ is a linearly compact in the $K$-topology and $\text{Soc}(rK)$ is essential in $rK$.

If these conditions hold, then $A$ is linearly compact in the $K$-topology and moreover, using the notations of 9., it is

$$rK \cong \bigoplus_{\lambda \in \Lambda} t_{\mathcal{F}}(E_R(\sigma(S_\lambda))) = \bigoplus_{\lambda \in \Lambda} \left[ t_{\mathcal{F}}(E_R(S_\lambda)) \right]^{(\nu_\lambda)}$$

and

$$K_A \cong \bigoplus_{\lambda \in \Lambda} t_{\mathcal{G}}(E_A(\sigma(S^*_\lambda))) = \bigoplus_{\lambda \in \Lambda} \left[ t_{\mathcal{G}}(E_A(S^*_\lambda)) \right]^{(\mu_\lambda)}.$$ 

PROOF. (a) $\Rightarrow$ (b) follows by Proposition 4, since, as we remarked in 9., $rK$ is a cogenerator of $\mathcal{G}_{\mathcal{F}}$.

(b) $\Rightarrow$ (a) follows by Proposition 2.

(a) $\Rightarrow$ (c). By (b) $R$ is linearly compact in the $K$-topology and since $rK_A$ is faithfully balanced with both $rK$ and $K_A$ s.q.i., $\text{Soc}(rK)$ is essential in $rK$, as we recalled in 9.

(c) $\Rightarrow$ (b). First of all, let us prove that $rK \leq \bigoplus_{\lambda \in \Lambda} t_{\mathcal{F}}(E(S_\lambda))^{(\nu_\lambda)}$.

Let $x \in K$. $Rx$ is linearly compact discrete and hence $\text{Soc}(Rx)$ is a direct sum of a finite number of left simple $R$-modules $S_1, \ldots, S_n$. By hypothesis, $\text{Soc}(rK)$ is essential in $rK$. Hence $\text{Soc}(Rx)$ is essen-
tial in $Rx$. It follows that

$$Rx \leq \bigoplus_{i=1}^{n} t_{\mathcal{F}}(E(S_i))$$

and hence the claimed inclusion is proved.

Let us prove that $R$ separates points and submodules of $K_A$. Let $L < K_A$ and let $x \in K$. Assume that $\text{Ann}_R(x) \supseteq \text{Ann}_R(L)$. Note that, by (1), $R \setminus \text{Ann}_R(x) \cong Rx$ is finitely embedded. Hence, by Lemma 8, there is a finite subset $F \subseteq L$ such that $\text{Ann}_R(x) \supseteq \bigcap_{l \in F} \text{Ann}_R(l)$. Thus, by Lemma 7, $x$ belongs to the submodule of $K_A$ spanned by $F$ and hence $x \in L$.

Let us assume that the equivalent conditions (a), (b) and (c) hold. We have already seen in the proof of (c) $\Rightarrow$ (b) that

$$\rho_K \leq \bigoplus_{\lambda \in \Lambda} [t_{\mathcal{F}}(E(S_\lambda))]^{(\sigma_2)}.$$

Obviously, it is clear that for every $\lambda \in \Lambda$,

$$[t_{\mathcal{F}}(E(S_\lambda))]^{(\sigma_2)} \leq t_{\mathcal{F}}(E(\sigma(S_\lambda))).$$

Since $\rho_K$ is s.q.i., $\rho_K$ is an injective cogenerator of $\mathcal{V}_{\mathcal{F}}$ (cf. [2], Theorem 6.7). Thus it is straightforward to prove that $\bigoplus_{\lambda \in \Lambda} t_{\mathcal{F}}(E(\sigma(S_\lambda))) \leq \rho_K$. Hence we get the following chain of inclusions:

$$\rho_K \leq \bigoplus_{\lambda \in \Lambda} [t_{\mathcal{F}}(E(S_\lambda))]^{(\sigma_2)} \leq \bigoplus_{\lambda \in \Lambda} t_{\mathcal{F}}(E(\sigma(S_\lambda))) \leq \rho_K$$

and therefore the first chain of inclusions is proved.

In view of remarks in 9. and by symmetry, the analogous equalities hold for $K_A$.

11. COROLLARY. Let $\rho_{K_A}$ be a faithfully balanced bimodule such that both $\rho_K$ and $K_A$ are s.q.i. Let $\rho_3$ be the set of left maximal ideals of $R$ which are open in the $K$-topology of $R$ and let $J(R)$ be the Jacobson radical of $R$. Then

$$J(R) = \bigcap \{P : P \in \rho_3\}.$$

In particular, $J(R)$ is closed in the $K$-topology of $R$. 
PROOF. Let \( Z_A \) denote the socle of \( K_A \). As we recalled in (9.), it is 
\( \text{Ann}_R(Z_A) = \bigcap \{ P : P \in \mathcal{R} \} = J(R) \).

Let \( a \in \text{Ann}_R(Z_A) \) and, by way of contradiction, assume that 
\( a \notin J(R) \). Thus there is a left maximal ideal \( Q \) of \( R \) such that \( a \notin Q \).

Hence \( Ra + Q = R \) and therefore \( 1 = ra + q \), where \( r \in R \) and \( q \in Q \).
Since \( a \in \text{Ann}_R(Z_A) \), for every \( x \in Z_A \) it is \( qx = (1 - ra)x = x \). Thus,
since \( Z_A \) is essential in \( K_A \), \( q \), as endomorphism of \( K_A \), is injective
and \( \text{Im}(q) = K_A \). Let \( q' : \text{Im}(q) \to K \) be the left inverse of the cores-
triction of \( q \) to \( \text{Im}(q) \). Since \( K_A \) is q.i., \( q' \) extends to an endomor-
phism of \( K_A \). Thus \( 1 \in Q \). Contradiction.

12. REMARK. Let \( R \) be a linearly compact ring with respect to
a left linear topology \( \tau \) and \( \mathcal{F} \) be the filter of open left ideals of \( R \).
Let \( \mathcal{F} = \bigoplus_{s \in S_F} t_{\mathcal{F}}(E(S)) \) be the minimal cogenerator of \( \mathcal{G}_F \) and denote
by \( \mathcal{F}^* \) the filter of left ideals which are open in the \( U \)-topology of \( R \),
i.e. in the Leptin topology of \( \tau, \tau^* \) (cf. Lemma 5). Clearly, a left
simple \( R \)-module belongs to \( \mathcal{G}_F \) if and only if it belongs to \( \mathcal{G}_F^* \).
Moreover for each simple left \( R \)-module \( S \), \( t_{\mathcal{F}}(E(S)) = t_{\mathcal{F}^*}(E(S)) \).
In fact, since \( \mathcal{F}^* \subseteq \mathcal{F} \), \( t_{\mathcal{F}^*}(E(S)) \leq t_{\mathcal{F}}(E(S)) \). On the other hand,
\[ t_{\mathcal{F}}(E(S)) \leq R U \in \mathcal{G}_{F^*} \]

In particular \( R U \) is also the minimal cogenerator of \( \mathcal{G}_{F^*} \).

PROOF OF THE MAIN THEOREM. \((a) \Rightarrow (b)\). Let \( \mu K \) be a cogener-
ator of \( \mathcal{G}_F \) and let \( A = \text{End}(\mu K) \). By Lemma 6, the \( K \)-topology
of \( R \) is equivalent to \( \tau \) and hence \( R \) is linearly compact in the \( K \)-topo-
logy too. Thus, since \( \mu K \) is a selfcogenerator, by Corollary 7.4 [2],
\( R = \text{End}(K_A) \) and therefore \( R K_A \) is faithfully balanced. By Proposi-
tion 2, \( K_A \) is q.i.

\((b) \Rightarrow (c)\) is trivial.

\((c) \Rightarrow (a)\). Since \( \mu K \) is faithful, the \( K \)-topology of \( R \) is Haus-
dorff. By Lemma 6, \( \tau \) is equivalent to the \( K \)-topology of \( R \) and hence
\( \tau \) is Hausdorff too. Thus, by Proposition 4, \( R \) is linearly compact.

\((d) \Rightarrow (c)\) is trivial.

\((a) \Rightarrow (d)\). Let us remark, first of all, that in view of Lemma 5,
\( R \) is linearly compact in the \( U \)-topology. Let us now proceed by
steps.
1) \( rU \) is q.i. Set \( E = E \left( \bigoplus_{s \in S} S \right) \). Let us prove that \( rU = t_\mathcal{F}(E) \).

From this the claim will follow for \( rU \) will be a fully invariant submodule of \( rE \). Since \( rU = \bigoplus_{s \in S} t_\mathcal{F}(E(S)) \), it is clear that \( rU \leq t_\mathcal{F}(E) \).

Conversely, let \( x \in t_\mathcal{F}(E) \). Then \( \text{Ann}_R(x) \in \mathcal{F} \) and hence \( Rx \cong R/\text{Ann}_R(x) \) is linearly compact discrete. Thus \( \text{Soc}(Rx) \) is a direct sum of a finite number of (non-isomorphic) left simple \( R \)-modules.

Since \( x \in E \), \( \text{Soc}(Rx) \) is essential in \( Rx \). Thus \( x \in t_\mathcal{F}(E(\text{Soc}(Rx))) \leq rU \).

2) \( rU \) is s.q.i. By Remark 12 and by Theorem 6.7 [2].

3) \( rU_\tau \) is faithfully balanced. Since \( R \) is linearly compact in the \( U \)-topology, it is complete. Thus, since \( rU \) is a selfcogenerator, by Corollary 7.4 [2], \( R = \text{End}(U_\tau) \).

4) \( U_\tau \) is s.q.i. Note that \( \text{Soc}(rU) \) is essential in \( rU \). Then the claim follows by Theorem 10.

13. COROLLARY. Let \( R \) be a left linearly compact ring with respect to a ring topology \( \tau \), let \( S^r \) be the set of open left maximal ideals of \( R \) and let \( J(R) \) be the Jacobson radical of \( R \). Then

\[
J(R) = \bigcap \{ P : P \in S^r \}.
\]

In particular, \( J(R) \) is closed in \( R \).

**Proof.** Follows by The Main Theorem, by Corollary 11 and by Remark 12.

The idea of the following application is due to Prof. A. Orsatti.

14. **Theorem (Leptin [1]).** Let \( R \) be a left linearly topologized ring with respect to a ring topology \( \tau \). Assume that \( R \) is linearly compact and that the Jacobson radical of \( R \), \( J(R) \), is zero. Then \( R \), endowed with the Leptin topology of \( \tau \), is topologically isomorphic to a topological product \( \prod_{\lambda \in \Lambda} \text{End}_{D_\lambda}(V_\lambda) \) where, for every \( \lambda \in \Lambda \), \( V_\lambda \) is a vector space over the division ring \( D_\lambda \) and \( \text{End}_{D_\lambda}(V_\lambda) \) is endowed with the finite topology.

(Zelinsky [5]). Moreover if \( \tau \) has two-sided ideals as a basis of neighbourhoods of zero, each \( V_\lambda \) has finite dimension over \( D_\lambda \).

**Proof.** Let \( \mathcal{F} \) be the filter of open left ideals of \( \tau \) and let \( rU \) be the minimal cogenerator of \( \mathcal{C}_\mathcal{F} \). Set \( A = \text{End}(rU) \). By The Main
Theorem. \( _1U_2 \) is faithfully balanced and both the modules \( _1U_2 \) and \( U_2 \) are s.q.i. Suppose that \( \text{Soc} (U_2) \) is strictly contained in \( U_2 \). Then, since \( U_2 \) is s.q.i. and \( R = \text{End} (U_2) \), there is a non zero element \( r \in R \) such that \( r(\text{Soc} (U_2)) = 0 \). Thus, by 9., by Remark 12 and by Corollary 13, \( r \) belongs to the Jacobson radical of \( R \). Hence \( \text{Soc} (U_2) = U_2 \). Since \( \text{Soc} (U_2) = \text{Soc} (R U) \) (cf. 9.), we get \( _1U_2 = \bigoplus_{\lambda \in \Lambda} S_\lambda \), where \( (S_\lambda)_{\lambda \in \Lambda} \) is a system of representatives of the isomorphism classes of the left simple \( R \)-modules of \( \mathcal{C}_\mathcal{F} \). Thus each \( S_\lambda \) is fully invariant in \( _1U_2 \) and hence \( A \) is canonically isomorphic to the ring product \( \prod_{\lambda \in \Lambda} D_\lambda \) where, for each \( \lambda \in \Lambda \), \( D_\lambda = \text{End}_R (S_\lambda) \) is a division ring. Of course such a product acts componentwise over \( U \) so that the action of \( A \) over each \( S_\lambda \) naturally identifies with that of \( D_\lambda \). Recall that, by 9., \( S_\lambda = \sigma (S_\lambda^*) \). Moreover, since \( R = \text{End} (U_2) \), each \( S_\lambda \) is fully invariant submodule of \( U_2 \). Therefore we get the natural algebraic isomorphisms

\[
\text{End} (U_2) \cong \prod_{\lambda \in \Lambda} \text{End}_A (S_\lambda) = \prod_{\lambda \in \Lambda} \text{End}_{D_\lambda} (S_\lambda).
\]

Now, since \( _1U_2 \) is a selfcogenerator, by Corollary 7.4 [2], \( \text{End} (U_2) \), endowed with the finite topology, is isomorphic to the completion of \( R \) in the \( U \)-topology. Since \( R \) is linearly compact in \( \tau \), the first statement follows easily by Lemma 5, as soon as we note that the finite topology of \( \text{End} (U_2) \) corresponds, through the isomorphisms \( (1) \), to the product topology of the finite topologies on the \( \text{End}_{D_\lambda} (S_\lambda), \lambda \in \Lambda \).

Assume now that \( \tau \) has two-sided ideals as a basis of neighbourhoods of 0. Fix \( \lambda \in \Lambda \) and let \( P \in \mathcal{P} \) such that \( R/P \cong S_\lambda \). Since \( P \in \mathcal{F} \), \( P \) contains an open two-sided ideal. Since \( \text{Ann}_R (S_\lambda) \) is the largest two-sided ideal contained in \( P \), it follows that \( \text{Ann}_R (S_\lambda) \in \mathcal{F} \). Let \( \{ e_i \}_{i \in I} \) be a basis of \( S_\lambda \) as a vector space over \( D_\lambda \). Then \( \text{Ann}_R (S_\lambda) = \bigcap_{i \in I} \text{Ann}_R (e_i) \). Since \( (\text{Ann}_R (e_i))_{i \in I} \) is a family of open coprimary left ideals of \( R \) and \( R \) is linearly compact, it is easy to check that the diagonal map \( R/\text{Ann}_R (S_\lambda) \to \prod_{i \in I} R/\text{Ann}_R (e_i) \) of the canonical maps \( R/\text{Ann}_R (S_\lambda) \to R/\text{Ann}_R (e_i) (i \in I) \) is an isomorphism. Since \( R/\text{Ann}_R (S_\lambda) \) is linearly compact discrete, \( I \) must be finite.

15. Remarks. 1) In the hypothesis of Theorem above, if \( \tau \) is the discrete topology, then \( R \) is semisimple artinian ([5]). In fact, since \( _1U_2 \) is linearly compact discrete (cf. [3], Th. 1), \( A \) is finite.
2) In the hypothesis of Theorem above, if \( \tau \) has two-sided ideals as a basis of neighbourhoods of zero, then \( \tau = \tau^* \). In fact let \( L \) be an open two-sided ideal of \( R \). Then \( R/L \) is a discrete linearly compact ring with zero Jacobson radical. By 1) above, \( R/L \) is artinian. Thus \( L \) is cofinite.

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