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On the Behaviour of the Surfaces of Equilibrium in the Capillary Tubes when Gravity Goes to Zero.

MICHELE EMMER (*) (**)

Introduction.

In a previous work I have studied by a variational method the problem of the surfaces of equilibrium of a fluid in a capillary tube considering also the gravity force [1].

The functional of the energy is the following:

$$(0.1) \quad \mathcal{F}(f) = \int_{\Omega} \sqrt{1 + |Df|^2} + \frac{1}{2} \frac{\rho g}{\sigma} \int_{\Omega} f^2 dx - \nu \int_{\partial\Omega} f dH_{n-1}$$

where Ω is an open and bounded set in R^n , and

$$(0.2) \quad k = \frac{\rho g}{\sigma}$$

is the capillary constant, with $\rho =$ density difference across the free surface, $g =$ gravitational acceleration, $\sigma =$ surface tension; ν is also a constant depending only on physical conditions. The first integral in (0.1) represents the surface tension, the second the gravity and the third the force of adhesion to the wall of the tube.

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Moreover $f(x)$ is the free surface of the fluid in the tube whose section is Ω .

In [1] I have proved a theorem of existence and unicity for the minimum of the functional $\mathcal{F}(f)$ in the class of $BV(\Omega)$ functions with the following hypotheses:

- i) $\partial\Omega$ lipschitz;
- ii) $|\nu| < 1/\sqrt{1+L^2}$

where L is the lipschitz constant of $\partial\Omega$.

I have also proved a regularity theorem for the solution of the problem.

I recall these results in the following chapter.

In this work I will consider the behaviour of the surfaces of equilibrium in a capillary tube when the gravity goes to zero.

For each $\varepsilon > 0$ let f_ε minimize the functional

$$(0.3) \quad \mathcal{F}_\varepsilon(f) = \int_{\Omega} \sqrt{1 + |Df|^2} - \nu \int_{\partial\Omega} f \, dH_{n-1} + \varepsilon \int_{\Omega} f^2 \, dx .$$

REMARK 0.1. Obviously we have

$$(0.4) \quad \lim_{\varepsilon \rightarrow 0^+} \mathcal{F}_\varepsilon(f) = \mathcal{F}_0(f)$$

where

$$(0.5) \quad \mathcal{F}_0(f) = \int_{\Omega} \sqrt{1 + |Df|^2} - \nu \int_{\partial\Omega} f(x) \, dH_{n-1} .$$

It easily follows

$$(0.6) \quad \inf \{ \mathcal{F}_0(f); f \in BV(\Omega) \} = -\infty .$$

Hence it has no meaning to look for a function minimizing the functional $\mathcal{F}_0(f)$ without any other condition. Many authors have studied by different methods this problem for the functional $\mathcal{F}_0(f)$ with a volume constraint, that is with the condition

$$(0.7) \quad \int_{\Omega} f(x) \, dx = V .$$

(See [2], [3], [4], [5], [6], [7], [8], [9], [10], [11].)

In this work instead of considering a volume constraint I shall examine the behaviour of the free surface when the gravity goes to zero.

The intuitive fact that the surfaces f_ε rise in the tube as the gravity decreases, is first confirmed by the following remark:

$$(0.8) \quad \int_{\Omega} f_\varepsilon(x) dx = \frac{1}{2\varepsilon} \nu H_{n-1}(\partial\Omega).$$

So it obviously follows that

$$(0.9) \quad \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} f_\varepsilon(x) dx = +\infty$$

that is the volume of the fluid continues to increase.

The proof of the previous well known remark is recalled in the following chapter.

To prove the monotonicity of the family f_ε I will use a strong maximum principle for the minimal surfaces operator, analogous to the results of P. Concus and R. Finn [12], and a comparison theorem between the solutions of the minimum problem for the functional $\mathcal{F}_\varepsilon(f)$ with two different gravities.

Finally I can prove that

$$(0.10) \quad \lim_{\varepsilon \rightarrow 0^+} f_\varepsilon(x) = +\infty \quad \forall x \in \bar{\Omega}.$$

that is the liquid rises at every point of Ω ; in other words I can say that when the gravity goes to zero we do not obtain a limit surface.

To prove (0.10) I use a technique introduced by E. Gonzalez, U. Massari and I. Tamanini [13] in the study of sets minimizing perimeter and containing an assigned volume.

In my problem I do not consider a volume constraint but a mean curvature term so the technique is simplified.

I prove the result in various steps; first I suppose that (0.11) is not satisfied, that is it exists a number $L \in \mathbb{R}$ such that

$$(0.11) \quad \inf_{x \in \bar{\Omega}} f_\varepsilon(x) \leq L, \quad \forall \varepsilon > 0.$$

Then I prove Lemma 3.1 which states

$$(0.12) \quad \lim_{\varepsilon \rightarrow 0^+} v_\varepsilon = 0$$

where

$$(0.13) \quad v_\varepsilon = H_{n+1}(F_\varepsilon \cap (-\infty, L+1))$$

and

$$(0.14) \quad F_\varepsilon = \{(x, f_\varepsilon(x), x \in \bar{\Omega}, x > f_\varepsilon(x))\}.$$

Then by defining

$$(0.15) \quad a_0 = L + \frac{1}{2}, \quad b_0 = L + 1$$

it follows that for a sufficiently small $\varepsilon > 0$ there exists a point $\tau^* \in [a_0, b_0]$ such that (Proposition 3.2)

$$(0.16) \quad \int_{\tau=\tau^*} \varphi_{F_\varepsilon} dH_n = 0.$$

To prove Proposition 3.2 I first prove a technical Lemma.

I shall study in a following paper the problem of the behaviour of the surfaces of equilibrium in a capillary tube when the gravity goes to zero together with a volume constraint.

I want to thank M. Miranda and E. Gonzalez for the useful talks on the subject.

1. Let

i) Ω be an open and bounded set in R^n ($n > 2$) and $\partial\Omega$ be his boundary;

ii) $\partial\Omega$ be locally lipschitz, i.e. for every $x \in \partial\Omega$ there exists a sphere $B_\rho(x)$ such that $\partial\Omega \cap B_\rho(x)$ is the graph of a function $u: R^{n-1} \supset \mathcal{A} \rightarrow R$ where \mathcal{A} is an open set and u a lipschitz function.

iii) L be the maximum of the lipschitz constants for the functions u .

The functional will be considered in the natural space for variational problems of this kind:

$BV(\Omega)$ i.e. the space of functions $f \in L^1(\Omega)$ whose derivatives $D_1f, \dots, D_n f$ in the sense of distributions are finite Radon measures on Ω . For functions $f \in BV(\Omega)$ it is possible by a result of M. Miranda [14] to define the trace on $\partial\Omega$ and $f|_{\partial\Omega}$ is a function in $L^1(\Omega)$.

The symbol $\int_{\Omega} \sqrt{1 + |Df|^2}$ will indicate the total variation of the vector measures whose components are the Lebesgue measure and the measures $D_1f, \dots, D_n f$, i.e.

$$(1.1) \quad \int_{\Omega} \sqrt{1 + |Df|^2} = \sup_{\mathbf{R}^n} \left\{ \int_{\mathbf{R}^n} \left(f(x) \sum_{i=1}^n D_i g_i(x) + g_{n+1}(x) \right); \right. \\ \left. g \in C_0^1(\Omega); |g| \leq 1 \right\}.$$

Let us suppose the tube be made of an omogeneous material; this fact implies that ν is a constant. Physically ν represents the cosine of the angle between the exterior normal to the walls of the capillary tube and the normal to the free surface of the liquid [15].

Then it is clear that $\nu \in [-1, 1]$.

Let us consider the case $\nu > 0$ and therefore $f(x) > 0$. For $\varepsilon \in R$, $\varepsilon > 0$ let us consider the new functional

$$(1.2) \quad \mathcal{F}_{\varepsilon}(f) = \int_{\Omega} \sqrt{1 + |Df|^2} - \nu \int_{\partial\Omega} f dH_{n-1} + \varepsilon \int_{\Omega} f^2 dx$$

where the last integral represents the gravitational potential energy.

In [1] the author has demonstrated, using the following estimate,

$$(1.3) \quad \int_{\partial\Omega} |f| dH_{n-1} \leq \sqrt{1 + L^2} \int_{\Omega} |Df| + c(\Omega) \int_{\Omega} |f| dx$$

where $c(\Omega)$ is a constant depending on the geometry of Ω , the following result:

THEOREM [1]. i) If $0 < \nu \leq 1/\sqrt{1 + L^2}$ then

$$(1.4) \quad \inf_{BV(\Omega)} \mathcal{F}_{\varepsilon}(f) > -\infty.$$

ii) If $0 < \nu < 1/\sqrt{1 + L^2}$, then every minimizing sequence is compact. Moreover the functional $\mathcal{F}_\varepsilon(f)$ is lower semicontinuous.

Hence for every $\varepsilon > 0$ there exists a solution for the problem

$$(1.5) \quad \inf_{BV(\Omega)} \mathcal{F}_\varepsilon(f) = \mathcal{F}_\varepsilon(f_\varepsilon).$$

In the same work [1] it was also demonstrated the uniqueness and regularity of the solution using the regularity methods of U. Masari [16]. L. Pepe [17] has then proved that the solution is analytic in Ω in every dimension (see also [18]).

For other conditions on the domain for the existence of solutions see also I. Tamanini [19], R. Finn-C. Gerhardt [20].

For a discussion of the physical meaning see P. Concus - R. Finn [12]. For an history of capillarity phenomena see [15], [21].

2. Let us now examine the properties of the family $\{f_\varepsilon\}$.

First of all, as I have already said in the Introduction, I notice that

$$(2.1) \quad \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} f_\varepsilon(x) dx = +\infty.$$

In fact from the Euler equation of f_ε we obtain

$$(2.2) \quad \int_{\Omega} f_\varepsilon(x) dx = \frac{\nu}{2\varepsilon} H_{n-1}(\partial\Omega)$$

and (2.1) follows.

REMARK 2.1. The integral (2.2) represents the volume of the liquid which rises in the capillary tube.

I will now prove a strong maximum principle for the minimal surfaces operator; I will use this principle to demonstrate the monotonicity of the sequence $\{f_\varepsilon\}$.

PROP. 2.1. Let M be the minimal surfaces operator. If we have

$$(2.3) \quad Mf \geq Mg \quad \text{in } \Omega,$$

$$(2.4) \quad \frac{\nu Df}{\sqrt{1 + |Df|^2}} = \frac{\nu Dg}{\sqrt{1 + |Dg|^2}} \quad \text{in } \partial\Omega$$

then we necessarily obtain either

$$(2.5) \quad f(x) \equiv g(x) + c \quad c > 0 \text{ in } \Omega$$

or

$$(2.6) \quad f(x) \leq g(x) \quad \text{in } \Omega .$$

PROOF. From (2.3) and (2.4) we have, for every function $\phi \geq 0$, $\phi \in C_0^1(\Omega)$

$$(2.7) \quad \int_{\Omega} \frac{Df D\psi}{\sqrt{1 + |Df|^2}} dx \leq \int_{\Omega} \frac{Dg D\psi}{\sqrt{1 + |Dg|^2}} dx$$

and then

$$(2.8) \quad \int_{\Omega} \left(\frac{Df}{\sqrt{1 + |Df|^2}} - \frac{Dg}{\sqrt{1 + |Dg|^2}} \right) D\psi dx \leq 0 .$$

Let us choose now $\phi = (f - g)V_0$. We consider the case $f(x) > g(x)$; then we have

$$(2.9) \quad \int_{\Omega} \left(\frac{Df}{\sqrt{1 + |Df|^2}} - \frac{Dg}{\sqrt{1 + |Dg|^2}} \right) D(f - g) dx \leq 0 .$$

and, by the convexity of the area functional, we obtain

$$(2.10) \quad Df = Dg$$

and then

$$(2.11) \quad f(x) = g(x) + c, \quad c > 0$$

the connected components of Ω . Moreover

$$(2.12) \quad f(x) \equiv g(x) + c \quad c > 0 \text{ in } \Omega$$

as the set where (2.11) is valid must be simultaneously closed and open.

Let us now prove that when the gravity decreases the free surfaces of the fluid rise inside the tube, that is that the family of functions $\{f_\varepsilon\}$ is monotone.

PROPOSITION 2.2. Let f_1 be the solution of the problem $\inf \mathcal{F}_{\varepsilon_1}(f)$ and f_2 be the solution of the problem $\inf \mathcal{F}_{\varepsilon_2}(f)$. Let $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ be such that

$$(2.8) \quad \varepsilon_1 > \varepsilon_2 > 0$$

then

$$(2.9) \quad f_1(x) < f_2(x) \quad \text{in } \Omega .$$

DIM. We know from Proposition 2.1 that we can have

$$(2.10) \quad \begin{cases} f_1(x) = f_2(x) + c, & c > 0 \text{ in } \Omega, \\ Df_1(x) = Df_2(x) & \text{in } \Omega. \end{cases}$$

Now, because $f_1(x)$ minimizes the functional $\mathcal{F}_{\varepsilon_1}(f)$, we obtain

$$(2.11) \quad \int_{\Omega} \sqrt{1 + |Df_1|^2} dx + \varepsilon_1 \int_{\Omega} f_1^2 dx - \nu \int_{\partial\Omega} f dH_{n-1} \leq \\ \leq \int_{\Omega} \sqrt{1 + |Df_2|^2} dx + \varepsilon_1 \int_{\Omega} f_2^2 dx - \nu \int_{\partial\Omega} f dH_{n-1} .$$

In the same way we have

$$(2.12) \quad \int_{\Omega} \sqrt{1 + |Df_2|^2} dx + \varepsilon_2 \int_{\Omega} f_2 dx - \nu \int_{\partial\Omega} f dH_{n-1} \leq \\ \leq \int_{\Omega} \sqrt{1 + |Df_1|^2} dx + \varepsilon_2 \int_{\Omega} f_1 dx - \nu \int_{\partial\Omega} f_1 dH_{n-1} .$$

From (2.10), (2.11) we obtain

$$(2.13) \quad \varepsilon_1 \int_{\Omega} (f_2 + c)^2 dx - \nu \int_{\partial\Omega} (f_2 + c) dH_{n-1} \leq \varepsilon_1 \int_{\Omega} f_2^2 dx - \nu \int_{\partial\Omega} f_2 dH_{n-1}$$

and from (2.10), (2.12) it follows

$$(2.14) \quad \varepsilon_2 \int_{\Omega} f_2^2 dx - \nu \int_{\partial\Omega} f_2 dH_{n-1} \leq \varepsilon_2 \int_{\Omega} (f_2 + c)^2 dx - \nu \int_{\partial\Omega} (f_2 + c) dH_{n-1} .$$

Then by (2.13) we obtain

$$(2.15) \quad 2\varepsilon_1 C \int_{\Omega} f_2 dx + \varepsilon_1 C^2 \text{mis } \Omega - \nu CH_{n-1}(\partial\Omega) \leq 0$$

and by (2.14)

$$(2.16) \quad 2\varepsilon_2 C \int_{\Omega} f_2 dx + \varepsilon_2 C^2 \text{mis } \Omega - \nu CH_{n-1}(\partial\Omega) \geq 0.$$

Then, as $c > 0$, $\varepsilon_1 > \varepsilon_2 > 0$ we have a contradiction. So the only possibility is the following:

$$(2.17) \quad f_2(x) \geq f_1(x) \quad \text{in } \Omega.$$

So we have demonstrated that

$$(2.18) \quad f_\varepsilon(x) \text{ is increasing as } \varepsilon \rightarrow 0^+$$

together with

$$(2.19) \quad \int_{\Omega} f_\varepsilon(x) dx \rightarrow +\infty \quad \text{as } \varepsilon \rightarrow 0^+.$$

3. I can now prove that not only the volume of the liquid rises but also that the free surface of the liquid rises at every point of Ω .

As I have already said in the Introduction, in the proof I will use a technique introduced by E. Gonzalez, U. Massari and I. Tamani [13].

The main result is the following:

THEOREM 3.1. In the previous hypothesis we obtain the following behaviour for the free surface

$$(3.1) \quad \lim_{\varepsilon \rightarrow 0^+} f_\varepsilon(x) = +\infty, \quad \forall x \in \bar{\Omega}.$$

We shall prove the theorem by several steps.

Let us suppose that there exists a number $L \in \mathbb{R}$ such that

$$(3.2) \quad \inf_{x \in \bar{\Omega}} f_\varepsilon(x) < L, \quad \forall \varepsilon > 0.$$

Let us define

$$(3.3) \quad \begin{cases} F_\varepsilon = \{ (x, f_\varepsilon(x), x \geq f_\varepsilon(x), x \in \overline{\Omega} \}, \\ E_\varepsilon = \{ (x, f_\varepsilon(x), x \leq f_\varepsilon(x), x \in \overline{\Omega} \}. \end{cases}$$

The first step is to prove a Lemma which gives an estimate about the overgraph of f_ε when ε goes to zero. More precisely

LEMMA 3.1. Let

$$(3.4) \quad v_\varepsilon = H_{n+1}(F_\varepsilon \cap (-\infty, L + 1))$$

then

$$(3.5) \quad \lim_{\varepsilon \rightarrow 0^+} v_\varepsilon = 0 .$$

DIM.

OBS. 3.1. It is obvious by the hypothesis (3.2) that

$$(3.6) \quad v_\varepsilon > 0 .$$

Let us define

$$(3.7) \quad \tilde{f}_\varepsilon(x) = \begin{cases} f_\varepsilon(x) & \text{if } f_\varepsilon(x) > L + 1 , \\ L + 1 & \text{if } f_\varepsilon(x) \leq L + 1 . \end{cases}$$

As $f_\varepsilon(x)$ minimizes the functional $\mathcal{F}_\varepsilon(f)$ we obtain

$$(3.8) \quad \int_{\Omega} \int_0^{L+1} |D\varphi_{E_\varepsilon}| - \nu \int_{\partial\Omega} \int_0^{L+1} \varphi_{E_\varepsilon} dH_n \leq \int_{\partial\Omega} \varphi_{F_\varepsilon} dH_n - \\ - \nu(|\partial\Omega|(L + 1)) + \varepsilon \int_{\Omega} \int_0^{L+1} \varphi_{F_\varepsilon}(x, t) t \, dx \, dt$$

where $|\partial\Omega| = H_{n-1}(\partial\Omega)$, $\varphi_A(x)$ is the characteristic function of a set A , $\int_A |D\varphi_E|$ is the perimeter of the set E in the open set A , $P(E, A)$ [22].

Now as

$$(3.9) \quad \nu \left(|\partial\Omega|(L + 1) - \int_{\partial\Omega} \int_0^{L+1} \varphi_{E_\varepsilon} dH_n \right) = \nu \int_{\partial\Omega} \int_0^{L+1} \varphi_{E_\varepsilon} dH_n$$

and also

$$(3.10) \quad \int_{\Omega} \int_0^{L+1} \varphi_{F_\varepsilon}(x, t) t \, dx \, dt \leq (L+1) v_\varepsilon$$

we have

$$(3.11) \quad \int_{\Omega} \int_0^{L+1} |D\varphi_{E_\varepsilon}| + \nu \int_{\partial\Omega} \int_0^{L+1} \varphi_{F_\varepsilon} \, dH_n \leq \int_{t=L+1} \varphi_{F_\varepsilon} \, dH_n + \varepsilon(L+1) v_\varepsilon.$$

Let now $B(R, c)$ be a sphere whose radius is R and the center is $(0, \dots, c)$, $c > 0$ and such that

$$(3.12) \quad \begin{cases} H_n(B(R, c) \cap \{t = L+1\}) = \int_{t=L+1} \varphi_{F_\varepsilon}(x) \, dH_n, \\ H_{n+1}(B(R, c) \cap \{t < L+1\}) = v_\varepsilon. \end{cases}$$

REMARK 3.2.

$$(3.13) \quad \int_{t=L+1} \varphi_{F_\varepsilon}(x) \, dH_n \leq H_n(Q).$$

By conditions (3.12) on the sphere we have

$$(3.14) \quad \int_{\Omega} \int_0^{L+1} |D\varphi_{E_\varepsilon}| + \int_{\partial\Omega} \int_0^{L+1} \varphi_{F_\varepsilon} \, dH_n = \int_{\bar{\Omega} \times (0, L+1)} |D\varphi_{F_\varepsilon}| \geq \int_{t < L+1} |D\varphi_{B(R, c)}|.$$

Moreover it is obvious that

$$(3.15) \quad \int_{\Omega \times (0, L+1)} |D\varphi_{E_\varepsilon}| \geq \int_{t=L+1} \varphi_{F_\varepsilon} \, dH_n$$

then, by (3.12)

$$(3.16) \quad \int_{\Omega \times (0, L+1)} |D\varphi_{E_\varepsilon}| \geq H_n(B(R, c) \cap \{t = L+1\}).$$

Moreover by (3.11), (3.14) we obtain

$$(3.17) \quad \nu \int_{\partial\Omega} \int_0^{L+1} \varphi_{F_\varepsilon} dH_n + \nu \int_{\Omega} \int_0^{L+1} |D\varphi_{E_\varepsilon}| + (1-\nu) \int_{\Omega} \int_0^{L+1} |D\varphi_{E_\varepsilon}| \geq \\ \geq \nu \int_{t < L+1} |D\varphi_{B(R,c)}| + (1-\nu) \int_{\Omega} \int_0^{L+1} |D\varphi_{E_\varepsilon}|.$$

By (3.15) we also obtain, recalling (3.12),

$$(3.18) \quad \nu \int_{t < L+1} |D\varphi_{B(R,c)}| - H_n(B(R,c) \cap \{t = L+1\}) + \\ + H_n(B(R,c) \cap \{t = L+1\}) \leq \\ \geq H_n(B(R,c) \cap \{t = L+1\}) + \varepsilon(L+1)v_\varepsilon.$$

Then finally

$$(3.19) \quad 0 \leq \frac{\int |D\varphi_{B(R,c)}| - H_n(B(R,c) \cap \{t = L+1\})}{v_\varepsilon} \leq \varepsilon \frac{L+1}{\nu}.$$

In the end we obtain

$$(3.20) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{\int |D\varphi_{B(R,c)}| - H_n(B(R,c) \cap \{t = L+1\})}{v_\varepsilon} = 0.$$

By the definition of v_ε (3.12) we have

$$(3.21) \quad \lim_{\varepsilon \rightarrow 0^+} v_\varepsilon = 0.$$

So Lemma 3.1 is proved.

We must now prove a technical Lemma. Let us first define

$$(3.22) \quad a_0 = L + \frac{1}{2}, \quad b_0 = L + 1.$$

Let us consider three points t_1, t_2, t_3 s.t.

$$(3.23) \quad a_0 < t_1 < t_2 < t_3 < b_0$$

and let us define

$$(3.24) \quad \begin{cases} v_1 = |F_\varepsilon \cap (t_1, t_2)|, \\ v_2 = |F_\varepsilon \cap (t_2, t_3)|, \\ v = v_1 + v_2, \end{cases}$$

where

$$(3.25) \quad \begin{cases} (t_1, t_2) = \bar{\Omega} \times (t_1, t_2), \\ (t_2, t_3) = \bar{\Omega} \times (t_2, t_3). \end{cases}$$

Let

$$(3.26) \quad m = \max_{i=1,2,3} \int_{t=t_i} \varphi_{F_\varepsilon} dH_n.$$

We now demonstrate the following:

LEMMA 3.2. There exists a constant c_1 such that

$$(3.27) \quad v_1 \wedge v_2 < c_1(m + b_0 v)^N$$

where $N = (n + 1)/n$.

PROOF. Let

$$(3.28) \quad \begin{cases} F_1 = F_\varepsilon \cap (t_1, t_2), \\ F_2 = F_\varepsilon \cap (t_2, t_3). \end{cases}$$

We now consider two spheres $B_1, B_2 \subset R^{n+1}$ s.t.

$$(3.29) \quad |B_i| = |F_i| = v_i, \quad i = 1, 2.$$

By the isoperimetric property of the sphere we obtain

$$(3.30) \quad \int_{R^{n+1}} |D\varphi_{B_i}| \leq \int_{R^{n+1}} |D\varphi_{F_i}|, \quad i = 1, 2$$

i.e.

$$(3.31) \quad (n + 1) \omega_{n+1}^{1/(n+1)} v_i^{n/(n+1)} \leq \int_{(t_i, t_{i+1})} |D\varphi_{F_i}| + \int_{t=t_i} \varphi_{F_\varepsilon} dH_n + \int_{t=t_{i+1}} \varphi_{F_\varepsilon} dH_n, \quad i = 1, 2.$$

By adding the two relationships (3.31) for $i = 1, 2$, we have

$$(3.32) \quad \int_{(t_1, t_2)} |D\varphi_{F_\varepsilon}| \geq (n+1)\omega_{n+1}^{1/(n+1)}(v_1^{n/(n+1)} + v_2^{n/(n+1)}) - \\ - \int_{t=t_1} \varphi_{F_\varepsilon} dH_n - \int_{t=t_2} \varphi_{F_\varepsilon} dH_n - 2 \int_{t=t_2} \varphi_{F_\varepsilon} dH_n .$$

REMARK 3.3. The set F_ε is the complementary set of E_ε , then by the minimizing property of E_ε we obtain that F_ε realizes the minimum for the following functional:

$$(3.33) \quad \int_{\Omega \times (0, +\infty)} |D\varphi_F| + \nu \int_{\partial\Omega} \varphi_F dH_{n-1} - \varepsilon \int_{\Omega \times (0, +\infty)} t\varphi_F dx dt$$

among all the overgraphs F such that $F \Delta F_\varepsilon \subset \subset \Omega \times (0, +\infty)$.

Then by Remark 3.3 we obtain

$$(3.34) \quad \int_{\Omega \times (t_1, t_2)} |D\varphi_{F_\varepsilon}| - \varepsilon \int_{t=t_2} t\varphi_{F_\varepsilon} dx dt + \nu \int_{\partial\Omega \times (t_1, t_2)} \varphi_{F_\varepsilon} dH_n \leq \int_{t=t_1} \varphi_{F_\varepsilon} dH_n + \int_{t=t_2} \varphi_{F_\varepsilon} dH_n$$

and then

$$(3.55) \quad \int_{\Omega \times (t_1, t_2)} |D\varphi_{F_\varepsilon}| + \nu \int_{\partial\Omega \times (t_1, t_2)} \varphi_{F_\varepsilon} dH_n \leq \int_{t=t_1} \varphi_{F_\varepsilon} dH_n + \int_{t=t_2} \varphi_{F_\varepsilon} dH_n + \varepsilon \int_{(t_1, t_2)} t\varphi_{F_\varepsilon} dx dt .$$

Now by the fact that

$$(3.36) \quad \int_{\Omega \times (t_1, t_2)} |D\varphi_{F_\varepsilon}| + \nu \int_{\partial\Omega \times (t_1, t_2)} \varphi_{F_\varepsilon} dH_n = \int_{\bar{\Omega} \times (t_1, t_2)} |D\varphi_{F_\varepsilon}|$$

we also have

$$(3.37) \quad \int_{(t_1, t_2)} |D\varphi_{F_\varepsilon}| \leq \int_{t=t_1} \varphi_{F_\varepsilon} dH_n + \int_{t=t_2} \varphi_{F_\varepsilon} dH_n + \varepsilon \int_{(t_1, t_2)} t\varphi_{F_\varepsilon} dx dt .$$

From (3.31), (3.37), and also considering that $0 < \nu < 1$, we

obtain

$$(3.38) \quad \int_{t=t_1} \varphi_{F_\varepsilon} dH_n + \int_{t=t_2} \varphi_{F_\varepsilon} dH_n + \varepsilon \int_{(t_1, t_2)} t \varphi_{F_\varepsilon} dx dt \geq \\ \geq \nu(n+1) \omega_{n+1}^{1/(n+1)} (v_1^{n/(n+1)} + v_2^{n/(n+1)}) - \nu \int_{t=t_1} \varphi_{F_\varepsilon} dH_n - 2\nu \int_{t=t_2} \varphi_{F_\varepsilon} dH_n - \nu \int_{t=t_2} \varphi_{F_\varepsilon} dH_n$$

and then

$$(3.39) \quad (1 + \nu) \left\{ \int_{t=t_1} \varphi_{F_\varepsilon} dH_n + \int_{t=t_2} \varphi_{F_\varepsilon} dH_n \right\} + \\ + 2\nu \int_{t=t_2} \varphi_{F_\varepsilon} dH_n \geq \nu(n+1) \omega_{n+1}^{1/(n+1)} (v_1^{n/(n+1)} + v_2^{n/(n+1)}) - \varepsilon \int_{(t_1, t_2)} t \varphi_{F_\varepsilon} dx dt.$$

Obviously we have

$$(3.40) \quad (n+1) \omega_{n+1}^{1/(n+1)} (v_1^{n/(n+1)} + v_2^{n/(n+1)}) \geq (n+1) \omega_{n+1}^{1/(n+1)} 2(v_1 \wedge v_2)^{n/(n+1)}.$$

Moreover $t < L + 1$; then, by considering (3.29), (3.40) and the definition of v , we obtain

$$(3.41) \quad 2\nu(n+1) \omega_{n+1}^{1/(n+1)} (v_1 \wedge v_2)^{n/(n+1)} \leq (L+1)v + (1+\nu)4m$$

and finally

$$(3.42) \quad (v_1 \wedge v_2) \leq c_1 ((L+1)v + m)^{n/(n+1)}$$

where

$$(3.43) \quad c_1 = c_1(n, \nu).$$

So we have proved Lemma 3.3.

I can now prove Theorem 3.1; by the regularity of the functions f_ε it is sufficient to prove the following:

PROPOSITION 3.1. Let $\varepsilon > 0$ be sufficiently small, then there exists $t^* \in [a_0, b_0]$ such that

$$(3.44) \quad \int_{t=t^*} \varphi_{F_\varepsilon} dH_n = 0.$$

PROOF. We shall construct the point t^* as the limit of two sequences $\{a_j\}$ and $\{b_j\}$, the first one increasing and the second one decreasing towards t^* and also such that

$$(3.45) \quad \lim_{j \rightarrow +\infty} \int_{t=a_j} \varphi_{F_\varepsilon} dH_n = \lim_{j \rightarrow +\infty} \int_{t=b_j} \varphi_{F_\varepsilon} dH_n = 0.$$

In order to construct the two sequences $\{a_j\}$ and $\{b_j\}$ we shall use an iterative method starting with $a_0 = L + \frac{1}{2}$, $b_0 = L + 1$.

Let us now suppose to have constructed a_j and b_j with $j \geq 0$ and $a_0 \leq a_j < b_j \leq b_0$; let us define

$$(3.46) \quad \begin{cases} l_j = b_j - a_j, \\ v_j = |F_\varepsilon \cap (a_j, b_j)|. \end{cases}$$

Let now be $h_j \in (0, l_j/3)$; we can find three points t_i^j ($i = 1, 2, 3$) s.t.

$$(3.47) \quad \begin{cases} t_1^j \in (a_j, a_j + h_j), \\ t_2^j \in \left(a_j + \frac{l_j}{2} - \frac{h_j}{2}, a_j + \frac{l_j}{2} + \frac{h_j}{2} \right), \\ t_3^j \in (b_j - h_j, b_j), \end{cases}$$

and moreover

$$(3.48) \quad \int_{t=t_i^j} \varphi_{F_\varepsilon} dH_n \leq \frac{v_j}{h_j} \quad (i = 1, 2, 3).$$

Let us define

$$(3.49) \quad m_j = \max_{i=1,2,3} \int_{t=t_i^j} \varphi_{F_\varepsilon} dH_n,$$

$$(3.50) \quad v_i^j = |F_\varepsilon \cap (t_i^j, t_{i+1}^j)| \quad (i = 1, 2).$$

Obviously from (3.48) we have

$$(3.51) \quad m_j \leq \frac{v_j}{h_j}.$$

Let

$$(3.52) \quad v_{j+1} = v_1^j \wedge v_2^j.$$

By Lemma 3.2 we have

$$(3.53) \quad v_{j+1} \leq C_1(m_j + b_0 v_j)^N \leq C_2 \left(\frac{v_j}{h_j} \right)^N$$

as $h_j < 1$. Let us now define the points a_{j+1}, b_{j+1}

$$(3.54) \quad (a_{j+1}, b_{j+1}) = \begin{cases} (t_1^j, t_2^j) & \text{if } v_1^j \leq v_2^j, \\ (t_2^j, t_1^j) & \text{otherwise.} \end{cases}$$

Now we want to estimate the quantities (3.46), (3.49), (3.50). We have

$$(3.55) \quad 2^j l_j \geq l_0 - 3 \sum_{i=0}^{j-1} 2^i h_i.$$

Moreover by (3.53) we obtain

$$(3.56) \quad v_{j+1} \leq C_2^{\sum_{i=0}^j N^i} \frac{v_0^{N^{j+1}}}{\prod_{i=0}^j h_i^{N^{j+1-i}}}$$

then by (3.51), (3.56) we obtain

$$(3.57) \quad m_{j+1} \leq \frac{\tilde{C} v_0^{N^{j+1}}}{\prod_{i=0}^{j+1} h_i^{N^{j+1-i}}} = \left(\frac{\tilde{C} v_0}{\prod_{i=0}^{j+1} h_i^{N^{-i}}} \right)^{N^{j+1}}.$$

Let us now choose

$$(3.58) \quad h_j = \frac{l_0}{k} \left(\frac{1}{4} \right)^j.$$

It remains now to fix k such that

$$(3.59) \quad 0 < h_j < l_j/3.$$

By the estimate (3.55) we have

$$(3.60) \quad 3h_j \leq \frac{l_0 - 3 \sum_{i=0}^{j-1} 2^i h_i}{2^j}$$

then, remembering (3.58)

$$(3.61) \quad k \geq 3 + 6 \left(\frac{2^j - 1}{2^j} \right), \quad \forall j$$

$$(3.62) \quad k \geq 9.$$

Let us put $k = 9$ and then

$$(3.63) \quad h_i = \frac{l_0}{k} \left(\frac{1}{4} \right)^i.$$

We have now to prove that

$$(3.64) \quad \lim_{j \rightarrow +\infty} m_j = 0.$$

Now by (3.57), (3.63) we have

$$(3.65) \quad m_{j+1} \leq \left(\frac{\tilde{C} v_0}{(l_0/9)^{n+1}} \cdot \frac{1}{(4)^{n(n+1)}} \right)^{N^{j+1}}$$

and then

$$(3.66) \quad m_{j+1} \leq \left(\bar{C} \frac{v_0}{l_0^{n+1}} \right)^{N^{j+1}}$$

where

$$(3.67) \quad v_0 = |F_\varepsilon \cap (a_0, b_0)| = |F_\varepsilon \cap (L + \frac{1}{2}, L + 1)|$$

then

$$(3.68) \quad v_0 \leq v_\varepsilon.$$

Remembering the Lemma 3.1 we have

$$(3.69) \quad \lim_{\varepsilon \rightarrow 0^+} v_0 = 0$$

and then

$$(3.70) \quad \lim_{j \rightarrow +\infty} m_j = 0 .$$

So Theorem 3.1 is proved.

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